HYPERCYCLIC TUPLE $C_0$-SEMIGROUPS OF OPERATORS

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In this paper, we introduce the notions of hypercyclic and locally topologically hypercyclic multi-parameter $C_0$-semigroups of operators on separable Banach spaces. Then we show that every finite dimensional Banach space admits a hypercyclic multi-parameter $C_0$-semigroup. Next, the hypercyclicity of tensor product of one-parameter $C_0$-semigroups as a multi-parameter $C_0$-semigroup will be discussed. Finally, locally topologically transitive two-parameter $C_0$-semigroups are studied.

1. Introduction and preliminaries

A continuous linear operator $T$ on a Banach space $X$ is called hypercyclic if it has a hypercyclic vector $x \in X$, i.e. there is a vector $x \in X$ such that $\text{Orb}(T, x) := \{T^n x : n \in \mathbb{N} \cup \{0\}\}$ is dense in $X$.

Ansari [1] and Bernal- Gonzalez [2] showed that every infinite-dimensional separable Banach space admits a hypercyclic operator. This result was also extended to the non-normable Fréchet case by Bonet and Peris [3]. For more details about hypercyclic operators see the surveys [4, 11, 12]. It is well known that there is no hypercyclic operator on a finite dimensional Banach space. Hypercyclic tuples of operators was introduced by N. S. Feldman [10]. A finite sequence $T = (T_1, T_2, ..., T_n)$ of commuting continuous linear operators on a locally convex space $X$ is called hypercyclic if there is a vector $x \in X$ whose orbit under $T$, i.e. $\text{Orb}(T, x) := \{T_k^1 x, T_k^2 x, ..., T_k^n x : k \in \mathbb{N}, i = 1, 2, ..., n\}$ is dense in $X$. In [10] Feldman showed that hypercyclic n-tuples can arise in finite dimensions when $n > 1$; something that does not happen for single operators.

In the continuous case, a one-parameter family $T = \{T(t)\}_{t \geq 0}$ of continuous linear operators on $X$, is a strongly continuous semigroup (or $C_0$-semigroup) of operators if $T(0) = I$, $T(t)T(s) = T(t+s)$, for all $t, s \geq 0$, and $\lim_{t \to 0} T_t x = x$ for all $x \in X$. For more information on $C_0$-semigroups refer to the books [9, 19]. A $C_0$-semigroup $T = \{T(t)\}_{t \geq 0}$ is said to be hypercyclic if $\text{Orb}(T, x) := \{T(t) x : t \geq 0\}$ is dense in $X$ for some $x \in X$. Desch, Schappacher and Webb in [7] initiated the investigation of hypercyclic semigroups. So far, several specific examples of hypercyclic strongly continuous semigroups have been studied (see for example [6, 7, 8, 17, 20]). Also, tensor product of hypercyclic semigroups was studied in [22]. It is well-known that there is no hypercyclic $C_0$-semigroups of operators on finite dimensional Banach spaces.

Multi-parameter $C_0$-semigroups of operators were investigated in [15, 16, 14]. Any homomorphism $W$ from the semigroup $(\mathbb{R}_+^d, +)$ into $B(X)$, the space of all bounded linear operators on the Banach space $X$, with $W(0) = I$ is called a $d$-parameter semigroup of operators on $X$ where $\mathbb{R}_+^d = \{(t_1, t_2, ..., t_n) : t_i \geq 0, i = 1, 2, ..., n\}$. Also, if the mapping $t \mapsto$...
$W(t)x$ is continuous at $0 \in \mathbb{R}_+^d$, for any $x \in X$, then $W$ is said to be strongly continuous or $d$-parameter $C_0$-semigroup. Ergodic properties of $d$-parameter semigroups has been studied in [21].

