ON HOMOLOGICAL NOTIONS OF BANACH ALGEBRAS RELATED TO A CHARACTER

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In this paper, we countinue our work in [16]. *We show that* $L^1(G,w)$ *is* ϕ_0*-biprojective if and only if* G *is compact, where* ϕ_0 *is the augmentation character. We introduce the notions of character Johnson amenability and character Johnson contractibility for Banach algebras. We show that* $\ell^1(S)$ *is pseudo-amenable if and only if* $\ell^1(S)$ *is character Johnson-amenable, provided that* S *is a uniformly locally finite band semigroup. We give some conditions whether* ϕ*-biprojectivity (*ϕ*-biflatness) of* $\ell^1(S)$ *implies the finiteness (amenability) of* S*, respectively.*

Keywords: Beurling algebras, semigroup algebras, ϕ-biprojective, ϕ-contractible, amenability.

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1. Introduction

Helemskii studied Banach algebras via the Banach homology theory. In order to his investigation, he defined biflat and biprojective Banach algebras. Indeed, A is called biflat (biprojective), if there exists a bounded A-bimodule morphism $\rho : A \to (A \otimes_p A)^{**}$ ($\rho : A \to A \otimes_p A$) such that $\pi^{**} \circ \rho$ is the canonical embedding of A into A^{**} (ρ is a right inverse for π), respectively, see [15]. He showed that $L^1(G)$ is a biflat Banach algebra if and only if G is amenable and also $L^1(G)$ is biprojective if and only if G is compact, see [8].

Recently, Kanuith et al. in [12] have been used this idea and defined a new notion of amenability for Banach algebra depended on a character of that Banach algebra. Indeed, for a character $\phi \in \Delta(A)$, they defined the new notion of left ϕ-amenability, that is, A is left ϕ-amenable Banach algebra if $\mathcal{H}^1(A, X^*) = \{0\}$, for every Banach A-bimodule X, provided that $a \cdot x = \phi(a)x$, for all $a \in A$ and $x \in X$. They also showed that the Fourier algebra $A(G)$ is ϕ-amenable for each $\phi \in \Delta(A)$. Hu *et al.* in [11] used the idea of virtual diagonal of Banach algebras and defined a parallel notion to left ϕ-amenability and called it left ϕ-contractibility. This theory has been under more investigations, Sangani Monfared in [18] defined the concept of character amenability which used every character of a Banach algebra to studying its properties. He showed that $L^1(G)$ is character amenable if and only if G is amenable. Recently Nasr-Isfahani *et al.* has been investigated the notions of left ϕ-amenability and left ϕ-contractibility in the Banach homology terms, see [14].

Motivated by these considerations, in order to find biflatness and biprojectivity related to a character the author with A. Pourabbas defined the notions of ϕ-biflatness, ϕ-biprojectivity and ϕ-Johnson amenability for Banach algebras, see [16]. They showed that for a locally compact group G, $L^1(G)$ is ϕ-biflat if and only if G is amenable. Also they showed that the Fourier algebra $A(G)$ is ϕ-biprojective if and only if G is discrete. For a discrete group G, they showed that $\ell^1(G)$ is ϕ-biprojective if and only if G is finite.

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The content of this paper is as follows, after recalling some definitions and background notations. We extend [16, Lemma 4.2] to Beurling algebras. We show that $L^1(G,w)$ is ϕ_0-biprojective if and only if G is compact, where ϕ_0 is the augmentation character.) We introduce character Johnson amenability and character Johnson contractibility for Banach algebras. We show that $\ell^1(S)$ is pseudo-amenable if and only if $\ell^1(S)$ is character Johnson-amenable, provided that S is a uniformly locally finite band semigroup. We give some conditions whether ϕ-biprojectivity (ϕ-biflatness) of $\ell^1(S)$ implies the finiteness (amenability) of S, respectively.

