CONCERNING A PROBLEM OF E. BOREL

C. PANA

It is widely known the special role of orthonormal basis in certain spaces of functions; they allow an elegant and efficient approach of analogic - digital conversion of signals.

In 1910 E. Borel considered a series of functions \( f_n \) from \( L^2(a,b) \) which will form an orthonormal basis and, moreover each function \( f_n \) will have at most two values.

The first example was given by Walsh. In this paper we have another description of Walsh functions and a new proof of the completeness of Walsh system. In the same time the connection between Rademacher and Haar functions used in the study of discrete signals is shown.

Keywords: orthonormal basis, wavelets, Walsh functions, Rademacher functions, Haar functions.

AMS: 42 C 10

Teacher, National College “Mircea cel Batrân”, Râmnicu Vâlcea, ROMANIA
Introduction

The Walsh functions are used in Electronics at connecting problems in data transmissions. This paper gives an original proof that Walsh functions form an orthonormal base in $L^2_{(0,1)}$.

1. Preliminaries

Let $H$ be a Hilbert space (complex); $(e_n)_{n\in\mathbb{N}}$ an orthonormal string in $H$ and $(a_n)_{n\in\mathbb{N}}$ a string of complex numbers. The series $\sum_n a_n e_n$ is known to converge in $H$ with the sum $s$ if and only if $(a_n)\in l^2$; moreover $a_n = \langle s, e_n \rangle$.

If $(e_n)$ is an orthonormal basis and if there is $t \in H$ so that $a_n = \langle t, e_n \rangle$ for any $n$, then $t = s$.

The set of indexes can be replaced with any other countable set. We also remind the following fact:

2. Lemma

Let $B = (e_n)$ an orthonormal string in $H$. The following 5 assertions are equivalent:

a) The subspace generated by $B$ is dense in $H$.

b) If $x \in H$ and $x \perp e_n$, $(\forall) n$ then $x = 0$.

c) $\forall x \in H$, $\sum_n |a_n|^2 = \|x\|^2$, where $a_n = \langle x, e_n \rangle$.

d) $\forall x \in H$, $x = \sum_n c_n e_n$, where $c_n = \langle x, e_n \rangle$.

e) $\forall x, y \in H$, $\langle x, y \rangle = \sum_n c_n \overline{d_n}$, where $c_n = \langle x, e_n \rangle$, $d_n = \langle y, e_n \rangle$.

Any string of vectors from $H$ satisfying one of these 5 conditions is called orthonormal basis in $H$. 
3. Examples

1) Let $H = L^2((0,1))$ with scalar product $\langle f, g \rangle = \int_0^1 f(x) \overline{g(x)} \, dx$. The string $e_n = \sqrt{2} \sin n\pi x \; (n \geq 1)$ is an orthonormal basis.

We prove that the assertion c) from the above lemma is true.

$$\sum_n |a_n|^2 = \|x\|^2 \text{ where } a_n = \langle x, e_n \rangle.$$

$$a_n = \langle x, e_n \rangle = \int_0^1 x \cdot \sqrt{2} \cdot \sin n\pi x \, dx \text{ which through integration through parts is }$$

$$(\frac{-1}{n\pi})^{n+1} \cdot \sqrt{2} \cdot \sum_n |a_n|^2 = \frac{2}{n\pi} \lim_{n \to \infty} \sum_{k=1}^n \frac{1}{k^2} = \frac{2}{\pi^2} \cdot \frac{\pi^2}{6} = \frac{1}{3} \|x\|^2$$

2) The discontinuous functions $\varphi_n : (0,1) \to \mathbb{R}, \varphi_n(t) = \operatorname{sgn}(\sin n\pi t), n \geq 1$ are considered. This string is not orthonormal in $L^2((0,1))$, but satisfies the condition b) from the previous lemma (if $f \in L^2((0,1))$ and $f \perp \varphi_n \; \forall n$ then $f = 0$ a.e.).

This string $(\varphi_n)$ has a substring that is $r_n : (0,1) \to \mathbb{R}, r_n = \varphi_{2n}, n \geq 1$ called Rademacher functions. The graphs of the first three functions Rademacher are shown in fig. 1.
It is easily proved that an orthogonal string is resulted in \( L^2_{(0,1)} \) but not an orthonormal basis (because \( \cos2\pi t \perp r_n, \forall n \) and it is not satisfied condition b) from the previous lemma).

The functions \( r_n \) only take 1 and -1 values. The definition domain of the function \( r_n \) can be divided in \( 2^{n-1} \) cycles of length \( \frac{1}{2^{n-1}} \), and on half of them \( r_n \) takes the value 1 and on the other half -1.

E. Borel considered, in 1910, the problem of finding a string of functions from \( L^2_{(0,1)} \) which will only take 2 values, but which will form an orthonormal base.

