U.P.B. Sci. Bull., Series A, Vol. 69, No. 4, 2007 ISSN 1223-7027

ON THE TIME/FREQUENCY SIMULTANEOUS ALIGNMENT OF THE SIGNALS COMPORTMENT

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În această lucrare, conceptul de "atom-frecvență" (T/F), elaborat de J. von Neumann şi D. Gabor, este punctat în termeni matematici; de asemenea, unele proprietăți ale transformatei Fourier cu fereastră sunt scoase în evidență (Proposition 2 şi Proposition 3); în final este prezentată o aplicație a undinelor pentru descrierea mecanismului fiziologic al urechii umane în timp-frecvență.

In this paper , the "atom-frequency" (T/F) concept, elaborated by J. von Neumann and D. Gabor, is pointed out in mathematical terms; also, some properties of Fourier transformation with window are marked out (Proposition 2 and Proposition 3); finally, we give an application of wavelets for (T/F) human ear physiological mechanism description.

Keywords: wavelets, „attenuated sine".

1. Introduction

If $u : R \to C$ is a $L^1(R)$-class function, ad-hoc called signal, then we assign its Fourier transformation $\hat{u} : R \to C$, which is a continuous and bounded function, defined by

$$\hat{u}(\omega) = \int_{-\infty}^{\infty} u(t), \tag{1}$$

this improper integral being convergent for any $\omega \in R$. The function $\hat{u}$ is called the frequency spectrum of the signal u, and $A(\omega) = |\hat{u}(\omega)|$, the frequency amplification of u.

An insufficiency of the classic Fourier transformation is constituted by the fact that we have to know the values of u for entire time axis (according to (1)) if we want to calculate the spectrum $u(\omega_0)$ in only one frequency $\omega_0 \in R$. Applying the Fourier inversion formula in adequate conditions (for example, if u is a continuous function and $u \subset L^1(R) \cup L^2(R))$,

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$$u(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(\omega)\, e^{i\omega t}\, d\omega, \tag{2}$$

it turns out that to determinate the sample $u(t_0)$ for a time moment $t_0 \in R$ it is necessary to know the spectrum $\hat{u}(\omega)$ in entire frequency band.

Thus, we consider the pairs (t_0, ω_0), simultaneous taken, and analyse the time/frequency comportment of some signals in their neighbourhood.

2. Time-Frequency Atoms

It is considered a plan related to an orthogonal fixed point where by abscissa is mentioned the generic time t, and by ordinate line, the generic frequency ω. To represent a signal in an orthogonal plane $t0\omega$ means to take simultaneous its time duration and its frequency ratio (human voice case). We fix a pair $(t_0, \omega_0) \in R \times R$.

Intuitively, an (T/F) atom around the point (t_0, ω_0) is any signal u (from $L^1(R) \cup L^2(R)$) with compact support, which contains t_0 (so u is null out of the support); moreover, $\hat{u}$ has to have compact support, which contains ω_0 (so frequencies $\hat{u}(\omega)$ are insignificant out of the support). According to the indeterminism principle, such a non-zero signal doesn't exist, no matter how small are the supports of u and $\hat{u}$.

John von Neumann called T/F atom any family of functions by the form $\{e^{i\omega_0 t} \cdot u(t - t_0)\}$, with $(t_0, \omega_0) \in R \times R$, where $u(t)$ is a fixed function (from $L^1(R) \cup L^2(R)$-class). Distributing the points (t_0, ω_0) uniformly in the $tO\omega$ plan , J. von Neumann has recommanded, in the Signals Theory, to use an orthonormated base in the Hilbert space $L^2(R)$ relative to the dot product $< f, g >= \int_{-\infty}^{\infty} f(t) \cdot \overline{g(t)} dt$, made of T/F atoms.