2. **Preliminaries**

We recall that if X is a Banach A-bimodule, then with the following actions X^* is also a Banach A-bimodule

$$(a\cdot f)(x)=f(x\cdot a),\quad (f\cdot a)(x)=f(a\cdot x)\qquad (a\in A, x\in X, f\in X^*).$$

Let A and B be Banach algebras. The projective tensor product of A and B is denoted by $A\otimes_p B$ and with the following multiplication is a Banach algebra

$$(a_1\otimes b_1)(a_2\otimes b_2)=a_1a_2\otimes b_1b_2\qquad (a_1,a_2\in A,\quad b_1,b_2\in B).$$

The Banach algebra $A\otimes_p A$ with the following actions is a Banach A-bimodule

$$a\cdot(b\otimes c)=ab\otimes c,\ (b\otimes c)\cdot a=b\otimes ca\qquad (a,b,c\in A).$$

Throughout, the character space of A is denoted by $\Delta(A)$. Let $\phi\in\Delta(A)$. Then ϕ has a unique extension to A^{**} denoted by $\tilde{\phi}$ and defined by $\tilde{\phi}(F)=F(\phi)$ for every $F\in A^{**}$. Clearly this extension remains to be a character on A^{**}. We denote $\pi_A: A\otimes_p A\to A$ for the product morphism which specified by $\pi_A(a\otimes b)=ab$.

Let A be a Banach algebra and X be a Banach A-bimodule. The n^{th} cohomology group of A with coefficients in X is denoted by $\mathcal{H}^n(A,X)$. In fact A is an amenable Banach algebra, if $\mathcal{H}^1(A,X^*)=\{0\}$ for every Banach A-bimodule X.

The Banach algebra A is called ϕ-biprojective (ϕ-biflat), if there exists a bounded A-bimodule morphism $\rho: A\to A\otimes_p A$ ($\rho: A\to (A\otimes_p A)^{**}$) such that

$$\phi\circ\pi_A\circ\rho(a)=\phi(a)\ (\tilde{\phi}\circ\pi_A^{**}\circ\rho(a)=\phi(a)),$$

respectively for every $a\in A$. A Banach algebra A is called ϕ-Johnson amenable(ϕ-Johnson contractible) if there exists $m\in(A\otimes_p A)^{**}$ ($m\in A\otimes_p A$) such that

$$a\cdot m=m\cdot a,\quad \tilde{\phi}\circ\pi_A^{**}(m)=1,(\phi\circ\pi_A(m)=1)\quad (a\in A),$$

respectively for every $a\in A$. For more details, we refer the readers to [16].

Let G be a locally compact group. A continuous map $w: G\to\mathbb{R}^+$ is called a weight function, if $w(e)=1$ and for every x and y in G, $w(xy)\leq w(x)w(y)$ and $w(x)\geq 1$. The Banach algebra of all measurable functions f from G into $\mathbb{C}$ with $||f||_w=\int|f(x)|w(x)dx<\infty$ and the convolution product is denoted by $L^1(G,w)$. The Banach algebra of all complex-valued, regular and Borel measures μ on G such that $||\mu||_w=\int_G w(x)d|\mu|(x)<\infty$ is denoted by $M(G,w)$. We write $M(G)$, whenever $w=1$. The map $\phi_0: L^1(G,w)\to\mathbb{C}$ which specified by

$$\phi_0(f)=\int_G f(x)dx$$

is called augmentation character, for more details see [3].

We recall that S is an inverse semigroup, if for each $s\in S$ there exists a unique element $s^*\in S$ such that $ss^*s=s$ and $s^*ss^*=s^*$ [10]. The set of idempotents of a semigroup S is denoted by $E(S)$. There exists a partial order on $E(S)$, indeed

$$s\leq t\Longleftrightarrow s=st=ts\qquad (s,t\in E(S)).$$

If S is an inverse semigroup, then there exists a partial order on S which is coincide with the partial order on $E(S)$. Indeed

$$s \leq t \Longleftrightarrow s = ss^*t \qquad (s, t \in S).$$

For the partially ordered set $(S, \leq)$, we denote $(x] = \{y \in S \,|\, y \leq x\}$. The set S is called locally finite (uniformly locally finite) if for every $x \in S$, we have $|(x]| < \infty$ ($\sup\{|(x]| | x \in S\} < \infty$), respectively.