The first example was made by Walsh in 1924 starting from the string \( (r_n) \), \( n \geq 1 \). He was the one to “completed” this string adding other functions. He considered the string \( (w_n) n \geq 1 \) of Walsh functions defined as the following:

\[ w_1 = 1 \]

then for any integer \( k \geq 1 \) we have a unique form in basis 2:
Concerning a problem of E. Borel

\[ k = 2^{n_1} + 2^{n_2} + \ldots + 2^{n_p} \quad \text{with} \quad n_1 > n_2 > \ldots > n_p \geq 0 \quad \text{and we define} \]

\[ w_{k+1} = r_{n_1+1} \cdot r_{n_2+1} \cdot \ldots \cdot r_{n_p+1}. \]

In the case of \( p = 1 \), so \( k = 2^{n_1} (n_1 \geq 0) \) we find again \( r_{n_1+1} \). In this way the functions Rademacher \( r_1, r_2, r_3 \ldots \) are among Walsh functions. It is obvious that the Walsh functions take only the values 1 and -1.

The following theorem belongs to Walsh but we have another proof:

4. Theorem

The functions Walsh \( (w_k), k \geq 1 \) form an orthonormal basis in \( L^2(0,1) \).

Proof: The product of two functions Walsh will be of the following form

\[ \left(r_{m_1}\right)^{a_1} \cdot \left(r_{m_2}\right)^{a_2} \cdot \ldots \cdot \left(r_{m_p}\right)^{a_p} \quad \text{where} \quad m_1 > m_2 > \ldots > m_p \geq 1 \quad \text{are integers and} \quad a_k \quad \text{are equal to} \quad 1 \quad \text{or} \quad 2. \]

If two functions coincide then \( a_k = 2 \) and \( r_{mk}^2 \equiv 1 \) a.e. so

\[ \int r_{mk}^2 = 1. \]

If \( f, g \) are distinct then in \( \langle f, g \rangle \) we can renumber the indexes and we only remember the functions Rademacher at one squared with indexes \( m_1, m_2, \ldots, m_q \). The product \( r_{m_2}(x) \cdot r_{m_q}(x) \) is constant with values 1 or -1 on each on the two semicycles \( 2^{m_2} \) or \( r_{m_2} \). A typical semicycle \( I_m \) is divided in \( 2^{m_1 - m_2} \) semicycles of \( r_{m_1} \) in which \( r_{m_1} \) is alternative +1 and -1. So

\[ \int_0^1 r_{m_1} \ldots r_{m_q} = \sum_{l \in m_2} \int (\text{const}) \cdot r_{m_1} \quad \text{so equal with } 0, \quad \text{because} \quad \int r_{m_1} = 0. \]

Let \( f \in L^2(0,1) \) and \( F(x) = \int_0^x f(t) \, dt \) so \( F(0) = 0 \).

Because after defining Walsh’s function we have
\[ w_1 = 1; \ w_2 = r_1; \ w_3 = r_2; \ w_4 = r_1 \cdot r_2; \ w_5 = r_3 \text{ we get:} \]
\[
 f \perp w_1 \Rightarrow \int_{0}^{1} f w_1 = 0 \text{ which is } F(1) = 0
\]
\[
 f \perp w_2 \Rightarrow \int_{0}^{1} f w_2 = 0 \iff \int_{0}^{1/2} f r_1 = 0 \iff \int_{0}^{1/2} f = 0 \iff \\
F\left(\frac{1}{2}\right) - F\left(\frac{1}{2}\right) = 0 \iff F\left(\frac{1}{2}\right) = 0
\]
\[
 f \perp w_3 \Rightarrow \int_{0}^{1} f w_3 = 0 \iff \int_{0}^{1/4} f r_2 = 0 \iff \int_{0}^{1/4} f + \int_{1/4}^{3/4} f - \int_{0}^{1/4} f = 0 \iff \\
F\left(\frac{1}{4}\right) - F\left(\frac{1}{2}\right) + F\left(\frac{3}{4}\right) - F\left(\frac{1}{2}\right) - F(1) + F\left(\frac{3}{4}\right) = 0 \\
\iff F\left(\frac{1}{4}\right) + F\left(\frac{3}{4}\right) = 0
\]
\[
 f \perp w_4 \Rightarrow \int_{0}^{1} f w_4 = 0 \iff \int_{0}^{1/4} f r_1 r_2 = 0 \iff \int_{0}^{1/4} f - \int_{1/4}^{3/4} f + \int_{3/4}^{1} f = \\
= F\left(\frac{1}{4}\right) - F\left(\frac{3}{4}\right) + F\left(\frac{1}{4}\right) - F(1) + F\left(\frac{3}{4}\right) = 0 \\
\iff F\left(\frac{1}{4}\right) - F\left(\frac{3}{4}\right) = 0
\]
\[
 f \perp w_5 \Rightarrow \int_{0}^{1} f w_5 = 0 \iff \int_{0}^{1} f r_3 = 0 \iff \\
F\left(\frac{1}{4}\right) = 0, F\left(\frac{3}{4}\right) = 0
\]
From relation (3) and (4) we have
\[
 F\left(\frac{1}{4}\right) = 0, F\left(\frac{3}{4}\right) = 0
\]
Concerning a problem of E. Borel