The concept of J-class $C_0$-semigroups of operators (or topologically transitive $C_0$-semigroups) also studied by Nasser in [18]. Recall that a $C_0$-semigroup $\{T(t)\}_{t \geq 0}$ on a normed space $X$ is called J-class if there exists $0 \neq x \in X$ such that $J_T(x) = X$, where

$$J_T(x) : = \{ y \in X : \text{ there exist a strictly increasing sequence}$$

$$(t_n)_{n \in \mathbb{N}} \subseteq [0,\infty) \text{ with } t_n \to 0 \text{ and a sequence}$$

$$(x_n)_{n \in \mathbb{N}} \text{ in } X \text{ such that } x_n \to x \text{ and } T(t_n)(x_n) \to y \}$$

In this paper, hypercyclicity and locally topologically transitivity of $d$-parameter $C_0$-semigroups of operators are investigated. In Section 2, some elementary hypercyclicity properties of $d$-parameter $C_0$-semigroups are studied. Finite dimensional Banach spaces and hypercyclic $d$-parameter $C_0$-semigroups on them will be discussed in Section 3. Indeed it will be shown that there are many hypercyclic $d$-parameter $C_0$-semigroups on $\mathbb{R}^n$ and $\mathbb{C}^n$ despite of they are finite dimensional. In Section 4, hypercyclicity of tensor product of one-parameter semigroups as $d$-parameter $C_0$-semigroups will be discussed. Finally, in the last section, locally topologically transitivity of $d$-parameter $C_0$-semigroups are investigated.

In the rest of the paper, $X$ is a separable Banach space.

2. Hypercyclic $d$-parameter $C_0$-semigroup

Recall that any homomorphism $W : (\mathbb{R}_+^d, +) \to B(X)$ with $W(0) = I$ is called a $d$-parameter semigroup on $X$, which is denoted by $(W, \mathbb{R}_+^d, X)$. The family $\{ W(t) \}_{t \in \mathbb{R}_+^d}$ is called strongly continuous (or $C_0$-semigroup) if the mapping $t \mapsto W(t)x : \mathbb{R}_+^d \to X$ is continuous for every $x \in X$. If $\{ e_i : i = 1, 2, ..., d \}$ is the standard basis of $\mathbb{R}_+^d$, then $u_i(t) := W(te_i)x$, $s \geq 0$, $x \in X$, is a $C_0$-one-parameter semigroup for which $u_iu_j = u_ju_i$ and $W(t_1t_2,...,t_n) = \prod_{i=1}^n u_i(t_i)$.

It is interesting to note that every one-parameter $C_0$-group $\{ T(t) \}_{t \in \mathbb{R}}$ can be considered as a two-parameter $C_0$-semigroup of the form $W(s,t) := T(s-t)$, $s, t \geq 0$. In this case, $u_1(t) = T(t)$, and $u_2(t) = T(-t)$, $t \geq 0$.

**Definition 2.1.** A $d$-parameter $C_0$-semigroup $(W, \mathbb{R}_+^d, X)$ is called

i) tuple-hypercyclic (or simply, hypercyclic) if there exists a $x \in X$ such that $\text{Orb}(W, x) := \{ W(t)x : t \in \mathbb{R}_+^d \}$ is dense in $X$. In this case, $x$ is called the tuple-hypercyclic vector of $W$.

ii) tuple-transitive (or transitive) if for every pair of non-empty open subsets $U, V$ of $X$, there is $t \in \mathbb{R}_+^d$ such that $W(t)(U) \cap V \neq \emptyset$.

We denote by $HC(W)$ the set of all hypercyclic vectors of $W$.

One can see that a $d$-parameter $C_0$-semigroup $W$ on $X$ is hypercyclic if and only if it is transitive. Also if $u_i(t) = W(te_i)$ is a hypercyclic one-parameter $C_0$-semigroup, for some $i = 1, 2, ..., d$, then $W$ is hypercyclic. However, we will show that the converse is not true in general.

**Remark 2.1.** Let $X$ be a real-Banach space, $\bar{X}$ be the complexification of $X$, $\{ T(t) \}_{t \geq 0}$ be a one-parameter $C_0$-semigroup on $X$ and $\{ \bar{T}(t) \}_{t \geq 0}$ be the complexification of $\{ T(t) \}_{t \geq 0}$.

If $\{ \bar{T}(t) \}_{t \geq 0}$ is hypercyclic, then so is $\{ T(t) \}_{t \geq 0}$ though its converse is not true in general. Suppose that $W$ is a $d$-parameter $C_0$-semigroup on $X$ and $\bar{W}$ is its complexification $d$-parameter $C_0$-semigroup on $\bar{X}$. One can see that hypercyclicity of $\bar{W}$ implies that $W$ is also hypercyclic. Conversely, if $W$ is a $d$-parameter hypercyclic $C_0$-semigroup on $X$, then $(W_1, \mathbb{R}_+^d, \bar{X})$ defined by

$$W_1(s, t)(x + iy) = W(s)x + iW(t)y, \ s, t \in \mathbb{R}_+^d, \ x, y \in X$$
is a 2d-parameter $C_0$-semigroup on $\overline{X}$.