3. ϕ-biprojectivity of Beurling algebras

Let A be a Banach algebra and let L be a closed ideal of A. We say that L is left essential as a Banach A-bimodule, if $\overline{AL} = L$.

Let A be a Banach algebra and $\phi \in \Delta(A)$. Suppose that $L \subseteq \ker \phi$ is a closed ideal of A. Clearly ϕ induces a character $\overline{\phi}$ on $\frac{A}{L}$, which is defined by $\overline{\phi}(x + L) = \phi(x)$ for every $x \in A$.

Proposition 3.1. *Let A be a Banach algebra and $\phi \in \Delta(A)$. Suppose that A is a ϕ-biprojective Banach algebra and $L \subseteq \ker \phi$ is a closed ideal of A which is left essential as a Banach A-bimodule. Then $\frac{A}{L}$ is $\overline{\phi}$-biprojective.*

Proof. Since A is a ϕ-biprojective Banach algebra, there exists a bounded A-bimodule morphism $\rho : A \to A \otimes_p A$ such that $\phi \circ \pi_A \circ \rho(a) = \phi(a)$ for every $a \in A$. Let $q : A \to \frac{A}{L}$ be the quotient map. Define $\rho_1 = id \otimes q \circ \rho : A \to A \otimes_p \frac{A}{L}$. Since L is an essential closed ideal of A, for every $l \in L$, we have

$$\rho_1(l) = id \otimes q \circ \rho(l) = id \otimes q \circ \rho(al') = id \otimes q(\rho(a) \cdot l') = 0,$$

where $l = al'$ for some $a \in A$ and $l' \in L$. Hence there exists an induced map (which still denoted by ρ_1) $\rho_1 : \frac{A}{L} \to A \otimes_p \frac{A}{L}$.

Now define $\rho_2 = q \otimes id_{\frac{A}{L}} \circ \rho_1 : \frac{A}{L} \to \frac{A}{L} \otimes_p \frac{A}{L}$. We will show that ρ_2 is a bounded $\frac{A}{L}$-bimodule morphism and $\overline{\phi} \circ \pi_{\frac{A}{L}} \circ \rho_2(x + L) = \overline{\phi}(x + L)$. Suppose that $x \in A$ and $\rho(x) = \sum_{i=1}^{\infty} a_i^x \otimes b_i^x$ for some sequences $(a_i^x)_i$ and $(b_i^x)_i$ in A. Then $\rho_2(x+L) = \sum_{i=1}^{\infty} a_i^x + L \otimes b_i^x + L$, so $\pi_{\frac{A}{L}} \circ \rho_2(x + L) = \sum_{i=1}^{\infty} a_i^x b_i^x + L$, therefore

$$\overline{\phi}(\sum_{i=1}^{\infty} a_i^x b_i^x + L) = \phi(\sum_{i=1}^{\infty} a_i^x b_i^x) = \phi \circ \pi_A \circ \rho(x) = \phi(x) = \overline{\phi}(x + L).$$

Now suppose that $a + L$ is an arbitrary element of $\frac{A}{L}$. Then $a + L \cdot \rho_2(x + L) = \sum_{i=1}^{\infty} aa_i^x + L \otimes b_i^x + L$. Since ρ is a left A-module morphism, ρ_1 is a left A-module morphism. Hence

$$\begin{aligned}
\rho_2(ax + L) = q \otimes id_{\frac{A}{L}} \circ \rho_1(ax + L) &= q \otimes id_{\frac{A}{L}}(a \cdot \rho_1(x + L)) \\
&= q \otimes id_{\frac{A}{L}}(\sum_{i=1}^{\infty} aa_i^x \otimes b_i^x + L) \\
&= \sum_{i=1}^{\infty} aa_i^x + L \otimes b_i^x + L \\
&= a + L \cdot \sum_{i=1}^{\infty} a_i^x + L \otimes b_i^x + L \\
&= a + L \cdot \rho_2(x + L).
\end{aligned}$$

Similarly one can show that ρ_2 is a right $\frac{A}{L}$-module morphism and the proof is complete. □

We recall that $m \in A \otimes_p A$ is a ϕ-Johnson contraction for A, if $a \cdot m = m \cdot a$ and $\phi \circ \pi_A(m) = 1$, where $a \in A$, for more details the reader referred to [16].