Proposition 1. Let be $u(t) = \frac{\sin \pi t}{\pi t}$ (for $t \neq 0$) and $u(0) = 1$. The T/F atoms

$$u_{lk}(t) = e^{2\pi l t} \cdot u(t - k)\ ;\ k, l \in Z, \tag{3}$$

(corresponding to the values $t_0 = k,\ \omega_0 = 2\pi l$), make an orthonormated base for $L^2(R)$.

Proof. Using the Parseval formula, it results immediately that $< u_{lk}, u_{pg} >= \delta_{lp} \cdot \delta_{kg}$.

We have a representation by the form $f(t) = \sum_{k,l} c_{lk} u_{lk}(t)$ for any signal $f \in L^2(R)$; the coefficients c_{lk} are immediately deduced from $c_{lk} =< f, u_{lk} >$, for any $k, l \in Z$. Therefore, any continual (analogical) signal f is identified by the sequence c_{lk}, which is an illustration of the delator phenomenon named analogical/digital conversion of the signals.

For the signal $u(t)$ from **Proposition 1** (called „attenuated sine"), we have

$$\hat{u}(\omega) = \begin{cases} 1, & \text{if } \omega \subset (\pi, \pi), \\ 0, & \text{in rest} \end{cases}$$

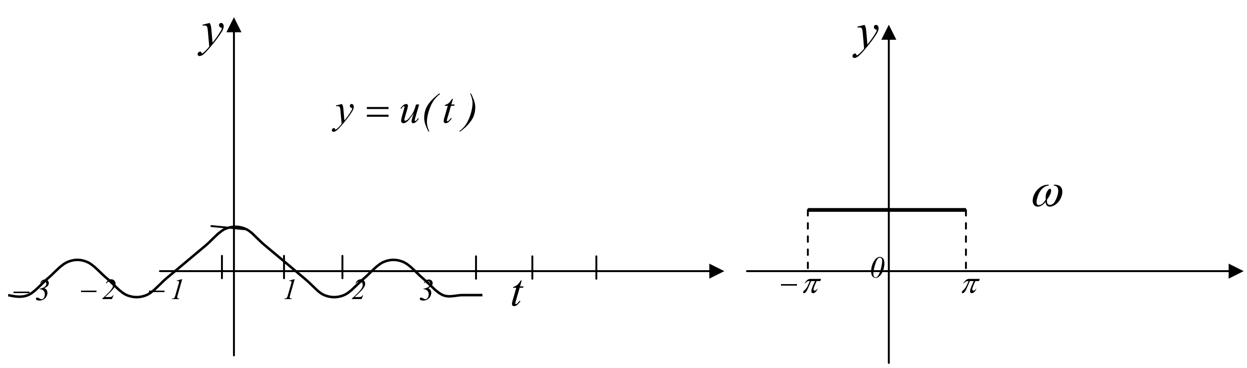
and for any $k, l \in Z$ fixed, it results that

$$\hat{u}_{lk}(\omega) = \int_{-\infty}^{\infty} u_{lk}(t) \cdot e^{-i\omega t} dt = \int_{-\infty}^{\infty} e^{2\pi i l t} \cdot u(t-k) \cdot e^{-i\omega t} dt;$$

making a change of variable, $t - k = \tau$, we obtain

$$\hat{u}_{lk}(\omega) = e^{-ik(\omega - 2\pi l)} \cdot \hat{u}(\omega - 2\pi l),$$

for any $k, l \in Z$. The graphics of the functions u and $\hat{u}$ are indicated in **Fig. 1; a), b).**



a) Fig. 1. b)

The signal $\hat{u}(t)$ has a good position in $t = 0$ and it is insignificant out of the system $[-1,1]$; $u(t-k)$ is the translation of u with k time units and it is well localized in the point k. Therefore, $u_{lk}(t)$ is well localized in k and it is insignificant out of the interval $[k-1,k+1]$, for $k \in Z$.