In general case, if $W_1$ and $W_2$ are two hypercyclic $d_1$-parameter and $d_2$-parameter $C_0$-semigroup on Banach spaces $X$ and $Y$, respectively, then $W(t_1, t_2) \in B(X \oplus Y)$, $t_1 \in \mathbb{R}^{d_1}$, $t_2 \in \mathbb{R}^{d_2}$, defined by $W(t_1, t_2)(x \oplus y) = W_1(t_1)x \oplus W_2(t_2)y$, is a $d_1 + d_2$-parameter hypercyclic $C_0$-semigroup on $X \oplus Y$. This together with the fact that $X \oplus Y \cong X \times Y$ implies that $W_1(t_1) \times W_2(t_2)$ is a tuple-hypercyclic on $X \times Y$.

As a consequence of this remark, one may construct a hypercyclic d-parameter $C_0$-semigroup on $L^p_{m_1}(\mathbb{R}^+; \mathbb{C}^d)$, the space of all functions $f = (f_1, \ldots, f_d) : \mathbb{R}^+ \rightarrow \mathbb{C}^d$ with $f_i \in L^p_{m_i}(\mathbb{R}^+)$, where $a = (a_1, \ldots, a_d)$ and $m_i(x) = e^{a_i|x|}$, $i = 1, 2, \ldots, d$. Trivially the equality $L^p_{m_1}(\mathbb{R}^+; \mathbb{C}^d) = \bigoplus_{i=1}^d L^p_{m_i}(\mathbb{R}^+)$ holds. Now it is well-known that the family $\{U_i(t)\}_{t \geq 0}$ defined on $L^p_{m_i}(\mathbb{R}^+)$ by $U_i(t)f(x) = f(x + t)$ is a hypercyclic one-parameter $C_0$-semigroup, $i = 1, \ldots, d$. Thus, $W(t_1, \ldots, t_d)$ defined on $L^p_{m_1}(\mathbb{R}^+; \mathbb{C}^d)$ by $W(t_1, \ldots, t_d)f(x) = (f_1(x + t_1), \ldots, f_d(x + t_d))$ is a hypercyclic d-parameter $C_0$-semigroup.

As another important class of tuple $C_0$-semigroups, we may consider hypercyclicity of the d-parameter translation semigroups on $L^p_{c}(\mathbb{R}^+)$.

Let $1 \leq p < \infty$ and $\nu : \mathbb{R}^+ \rightarrow \mathbb{R}$ be a locally integrable function. We consider the space of 2-variable weighted $p$-integrable functions defined by

$$L^p_c(\mathbb{R}^+) := \{ f : \mathbb{R}^+ \rightarrow C : f \text{ is measurable and } \| f \| < \infty \},$$

where $\| f \| := \left( \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} |f(x,y)|^p \nu(x,y) \, dx \, dy \right)^{\frac{1}{p}}$. For any $f \in L^p_c(\mathbb{R}^+)$, define $W(s,t)f(x) = f(x + s, y + t)$. The proof of the following lemma and proposition is similar to the proofs described in Example 7.4 and Example 7.10 [13].

**Lemma 2.1.** The family $\{W(s,t)\}_{s,t \geq 0}$ is a two-parameter $C_0$-semigroup on $L^p_c(\mathbb{R}^+)$ if and only if there exist $M > 0$, $w_1, w_2 \in \mathbb{R}$ such that

$$\nu(x,y) \leq Me^{w_1 s + w_2 t} \nu(x+s, y+t), \quad (s,t \geq 0).$$

In the following proposition, we assume that the weight $\nu$ satisfies (2.1) for any $x, y > 0$. Note that this is equivalent to

$$\nu(x,y) \leq Me^{w_1 (u-x) + w_2 (v-y)} \nu(u, v) \quad (u \geq x \geq 0, \quad v \geq y \geq 0).$$

**Proposition 2.1.** For the translation semigroup on the space $X = L^p_c(\mathbb{R}^+)$, the following assertions are equivalent:

(i) the translation semigroup is hypercyclic;

(ii) $\liminf_{\| (x,y) \| \rightarrow \infty} \nu(x,y) = 0$.