Let A be a Banach algebra and $\phi \in \Delta(A)$. A is left ϕ-contractible if and only if there exists an element m in A such that $am = \phi(a)m$ and $\phi(m) = 1$, see [11] and [14]. Note that the left ϕ-contractibility of a Banach algebra A is equivalent to property that; the Banach algebra $\mathbb{C}$ is a projective left Banach A-module with the following left action, $a \cdot z = \phi(a)z$ for every $a \in A$ and $z \in \mathbb{C}$ [14, Theorem 4.3].

Compare the following Theorem with [8, Theorem 5.13].

Theorem 3.1. *Let G be a locally compact group, let ω be a weight on G and let ϕ_0 be the augmentation character on $L^1(G, w)$. Then the following are equivalent*

(i) *$L^1(G, w)$ is ϕ_0-biprojective;*

(ii) *$L^1(G, w)$ is left ϕ_0-contractible;*

(iii) *G is compact.*

Proof. (i)$\Rightarrow$(ii) Set $A = L^1(G, w)$ and $L = \ker \phi_0$. Let A be ϕ_0-biprojective. Since A has a bounded approximate identity, L becomes a left essential Banach A-bimodule. Thus by the proof of previous Proposition there exists a bounded left A-module morphism

$$\rho_1 : \frac{A}{L} \to A \otimes_p \frac{A}{L}.$$

Since $\frac{A}{L} \cong \mathbb{C}$, hence we have $\rho_1 : \mathbb{C} \to A \otimes_p \mathbb{C} \cong A$ such that $\overline{\phi_0} \circ \pi_{A,\frac{A}{L}} \circ \rho_1(c) = \overline{\phi_0}(c)$, where $c \in \mathbb{C}$. Set $m = \rho(1) \in A$. Then $\phi_0(m) = \phi_0(\rho(1)) = \overline{\phi_0} \circ \pi_{A,\frac{A}{L}} \circ \rho(1) = 1$ and $a \cdot \rho(1) = \rho(a \cdot 1) = \phi_0(a)\rho(1)$, where $a \in A$. Hence A is left ϕ_0-contractible.

(ii)$\Rightarrow$(iii) Suppose that A is a left ϕ_0-contractible Banach algebra. Then there exists an element $m \in A$ such that $am = \phi_0(a)m$ and $\phi_0(m) = 1$, where $a \in A$. Let $g \in G$ be an arbitrary element and $f \in A \setminus L$. Hence

$$\phi_0(f)\delta_g * m = \delta_g * (f * m) = (\delta_g * f) * m = \phi_0(\delta_g * f)m = \phi_0(f)m.$$

Hence m is constant and belongs to A, which implies that $\int_G w(x)dx < \infty$. Therefore

$$|G| = \int_G w(e)dx < \infty,$$

so G is a compact group.

(iii)$\Rightarrow$(i) Let G be a compact group and consider a normalized left Haar measure. Then $m = 1 \otimes 1$ in $A \otimes_p A$ satisfies $a \cdot m = m \cdot a = \phi_0(a)m$ and $\phi_0 \circ \pi_A(m) = 1$, where $a \in A$. Thus A is ϕ_0-Johnson contractible. Hence [16, Lemma 3.2] gives ϕ_0-biprojectivity of A. $\square$

It is easy to see that every biprojective Banach algebra A is ϕ-biprojective for every $\phi \in \Delta(A)$, but the converse is not always true. On the other hand [15, Theorem 5.2.30] asserts that, if A is biprojective, then for every Banach A-bimodule X, $\mathcal{H}^n(A, X) = 0$, where $n \geq 3$. This question maybe asked "what will happen, if A is ϕ-biprojective?" at the following corollary we answer this question for the group algebras.