In the same way, $\hat{u}(\omega - 2\pi l)$ is null for $\omega - 2\pi l \notin [-\pi,\pi]$, hence $\hat{u}_{lk}(\omega)$ is null out of the interval $[-\pi + 2\pi l, \pi + 2\pi l]$, which means that $\hat{u}_{lk}(\omega)$ is well localized around the $2\pi l$ frequency.

Let us consider now rectangles from the $tO\omega$ plan, hachured like in Fig. 2 and centered in the points $(k, 2\pi l)$, with $k,l \in Z$. In this way, the time/frequency T/F plan, identified with the $tO\omega$ plan, is parried with rectangles, like in Fig. 2.

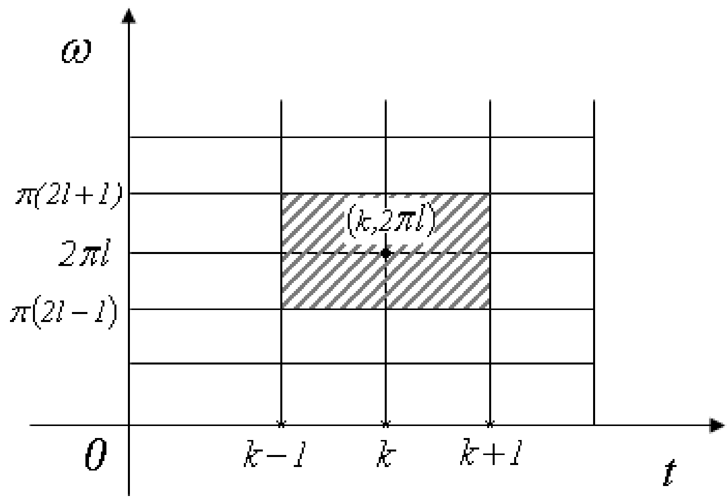


Fig. 2.

These kind of rectangles can be covered between themselves (so that they can't make a plan partition).

The orthonormated base u_{lk}, $l,k \in Z$ from the **Proposition 1** presents six disadvantages, which are connected with weak convergence in dots products calculus $< t., u_{lk} >$, necessary for signals digital representation. Also, the fact that all the T/F atoms have the same duration is an impediment in some type of applications (for example, in Geophysics or Radar). That's why we proposed some other kind of orthonormated bases for $L^2(R)$.

3. T/F Transformation

D. Gabor proposed the replacement of the discret values $k,l \in Z$ with continuous variables $\xi, \tau \in R$, considering T/F atoms by the type

$w_{\xi\tau}(t) = e^{i\xi t} \cdot w(t-\tau)$, where $w(t) \in L^2(R)$ is a signal with the norm $\|w\|_2 = \frac{1}{\sqrt{2\pi}}$ (called window). For any signal $u \in L^2(R)$, a function with two real variables, W_u, defined by

$$W_u(\tau,\xi) = < u, w_{\xi\tau} > = \int_{-\infty}^{\infty} u(t) \cdot e^{-i\xi t} \cdot \overline{w(t-\tau)} dt, \quad (4)$$

was called the T/F transformation of u, with the window w fixed (or equivalent of Fourier transformation with window).

It may be remarked the analogy with the relation (1). D. Gabor proved , knowing W_u (similarly with (2)), the recuparation of u formula, namely

$$u(t) = \iint_{R^2} W_u(\tau,\xi) \cdot w_{\xi\tau}(t) d\tau d\xi. \quad (5)$$

Note: If $w \equiv 1$ (constant function), we have $W_u(\tau,\xi) = \hat{u}(\xi)$ (Fourier classic transformation) and if $w = \tau$ (Dirac distribution), then $W_u(\tau,\xi) = u(\tau) \cdot e^{-i\xi\tau}$. In the two cases, w doesn't belong to the space $L^2(R)$.