The following is a hypercyclic criterion for d-parameter $C_0$-semigroups.

**Theorem 2.1** (Hypercyclic criterion). Let $X$ be a separable Banach space and $(W, \mathbb{R}^+_d, X)$ be a two-parameter semigroup on $X$. Let $Y, Z \subseteq X$ be dense subsets of $X$ and $(t_k)_{k \in \mathbb{N}} \subseteq \mathbb{R}^+_d$ where $t_k = (t_{k1}, t_{k2}, \ldots, t_{kd})$, and $S_k : Z \rightarrow E, s \geq 0$, be a family of linear mapping such that for some $j = 1, 2, \ldots, d$

i) $\lim_{k \rightarrow \infty} W(t_k)y = 0$ for all $y \in Y$;

ii) $\lim_{k \rightarrow \infty} S_{tk_j}z = 0$ for all $z \in Z$;

iii) $W(t)S_{tk_j}z = z$ for all $z \in Z, t \in \mathbb{R}^+_d$ and $k \in \mathbb{N}$,

then $(W, \mathbb{R}^+_d, X)$ is transitive and in particular hypercyclic.

**Proof.** Let $U, V$ be a pair of non-empty open subsets of $X$. Fix $y \in U \cap Y$ and $z \in V \cap Z$. By i) and iii), $\lim_{k \rightarrow \infty} W(t_k)y = 0$ and $W(t)S_{tk_j}z = z$. Put $y_k = y + S_{tk_j}z$. Trivially $\lim_{k \rightarrow \infty} W(t_k)y_k = z$. This implies that $W(t_k)(U \cap V \neq \emptyset)$ for sufficiently large $k \in \mathbb{N}$. \qed
3. Hypercyclic $d$-parameter $C_0$-semigroups on finite dimensional Banach spaces

It is well-known that there is no hypercyclic one-parameter $C_0$-semigroup on a finite dimensional Banach space. In this section, we show that there are many tuple-hypercyclic $C_0$-semigroups on $\mathbb{R}^n$ and $\mathbb{C}^n$. In a $n$-dimensional complex (or real) Banach space, a one-parameter $C_0$-semigroup is essentially uniformly continuous and so is of the form $\{e^{tA}\}_{t \geq 0}$, for some $A \in M_n(\mathbb{C})$ (respectively, $A \in M_n(\mathbb{R})$). Thus a $d$-parameter $C_0$-semigroup on this space is of the form

$$W(t_1, t_2, \ldots, t_d) = e^{t_1 A_1 + t_2 A_2 + \cdots + t_d A_d}$$

where $A_j \in M_n(\mathbb{R})$ and $A_i A_j = A_j A_i$, for $i, j = 1, \ldots, d$.

In the following proposition, we construct tuple-hypercyclic $C_0$-semigroups on $\mathbb{R}^n$, using the translation semigroups.

**Proposition 3.1.** For any $n \in \mathbb{N}$, there exists a hypercyclic $2n$-parameter $C_0$-semigroup on $\mathbb{R}^n$.

**Proof.** Let $\{e_i : i = 1, 2, \ldots, n\}$ be the standard basis for $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$, define $u_i(t)x = x + te_i$ and $v_i(t)x = x - te_i$, $i = 1, \ldots, n$. Trivially $\{u_i(t)\}_{i \geq 0}$ and $\{u_i(t)\}_{i \geq 0}$, $i = 1, 2, \ldots, n$, are commuting $C_0$-semigroup of operators on $\mathbb{R}^n$ and so

$$W(t_1, t_2, \ldots, t_n, s_1, s_2, \ldots, s_n)x := \prod_{i=1}^{n} u_i(t_i) v_i(s_i)x, \quad (x \in \mathbb{R}^n)$$

is a $2n$-parameter $C_0$-semigroup. Also, one can see that in this case, $\text{Orb}(W, (1, 1, \ldots, 1)) = \mathbb{R}^n$. Thus $W$ is a hypercyclic $2n$-parameter $C_0$-semigroup on $\mathbb{R}^n$ with the hypercyclic vector $(1, 1, \ldots, 1) \in \mathbb{R}^n$. \[\square\]

As a consequence of Proposition 3.1, and Remark 2.1, $\mathbb{C}^n$ admits a $4n$-parameter hypercyclic $C_0$-semigroup.