Corollary 3.1. *Let G be a locally compact group.*

(i) *If $L^1(G)$ is ϕ_0-biprojective, then for every Banach $L^1(G)$-bimodule X, $\mathcal{H}^n(L^1(G), X) = 0$, where $n \geq 3$.*

(ii) *$L^1(G)$ is ϕ_0-biprojective if and only if $\mathcal{H}^1(L^1(G), X) = 0$, for every Banach $L^1(G)$-bimodule X with $x \cdot a = \phi_0(a)x$ such that $a \in L^1(G)$ and $x \in X$.*

Proof. (i) Let $L^1(G)$ be ϕ_0-biprojective. Then by Theorem 3.1 G is compact and [15] shows that $L^1(G)$ is biprojective for every compact group G. Now using [15, Theorem 5.2.30] one can get the results.

(ii) holds by Theorem 3.1. $\square$

For a Banach algebra A, dbA denoted for the minimum values of $n \in \mathbb{Z}^+$ such that $A^\sharp$ has a projective resolution of length n, see [2, page 294]. Helemskii showed that for a biprojective Banach algebra A, $dbA \leq 2$, see [2, Theorem 2.8.56]. Also it is well-known that $L^1(G)$ is biprojective if and only if G is compact. Combine these facts and the previous corollary one can see that if $L^1(G)$ is ϕ_0-biprojective, then $dbL^1(G) \leq 2$.

4. ϕ-homological properties of semigroup algebras

We remind that S is a left (right) amenable semigroup if there exists an element $m \in \ell^1(S)^{**}$ such that

$$s \cdot m = m \quad (m \cdot s = m), \qquad ||m|| = m(\phi) = 1 \qquad (s \in S),$$

where ϕ is the augmentation character of $\ell^1(S)$, respectively. The semigroup S is called amenable, if it is both left and right amenable.

We recall that S is a band semigroup, if $S = E(S)$. A band semigroup S is called rectangular band if $xyx = x$, for every $x, y \in S$. In this case there exists an equivalence relation on S, in fact

$$a\mathfrak{R}b \Longleftrightarrow S^1aS^1 = S^1bS^1, \qquad (a, b \in S),$$

where $S^1 = S \cup \{1\}$ [10]. Let A be a Banach algebra and Λ be a semilattice. Suppose that $\{A_\lambda : \lambda \in \Lambda\}$ is a collection of closed subalgebra of A. If A is a ℓ^1-direct sum of A_λ as a Banach space and $A_{\lambda_1}A_{\lambda_2} \subseteq A_{\lambda_1\lambda_2}$, then A is called ℓ^1-graded of A_λ's and denoted by $\oplus_\lambda^{\ell^1} A_\lambda$.

We say that A is character-Johnson amenable (character-Johnson contracatible), if for every $\phi \in \Delta(A)$, A is ϕ-Johnson amenable (ϕ-Johnson contractible), respectively.

Theorem 4.1. *Suppose that S is a band semigroup. Let $\ell^1(S)$ be character Johnson-amenable. Then S is a semilattice, so is amenable.*

Proof. Let S be a band semigroup. Then by [10, Theorem 4.4.1] $S = \cup_{\lambda\in\Lambda} S_\lambda$, where S_λ is a rectangular band semigroup for every $\Lambda \in \Lambda$. Since $S_{\lambda_1}S_{\lambda_2} \subseteq S_{\lambda_1\lambda_2}$, we have $\ell^1(S) = \oplus_\lambda^{\ell^1} \ell^1(S_\lambda)$, here the index set Λ is a semilattice.