We fix $\tau \in R$ and $a > 0$. We choose a window $w : R \to R$ which has to have its support contained in the interval $[-\tau, -\tau + a]$. Then, for $\xi = \frac{2\pi n}{a}$, $n \in Z$, we have (according to (4))

$$W_u(\tau, \frac{2\pi n}{a}) = \int_{-\infty}^{\infty} u(t) \cdot \overline{w(t-\tau)} \cdot e^{-\frac{2\pi i n}{a}t} dt = \int_{0}^{a} u(t) \cdot \overline{w(t-\tau)} \cdot e^{-\frac{2\pi i n}{a}t} dt .$$

Proposition 2. Let be $u \in L^2(R)$ and c_n the Fourier complex coefficients of the function $u(t) \cdot \overline{w(t-\tau)}$ restricted to the interval $[0, a]$ and then extended to R by its periodicity. In these conditions ,

$$W_u(\pi, \frac{2\pi n}{a}) = a \cdot c_n, \text{ for any } n \in Z . \quad (6)$$

Proof. The demonstration results directly from definitions.

Therefore, knowing W_u, we can determinate the coefficients c_n using the relation (6); the signal $u(t)$ is recovered from its Fourier coefficients:

$$u(t) = \sum_{n \in Z} c_n \cdot e^{in\frac{2\pi}{a}t} .$$

In other words, choosing convenable windows, from data about the signal $u(t)$, we can find out local data about its T/F transformation, W_u, and conversely.

Now we fix a window w. For any fixed $t, \omega \in R$, we can consider the function $h_{t,\omega} : R \to C$ defined by

$$h_{t,\omega}(\tau) = \overline{w(\tau - t)} \cdot e^{i\omega\tau} .$$

For any vector $h \in L^2(R)$, we note with h^* the functional defined by $h^* =< x, h >$. With these notations , we can write:

Proposition 3. For any window w, we have the relation

$$\iint_{R^2} h_{t,\omega} \cdot h_{t,\omega}^* d\omega dt = I ,$$

where I is the identity on the Hilbert space $L^2(R)$.

Proof. Let us take $u \in L^2(R)$, arbitrary fixed. We have to prove that

$$u(\tau) = \iint_{R^2} h_{t,\omega}(\tau) \cdot h_{t,\omega}^*(u) d\omega dt . \qquad (7)$$

Be it $u_t(\tau) = u(\tau) \cdot w(\tau - t)$. Then $W_u(t,\omega) = \hat{u}_t(\tau)$, and, according to the relation (2), it results that $u_t(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} W_u(t,\omega) \cdot e^{i\tau\omega} d\omega$. If we multiplicate with $\overline{w(\tau - t)}$ and integrate related with t, we obtain

$$\int_{-\infty}^{\infty} u_t(\tau) \cdot \overline{w(\tau - t)} dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} W_u(t,\omega) \overline{w(\tau - t)} \cdot e^{i\tau\omega} d\omega,$$

which means

$$u(\tau) \cdot \|w\|_2^2 = \frac{1}{2\pi} \iint_{R^2} W_u(t,\omega) \cdot h_{t,\omega}(\tau) dt d\omega ;$$

we also know that $\|w\|_2 = \frac{1}{\sqrt{2\pi}}$. That’s why we can write now the following relation:

$$u(\tau) = \iint_{R^2} h_{t,\omega}(\tau) \cdot W_u(t,\omega) dt d\omega \ .$$

But

$$W_u(t,\omega) = \int_{-\infty}^{\infty} w(\tau - t) \cdot u(\tau) \cdot e^{-i\omega\tau} d\tau = \int_{-\infty}^{\infty} \overline{h_{t,\omega}(\tau)} \cdot u(\tau) d\tau =$$

$$=< u, h_{t,\omega} >= h_{t,\omega}^*(u) ,$$

and the relation (7) is now proved.