In the following proposition, we construct a tuple-hypercyclic $C_0$-semigroup on $\mathbb{R}^n$ with less parameters using rotations and homogeneities.

**Proposition 3.2.** Let $n \in \mathbb{N}$ be given. If $n$ is even, then there exists a hypercyclic $n$-parameter $C_0$-semigroup, and if $n$ is odd, then there exists a hypercyclic $n + 1$-parameter $C_0$-semigroup on $\mathbb{R}^n$.

**Proof.** First let $n$ be even and $m = \frac{n}{2}$. For $k = 1, 2, \ldots, m$ and $s, t \geq 0$, define

$$W_k(s, t) := e^{2s} I_n$$

where $I_n$ is the identity matrix on $\mathbb{R}^n$ and the matrix

$$\begin{pmatrix}
1 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & \cos t & \sin t & 0 & \ldots & 0 \\
0 & \ldots & 0 & -\sin t & \cos t & 0 & \ldots & 0 \\
0 & \ldots & 0 & 0 & 0 & 1 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & \ldots & \ldots & 0 & 0 & 1
\end{pmatrix} \quad (3.1)$$

begin from $2k - 1$th columns and row. One can easily see that $W_k(s, t)$ is a two-parameter $C_0$-semigroup on $\mathbb{R}^n$. Now, define $W : \mathbb{R}^+ \to M_n(\mathbb{R})$ as follows

$$W(s_1, t_1, s_2, t_2, \ldots, s_m, t_m) := \prod_{k=1}^{m} W_k(s_k, t_k). \quad (3.2)$$
It is not hard to see that $W$ is a $n$-parameter $C_0$-semigroup on $\mathbb{R}^n$. Also, letting $v := (1,0,1,0,...,1,0)^T$, one can show that $\text{Orb}(w,v) = \mathbb{R}^n$. Thus, $W$ is a hypercyclic $n$-parameter $C_0$-semigroup on $\mathbb{R}^n$.

For an odd number $n = 2m + 1$, let $W_k$ be defined as (3.1), $k = 1,2,...,m$. Now define $W_{m+1} : \mathbb{R}_+^{m+1} \to M_n(\mathbb{R})$ by

$$W_{m+1}(s,t)x := x + (s-t)e_n.$$  

If we define $W : \mathbb{R}_+^{n+1} \to M_n(\mathbb{R})$ by

$$W(s_1,t_1,s_2,t_2,...,s_m,t_m,s_{m+1},t_{m+1}) := \prod_{k=1}^{m+1} W_k(s_k,t_k),$$  \hspace{1cm} (3.3)$$

then $W$ is a hypercyclic $(n + 1)$-parameter $C_0$-semigroup on $\mathbb{R}^n$.

Applying Proposition 3.2 and Remark 2.1, if $n$ is even (respectively, odd), then there exists a $2n$-parameter (respectively, $2n + 2$-parameter) $C_0$-semigroup on $\mathbb{C}^n$.

**Remark 3.1.** (1) It is well-known that if $\{T_t\}_{t \geq 0}$ is a hypercyclic $C_0$-Semigroup of operators then each $T_t$ $(t > 0)$ is hypercyclic as a single operator (see [5]). This is not true for the $d$-parameter case. Indeed in Section 3, it will be proved that finite dimensional Banach spaces admit a tuple-hypercyclic $C_0$-semigroup although there is no hypercyclic operator on these spaces.

(2) As another consequence of existences of a tuple-hypercyclic $C_0$-semigroup on finite dimensional Banach spaces, it can be concluded that if $W$ is a hypercyclic $C_0$-semigroup on a finite dimensional Banach space $X$, then $W^*(t)$, $t \in \mathbb{R}_+^d$, could has an eigenvalue.