\[ \int_{0}^{1/8} f - \int_{1/8}^{1/4} f + \int_{1/4}^{3/8} f - \int_{3/8}^{1/2} f + \int_{1/2}^{5/8} f - \int_{5/8}^{3/4} f + \int_{3/4}^{7/8} f - \int_{7/8}^{1} f = 0 \]

\[ \Rightarrow F\left(\frac{1}{8}\right) - F\left(\frac{1}{4}\right) + F\left(\frac{1}{8}\right) + F\left(\frac{3}{8}\right) - F\left(\frac{1}{4}\right) - F\left(\frac{1}{2}\right) + F\left(\frac{3}{8}\right) + F\left(\frac{5}{8}\right) - F\left(\frac{1}{2}\right) - F\left(\frac{3}{4}\right) + \]

\[ + F\left(\frac{5}{8}\right) + F\left(\frac{7}{8}\right) - F\left(\frac{3}{4}\right) - F\left(1\right) + F\left(\frac{7}{8}\right) = 0 \Leftrightarrow \]

\[ \Leftrightarrow F\left(\frac{1}{8}\right) + F\left(\frac{3}{8}\right) + F\left(\frac{5}{8}\right) + F\left(\frac{7}{8}\right) = 0 \] (6)

Than from \( f \perp w_6, w_7, w_8 \) it results \( F\left(\frac{1}{8}\right) = F\left(\frac{3}{8}\right) = F\left(\frac{5}{8}\right) = F\left(\frac{7}{8}\right) = 0 \).

Than through incomplete induction it results that \( F \) is cancelled on

\[ S = \left\{ \frac{2k + 1}{2^n}, k, n \in \mathbb{N}, 2k + 1 \leq 2^n - 1 \right\} \]

Because \( S \) is dense in \((0,1)\) and \( F \) is continuous \( \Rightarrow F = 0 \) on \((0,1)\) so \( f = 0 \) a.e. in \((0,1)\).

5. Observation

In many papers the Walsh functions \( (w_k), k \geq 1 \) are ordered differently so that each \( w_k \) could have exactly \( k + 1 \) “crossing in 0” in the interval \((0,1)\).

The graphs of the first 3 Walsh functions are shown in fig. 2.

![Graphs of the first 3 Walsh functions](image-url)
For the sinc signals it is fundamental that notion of frequency. Let the family of functions \( (\sin 2\pi \omega t), \omega \in \mathbb{R}^* \) for any \( \omega \) the function \( t \rightarrow \sin 2\pi \omega t \), has the period \( \frac{1}{\omega} \) and in any semi-open interval of length \( \frac{1}{\omega} \), \( \sin 2\pi \omega t \) has \( 2\omega \) zeroes.

The index \( \omega \) appears as being equal with half the number of zeroes of the signal \( \sin 2\pi \omega t \) in a time unit.

For the Walsh functions, the notion of sequence is defined as being half the number of changes of signal in time unit.

Haar build an orthonormal base for \( L^2_{(0,1)} \), formed with functions having at most three values, which approximates uniformly any continuous function \( f : [0,1] \rightarrow \mathbb{R} \).

Functions Haar have a systematic application in the wavelets theory.

The definition of functions \( h_n : (0,1) \rightarrow \mathbb{R}, n \geq 1 \) is the following:
Concerning a problem of E. Borel

\[ h_1 = 1; \quad h_{2^k + l}(x) = \begin{cases} 
2^{k/2}, & \text{if } x \in \left[ \frac{l-1}{2^k}, \frac{l-2}{2^k} \right] \\
-2^{k/2}, & \text{if } x \in \left[ \frac{l-2}{2^k}, \frac{l}{2^k} \right] \\
0, & \text{otherwise} \end{cases} \]

for \( l = 1, 2, \ldots, 2^k; \ k \geq 0 \)

The definition domain of each function Haar divided in \( 2^k \) cycles of length \( \frac{1}{2^k} \).

The graph of a Haar function is shown in figure 3.

![Graph of Haar function](image)

Fig. 3

It can be proven that the string \( (h_n)_{n\geq1} \) forms an orthonormal basis in \( L^2_{(0,1)} \).
Conclusions

I have studied a series of functions \( f_n \) from \( L^2(a,b) \) which will form an orthonormal basis and, moreover, each function \( f_n \) will have at most two values. The results have applications in wavelet’s theory.

I have given a new proof for the fact that the string \( \{w_k\}_{k=1}^{\infty} \) of Walsh’s functions is a Hilbert base in \( L^2(0,1) \).

In this article I offered a rather more simple proof than the one in paper [1].

REFERENCES