Set $I = \oplus_{\lambda\leq\lambda_0}^{\ell^1} \ell^1(S_\lambda)$, where $\lambda_0 \in \Lambda$ is fixed. One can easily see that I is a closed ideal of $\ell^1(S)$. Since $\ell^1(S_{\lambda_0})$ is a homomorphic image of I. For every $\phi \in \Delta(\ell^1(S_{\lambda_0}))$ we take $\phi \circ \eta$ as a character on I, which we denote it by ϕ_I, where $\eta : I \to \ell^1(S_{\lambda_0})$ is a homomorphism with a dense range. It is easy to see that ϕ_I can be extend to $\ell^1(S)$ which is denoted by ϕ_S

Moreover, there exists an isomorphism between S_{λ_0} and $L \times R$, where L and R are denoted for a left-zero semigroup and a right-zero semigroup, respectively [10, Theorem 1.1.3]. Also we have

$$\ell^1(S_{\lambda_0}) \cong \ell^1(L \times R) \cong \ell^1(L) \otimes_p \ell^1(R).$$

Take $\phi = \phi_0 \otimes \sigma_0 \in \Delta(\ell^1(S_{\lambda_0}))$, where ϕ_0 and σ_0 are the augmentation characters on $\ell^1(L)$ and $\ell^1(R)$, respectively. Consider ϕ_I and ϕ_S corresponding to ϕ as before. Since $\ell^1(S)$ is character Johnson-amenable, by [16, Proposition 2.2] $\ell^1(S)$ is left ϕ_S-amenable and right ϕ_S-amenable. Since $\phi_S|_{\ell^1(S_{\lambda_0})} \neq 0$, we have $\phi_I \neq 0$, so by [12, Lemma 3.1] I is left ϕ_I-amenable and right ϕ_I-amenable. But, since $\ell^1(S_{\lambda_0})$ is a homomorphic image of I, by [12, Proposition 3.5] $\ell^1(S_{\lambda_0})$ is left ϕ-amenable and right ϕ-amenable. Hence by [12, Theorem 3.3] $\ell^1(L)$ is left ϕ_0-amenable and $\ell^1(R)$ is right σ_0-amenable. So [12, Theorem 1.4] shows that there exists a net $(m_\alpha)_\alpha$ in $\ell^1(L)$ such that

$$am_\alpha - \phi_0(a)m_\alpha \xrightarrow{||\cdot||} 0, \quad \phi(m_\alpha) = 1. \tag{1}$$

Replace $a_1 = \delta_{s_1}$ and $a_2 = \delta_{s_2}$ in (1) instead of a, respectively for every $s_1, s_2 \in L$. One can see that $m_\alpha \to \delta_{s_1}$ and $m_\alpha \to \delta_{s_2}$, which implies that L and similarly R are singleton, then

S_{λ_0} is singleton and therefore with the same argument we can show that S_λ is singleton for every $\lambda \in \Lambda$. Hence $S = \cup_{\lambda\in\Lambda} S_\lambda$ is isomorphic to Λ. Since every semilattice is commutative, S is amenable and the proof is complete. □

We recall that A is a pseudo-amenable Banach algebra, if there exists a (not necessarily bounded) net $(m_\alpha)_\alpha$ in $A \otimes_p A$ such that $a \cdot m_\alpha - m_\alpha \cdot a \xrightarrow{||\cdot||} 0$ and $\pi_A(m_\alpha)a \xrightarrow{||\cdot||} a$, for every $a \in A$, see [7].
Using [4, Corollary 3.5] and previous Theorem, we get the following corollary.

Corollary 4.1. *Let S be a uniformly locally finite band semigroup. Then $\ell^1(S)$ is pseudo-amenable if and only if $\ell^1(S)$ is character Johnson-amenable.*

Note that in the general case, the pseudo-amenability is not equivalent with the character Johnson-amenability. To see this we give the following example.

Example 4.1. *Suppose that G is a compact infinite group. Then by* [7, Proposition 4.2] *$L^1(G)^{**}$ is not pseudo-amenable. The set of all continuous character $\rho : G \to \mathbb{T}$ is denoted by $\widehat{G}$. It is well-known that every character $\phi \in \Delta(L^1(G))$ is of the form*