Corollary. For any signal $u \in L^2(R)$, its energy $E(u) = \|u\|_2^2$ is given by the formula

$$E(u) = \iint_{R^2} |W_u(t,\omega)|^2 \, dt d\omega .$$

Proof. We have $|W_u(t,\omega)|^2 = \overline{W_u(t,\omega)} \cdot W_u(t,\omega) = \overline{h^*_{t,\omega}(u)} \cdot h^*_{t,\omega}(u)$; so, $\iint_{R^2} |W_u(t,\omega)|^2 \, dt d\omega = \|u\|^2$.

The size $|W_u(t,\omega)|^2$ has the next phisycal interpretation: it is the energy density of the signal u related to the time unit in T/F plan.

Note: The T/F transformation works with a fixed duration of the window, meaning that we have to consider only $w(t-b)$ translations of the window if we want to calculate $W_u(b,\xi)$; this can be an inconvenient in T/F analysis of some signals $u(t)$ with high variations in short intervals of time (like in Geophysics, Radar, Human voice, etc.). This was one of the reasons which determinated us to propose more flexible windows, which can be translatated and also, delated (or contracted); this fact marked the appearance of the wavelet concept.

We fix a window-function $\Psi : R \to R$ so that $\Psi(t)$ and $t\Psi(t)$ belong to $L^2(R)$; moreover, $\hat{\Psi}(0) = 0$; the function Ψ was called wavelet. For any signal $u \in L^2(R)$, we define the transformation of u by the wavelet Ψ as the next function of two real variables a, b, with $a > 0$:

$$W_u(a,b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} u(t) \cdot \Psi\left(\frac{t-b}{a}\right) dt . \tag{8}$$

Applying the Parseval classic relation, we obtain

$$W_u(a,b) = \frac{\sqrt{a}}{2\pi} \int_{-\infty}^{\infty} \hat{u}(\omega) \cdot e^{-ib\omega} \cdot \overline{\hat{\Psi}(a\omega)} d\omega =$$

$$= \frac{\sqrt{a}}{\pi} \operatorname{Re} \int_{-\infty}^{\infty} \hat{u}(\omega) \cdot e^{-ib\omega} \cdot \overline{\hat{\Psi}(a\omega)} d\omega .$$

This relation shows the connection with the T/F transformation.

4. Application

Let x be the location of a sensitive cell in the spiral shell cortex of the human ear. We suppose that the audio signal received in x, $g_x(t)$, is the convolution of an acoustic signal $u(t)$ with a linear filter which depends by the location x, and has the transfer function $H_x(\omega)$. Then, in the frequency domain, we have the relation

$$\hat{g}_x(\omega) = \hat{u}(\omega) \cdot H_x(\omega). \tag{9}$$

Suppose that by the spiral shell geometry itself we have a delay in frequency, i.e. there exists a function G so that $H_x(\omega) = G(x - \ln \omega)$. If we note $a = e^{-x}$, results that $x = \ln \frac{1}{a}$ and we consider a wavelet $\Psi(t)$ with

$$\hat{\Psi}(\omega) = \overline{G\left(\ln \frac{1}{\omega}\right)}. \tag{10}$$

Then, using (9), results $\hat{g}_x(\omega) = \hat{\Psi}_x(\omega) \cdot \hat{u}(\omega)$ and from the inversion Fourier formula (2), we obtain the relation

$$g_x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \cdot \overline{\hat{\Psi}(a\omega)} \cdot \hat{u}(\omega) d\omega =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(\omega) \cdot \overline{\hat{\Psi}(a\omega) e^{-i\omega t}} d\omega = \frac{1}{a} \int_{-\infty}^{\infty} u(\tau) \cdot \Psi\left(\frac{\tau - t}{a}\right) d\tau.$$

Using again the Parseval relation, results that the reception at the sensorial cell localized in x is given by

$$g_x(t) = e^{-x/2} W(t, e^{-x}).$$

The construction of the $W(t)$ wavelet with the property (10) is still an open problem.

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