4. Tensor product of tuple-hypercyclic $C_0$-semigroups

Recall that for Banach spaces $X$ and $Y$, we denote their (algebraic) tensor product by $X \otimes Y$. Furthermore, let $\alpha$ be a tensor norm (or uniform cross-norm) on $X \otimes Y$. Then $\alpha$ is, in particular, a reasonable cross-norm on $X \otimes Y$, which implies that for any $x \in X$ and $y \in Y$ we have

$$\alpha(x \otimes y) = \|x\|_X \cdot \|y\|_Y.$$  

It is well known that

$$\pi(z) = \inf \left\{ \sum_{i=1}^{n} \|x_i\|_X \cdot \|y_i\|_Y : z = \sum_{i=1}^{n} x_i \otimes y_i, \quad (z \in X \otimes Y) \right\}$$

defines a tensor norm on $X \otimes Y$, which is called the projective tensor norm. Actually, this norm is the greatest reasonable cross-norm on $X \otimes Y$. For any norm $\alpha$ on $X \otimes Y$, we denote by $X \hat{\otimes}_\alpha Y$ the completion of the normed space $(X \otimes Y, \alpha)$.

For bounded operators $T : X \to X$, $S : Y \to Y$ and any uniform cross-norm $\alpha$, the product $T \otimes S$ is a bounded operator on $(X \otimes Y, \alpha)$ by definition of uniform cross-norm. The unique extension of $T \otimes S$ to $X \hat{\otimes}_\alpha Y$ is for simplicity, also denoted by $T \otimes S$.

Let $\{T(t)\}_{t \geq 0}$ and $\{S(s)\}_{s \geq 0}$ be two one-parameter $C_0$-semigroups on Banach spaces $X$ and $Y$, respectively. One can prove that the tensor product $T(t) \otimes S(s)$ is a two-parameter $C_0$-semigroup on $(X \otimes Y, \alpha)$ for any uniform cross-norm $\alpha$. The following theorem shows that the tensor product of two hypercyclic one-parameter $C_0$-semigroups is a tuple-hypercyclic $C_0$-semigroup.

**Theorem 4.1.** Let $\{T(t)\}_{t \geq 0}$ and $\{S(s)\}_{s \geq 0}$ be two hypercyclic one-parameter $C_0$-semigroups on Banach spaces $X$ and $Y$, respectively. Then the two-parameter $C_0$-semigroup $T(t) \otimes S(s)$ is tuple-hypercyclic on $X \hat{\otimes}_\alpha Y$ for any reasonable cross norm $\alpha$. 

Proof. Let $\alpha$ be a reasonable cross norm. Consider the norm $(x,y) := \sup\{\|x\|, \|y\|\}$ on $X \times Y$. The canonical bilinear map
$$\psi : X \times Y \rightarrow (X \otimes Y, \alpha)$$
$$(x,y) \mapsto x \otimes y$$
is continuous and $\|\psi\| \leq 1$. So for any $n \geq 1$, the mapping
$$\psi_n : X^n \times Y^n \rightarrow X \otimes Y$$
$$(x_1, \ldots, x_n, y_1, \ldots, y_n) \mapsto \sum_{k=1}^n \psi(x_k, y_k)$$is continuous with the norm
$$\|(x_1, \ldots, x_n, y_1, \ldots, y_n)\| := \max\{\|x_1\|, \|y_1\|, \ldots, \|x_n\|, \|y_n\|\}$$on $X^n \times Y^n$. We shall show that $T(t) \otimes S(s)$ is tuple-transitive. Let $U$ and $V$ be non-empty open subsets of $X \otimes \alpha Y$. As $X \otimes Y = \text{span}(\psi(X \times Y))$ is dense in $X \otimes \alpha Y$, we may find elements $u = \sum_{k=1}^n x_k \otimes y_k \in U$ and $v = \sum_{k=1}^n p_k \otimes q_k \in V$. Without loss of generality, we may assume $m = n$ (by extending one of the sums by zero elements if necessary). Continuity of $\psi_n$ implies that $\psi_n^{-1}(U)$ and $\psi_n^{-1}(V)$ are non-empty open subsets of $X^n \times Y^n$. From hypercyclicity of $\{T(t)\}_{t \geq 0}$ and $\{S(s)\}_{s \geq 0}$, we deduce that the diagonal semigroups $T^n(t) = T(t) \times \ldots \times T(t)$ and $S^n(s) = S(s) \times \ldots \times S(s)$ are hypercyclic on $X^n$ and $Y^n$, respectively (see Corollary 1.3, [22]). By Remark 2.1, this implies that $T^n(t) \times S^n(s)$ is hypercyclic. So there exist $t, s > 0$ such that
$$T^n(t) \times S^n(s)(\psi_n^{-1}(U)) \cap \psi_n^{-1}(V) \neq \emptyset.$$But
$$\psi_n(T^n(t) \times S^n(s)(\psi_n^{-1}(U)) \cap \psi_n^{-1}(V)) \subseteq T(t) \otimes S(s)(U) \cap V.$$Therefore $\{T(t) \otimes S(s)\}_{s,t \geq 0}$ is tuple-transitive two-parameter $C_0$-semigroup.