$$\phi_\rho(f) = \int_G \overline{\rho(x)} f(x)dx,$$

where dx is a left Haar measure on G, for more details, see [9, Theorem 23.7]. *It is also well-known that ϕ_ρ has a unique extension to $L^1(G)^{**}$, which denoted by $\tilde{\phi}_\rho$. Hence $\Delta(L^1(G)^{**})$ consists of all $\tilde{\phi}_\rho$, for every $\rho \in \widehat{G}$. Since G is compact, $\widehat{G} \subset L^\infty(G) \subseteq L^1(G)$. Define $m_\rho = \rho \otimes \rho \in L^1(G) \otimes_p L^1(G)$. Since two maps $a \mapsto a\rho$ and $a \mapsto \rho a$ are w^*-continuous on $L^1(G)^{**}$ for every $a \in L^1(G)^{**}$, one can easily see that $a \cdot m_\rho = m_\rho \cdot a$ and $\tilde{\phi}_\rho \circ \pi^{**}_{L^1(G)^{**}}(m_\rho) = 1$. Hence $L^1(G)^{**}$ is character Johnson-amenable.*

It is well-known that for an inverse semigroup S there exists an equivalence relation $\mathcal{R}$ on S, that is, for every $x, y \in S$, $x\mathcal{R}y$ if and only if there exits $e \in E(S)$ such that $es = et$. Consider $G_S = \frac{S}{\mathcal{R}}$, see [13].

Proposition 4.1. *Let S be an inverse semigroup. If $\ell^1(S)$ is character Johnson-amenable, then G_S is an amenable group.*

Proof. Since G_S is a quotient of S, then $\ell^1(G_S)$ is a homomorphic image of $\ell^1(S)$. Suppose that $\phi \in \Delta(\ell^1(G_S))$ and $p : \ell^1(S) \to \ell^1(G_S)$ is a dense range homomorphism. Since $\ell^1(S)$ is character Johnson amenable, $\ell^1(S)$ is $\phi\circ p$-Johnson amenable. Now by [16, Proposition 2.2], $\ell^1(S)$ is left $\phi\circ p$-amenable. Hence [12, Proposition 3.5] shows that $\ell^1(G)$ is left ϕ-amenable. Now by applying [1, Corollary 3.4] G_S must be amenable. □

Let G be a group and I be a non-empty set. Set $\mathcal{M}^0(G, I) = \{(g)_{ij} | g \in G, i, j \in I\} \cup \{0\}$, where $(g)_{ij}$ is denoted for $I \times I$ matrix with entry g in $(i, j)^{th}$-position and zero elsewhere. With the following multiplication $\mathcal{M}^0(G, I)$ is a semigroup

$$(g)_{ij}(h)_{kl} = \begin{cases} (gh)_{il} & \text{if} \quad j = k \\ 0 & \text{if} \quad j \neq k \end{cases}$$

This semigroup is called Brandt semigroup over G with index set I. It is well-known that for $S = \mathcal{M}^0(G, I)$, $G_S = G$.

Corollary 4.2. *Let G be a group and I be a non-empty set and also let $S = \mathcal{M}^0(G, I)$. If $\ell^1(S)$ is character Johnson amenable, then $\ell^1(S)$ is pseudo-amenable.*

Proof. Let $\ell^1(S)$ be character Johnson amenable. By previous Proposition $G_S = G$ must be amenable. Now apply [5, Corollary 3.8] to show that $\ell^1(S)$ is pseudo-amenable. □

Proposition 4.2. *Let S be an inverse semigroup. If $\ell^1(S)$ is character Johnson-contractible, then G_S is finite.*

Proof. Use the same argument as in the proof of pervious Proposition and the fact that $\ell^1(G_S)$ is left ϕ-contractible if and only if G_S is finite, see [1, Theorem 3.3]. □

Remark 4.1. *The results of the previous two propositions hold even if we replace the hypothesis "A is left ϕ-amenable (ϕ-contractible)" instead of "A is character Johnson amenable (character Johnson contractible)" respectively for every $\phi \in \Delta(A)$.*

Proposition 4.3. *Let S be a semigroup such that its center $Z(S)$ is non-empty. If $\ell^1(S)$ is ϕ-biflat, then S is amenable, where ϕ is the augmentation character on $\ell^1(S)$.*