5. J-class two-parameter $C_0$-semigroup

In this section, we study the locally topologically transitive two-parameter $C_0$-semigroups (or simply J-class two-parameter $C_0$-semigroups).

Definition 5.1. A two-parameter $C_0$-semigroup $\{W(s,t)\}_{s,t \geq 0}$ on a normed space $X$ is called J-class if there exists $0 \neq x \in X$, such that $J_W(x) = X$, where

$$J_W(x) := \{y \in X : \text{there exist strictly increasing sequences}$$

$$(s_n)_{n \in \mathbb{N}}, (t_n)_{n \in \mathbb{N}} \subseteq [0, \infty) \text{ with } s_n, t_n \rightarrow \infty \text{ and a sequence}$$

$$(x_n)_{n \in \mathbb{N}} \text{ in } X \text{ such that } x_n \rightarrow x \text{ and } W(s_n,t_n)(x_n) \rightarrow y\}.$$We put $A_W := \{x \in X : J_W(x) = X\}$.

The following is a topological characterization of J-class two-parameter $C_0$-semigroups. Our proof is a modification of the proof of a similar result in one-parameter case in [18].

Theorem 5.1. A Two-parameter $C_0$-semigroup $\{W(s,t)\}_{s,t \geq 0}$ on a Banach space $X$ is tuple J-class $C_0$-semigroup with a non-zero vector $x$ if and only if for every neighborhood $U$ of $x$ and open set $V \subset X$ there exist $s, t > 0$ such that $W(s,t)U \cap V \neq \phi$.

Proof. Assume that for every neighborhoods $U$ of $x$ and open set $V \subset X$ there exist $s, t > 0$ such that $W(s,t)U \cap V \neq \phi$. We show that $J_W(x) = X$. Let $y \in X$ be given and assume first that $y \notin \{W(s,t)x : s, t \geq 0\}$. For each $n \in \mathbb{N}$, there exists $s_n, t_n \geq 0$ with $W(s_n,t_n)(B_\frac{1}{n})(x) \cap B_\frac{1}{n}(y) \neq \phi$, which in particular means that there exists $x_n \in X$ with $\|x_n - x\| < \frac{1}{n}$ and $\|W(s_n,t_n)x_n - y\| < \frac{1}{n}$. This implies that $\lim_{n \to \infty} W(s_n,t_n)x_n = y$. We are going to show that $\lim_{n \in \mathbb{N}}$ and $\lim_{n \in \mathbb{N}}$ have a strictly increasing subsequences tending to infinity. Suppose in contrary that $\lim_{n \in \mathbb{N}}$ and $\lim_{n \in \mathbb{N}}$ are bounded. Then there are subsequences of $(s_n)_{n \in \mathbb{N}}$ and $(t_n)_{n \in \mathbb{N}}$ converging to some $t$ and $s$ respectively. Without lose of generality, we assume that $s_n \to t$ and $t_n \to s$. It follows from the continuity of the map $g : \mathbb{R}_+^2 \times X \to X$, $g(s,t,x) := W(s,t)x$, that $\lim_{n \to \infty} W(s_n,t_n)x_n = W(s,t)x = y$, which
Let suppose that \( t,s \) for fixed \( x \in X \) is a countable union of compact sets, in particular \( g_x(\mathbb{R}^2_+) = \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} g_x([0,n] \times [0,m]) \) and therefore is of the first category. Hence

\[
B_{\frac{1}{n}}(y) \setminus \{ W(s,t)x \mid s,t \geq 0 \} \neq \emptyset
\]

holds for each \( n \in \mathbb{N} \). Letting \( y_n \in B_{\frac{1}{n}}(y) \setminus \{ W(s,t)x \mid s,t \geq 0 \} \) and using the first part of the proof, there exists \( s_n, t_n > n \), such that \( ||W(t_n, s_n)x_n - y_n)|| < \frac{1}{n} \). Now we choose \( y_{n+1} \in B_{\frac{1}{n+1}}(y) \setminus \{ W(s,t)x \mid s,t \geq 0 \} \). Then again we choose \( t_{n+1} > \max\{t_n, n+1\} \), \( s_{n+1} > \max\{s_n, n+1\} \) and \( x_{n+1} \) with \( ||x_{n+1} - x|| < \frac{1}{n+1} \), such that

\[
||W(s_{n+1}, t_{n+1})x_{n+1} - y_{n+1}|| < \frac{1}{n+1}.
\]