Proof. Suppose that $\ell^1(S)$ is ϕ-biflat, where ϕ is the augmentation character on $\ell^1(S)$. Let $\rho : \ell^1(S) \to (\ell^1(S) \otimes_p \ell^1(S))^{**}$ be a bounded $\ell^1(S)$-bimodule morphism such that $\tilde{\phi} \circ \pi^{**}_{\ell^1(S)} \circ \rho(a) = \phi(a)$, for every $a \in \ell^1(S)$. Set $m_0 = \rho(\delta_{s_0})$, where $s_0 \in Z(S)$, it is easy to see that $\delta_s \cdot m_0 = m_0 \cdot \delta_s$ and $\tilde{\phi} \circ \pi^{**}_{\ell^1(S)}(m_0) = 1$. Then $\ell^1(S)$ is ϕ-Johnson amenable. Applying the same arguments as in the proof of [16, Proposition 2.2], one can show that $\delta_s \cdot m_0 = m_0 \cdot \delta_s = m_0$ and $\tilde{\phi} \circ \pi^{**}_{\ell^1(S)}(m_0) = 1$. Suppose that $m = \pi^{**}_{\ell^1(S)}(m_0) \in \ell^1(S)^{**}$. Hence we have $\delta_s m = m\delta_s = m$ and $\tilde{\phi}(m) = 1$. Hence S is an amenable semigroup, see [12, Theorem 1.1]. □

Proposition 4.4. *Let S be a semigroup such that $Z(S)$ is non-empty. If $\ell^1(S)$ is ϕ-biprojective and S has left or right unit, then S is finite, where ϕ is the augmentation character on $\ell^1(S)$.*

Proof. Suppose that $\ell^1(S)$ is ϕ-biprojective, where ϕ is the augmentation character on $\ell^1(S)$. Then there exists a bounded $\ell^1(S)$-bimodule morphism $\rho : \ell^1(S) \to \ell^1(S) \otimes_p \ell^1(S)$ such that $\phi \circ \pi_{\ell^1(S)} \circ \rho(a) = \phi(a)$, for every $a \in \ell^1(S)$.

Define $m = \pi_{\ell^1(S)} \circ \rho(\delta_{s_0})$, where $s_0 \in Z(S)$. Then we have $\delta_s m = m\delta_s = m$ and $\phi(m) = 1$. Now if e_r is a right unit for S, then for every $s \in S$ we have

$$m(s) = m(se_r) = \delta_s m(e_r) = m(e_r),$$

that is, $m \in \ell^1(S)$ is a constant function on S , so S must be finite. □

Remark 4.2. *There exists a biprojective semigroup algebra which is not character Johnson amenable. To see this let S be an infinite left zero semigroup, that is, $st = s$ for every $s, t \in S$. It is easy to see that*

$$fg = \phi_S(g)f, \quad f, g \in \ell^1(S),$$

where ϕ_S is the augmentation character on $\ell^1(S)$. Define $\rho : \ell^1(S) \to \ell^1(S) \otimes_p \ell^1(S)$ by $\rho(f) = f \otimes f_0$. It is easy to see that ρ is a bounded A-bimodule morphism which $\pi_{\ell^1(S)} \circ \rho(f) = f$ for every $f \in \ell^1(S)$. It follows that $\ell^1(S)$ is biprojective. Now using the same method as in the proof of 4.1 one can see that $\ell^1(S)$ is not character Johnson amenable. Note that in the previous Proposition the hypothesis "$Z(S) \neq \emptyset$" is necessary. It is easy to see that for a left zero semigroup S, $Z(S) = \emptyset$. Also one can show that for the augmentation character ϕ, $\ell^1(S)$ is ϕ-biprojective, but S is not finite.

Also note that the hypothesis "existence of left or right unit" is necessary. To see this let $S = \mathbb{N}$ with the product $m \cdot n = \min\{m, n\}$ $(m, n \in S)$ which is an infinite semigroup with no unit such that $Z(S) = S$ [16, Example 5.2] *and $\ell^1(S)$ is ϕ-biprojective, where ϕ is the augmentation character.*

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