It follows that \( (s_n)_{n \in \mathbb{N}} \) and \( (t_n)_{n \in \mathbb{N}} \) are strictly increasing sequences of positive real numbers tending to infinity and

\[
||W(s_n, t_n)x_n - y|| \leq ||W(s_n, t_n)x_n - y_n|| + ||y_n - y||
\]

\[
\leq \frac{1}{n} + \frac{1}{n} = \frac{2}{n}.
\]

This implies \( \lim_{n \to \infty} W(s_n, t_n)x_n = y \) and it shows the first direction. The other direction is obvious.

Trivially transitivity of \( \{ W(s, t) \}_{s,t \geq 0} \) implies its locally topologically transitivity for all \( x \in X \). Hence for any \( x \in HC(W) \), \( JW(x) = X \). As another consequence of Theorem 5.1, we get the following proposition.

**Proposition 5.1.** Let \( \{ T(t) \}_{t \geq 0} \) and \( \{ S(s) \}_{s \geq 0} \) be two \( J \)-class \( C_0 \)-semigroups on Banach spaces \( X \) and \( Y \), respectively. Then the two-parameter \( C_0 \)-semigroup \( \{ T(t) \times S(s) \}_{s,t \geq 0} \) is \( J \)-class on \( X \times Y \).

**Proof.** Suppose that \( JT(x) = X \) and \( JS(y) = Y \) for some \( 0 \neq x \in X \) and \( 0 \neq y \in Y \). We show that \( JT \times JS(x,y) = X \times Y \).

Let \( U \) be a non-empty open subset of \( x \times y \) and \( V \) be an arbitrary non-empty open subset of \( X \times Y \). So there exists non-empty open subsets \( U_1, U_2 \subseteq X \) and \( V_1, V_2 \subseteq Y \), such that \( (x,y) \in U_1 \times V_1 \subseteq U \) and \( U_2 \times V_2 \subseteq V \).

The semigroups \( \{ T(t) \}_{t \geq 0} \) and \( \{ S(s) \}_{s \geq 0} \) are \( J \)-class so there exists \( t, s > 0 \), such that

\[
T(t)U_1 \cap U_2 \neq \emptyset \text{ and } S(s)V_1 \cap V_2 \neq \emptyset.
\]

But

\[
T(t)U_1 \cap U_2 \times S(s)V_1 \cap V_2 \subset T(t) \times S(s)U \cap V,
\]

which completes the proof.

**Proposition 5.2.** Let \( X \) be a Banach space and \( \{ W(s,t) \}_{s,t \geq 0} \) be a two-parameter \( C_0 \)-semigroup. Suppose that \( (x_n)_{n \in \mathbb{N}} \) and \( (y_n)_{n \in \mathbb{N}} \) are two convergent sequences to \( x \) and \( y \), respectively, with the property that \( y_n \in JW(x_n) \). Then \( y \in JW(x) \). In particular, \( JW(x) \) and \( AW \) are closed subsets of \( X \).

**Proof.** Take \( y_n \) with \( n_1 \geq 1 \) large enough, such that \( ||y_{n_1} - y|| < 1 \). Since \( y_{n_1} \in JW(x_{n_1}) \), choose \( z_1 \) with \( ||z_1 - x_{n_1}|| < 1 \) and \( s_1, t_1 \geq 1 \) such that \( ||W(s_1, t_1)z_1 - y_{n_1}|| < 1 \). Next choose \( y_{n_2} \) with \( n_2 > n_1 \) and \( ||y_{n_2} - y|| < \frac{1}{2} \) and as above, since \( y_{n_2} \in JW(x_{n_2}) \), choose \( z_2 \), \( t_2 \geq \max\{2, t_1\} \) and \( s_2 \geq \max\{2, s_1\} \) with \( ||z_2 - x_{n_2}|| < \frac{1}{2} \) and \( ||W(s_2, t_2)z_2 - y_{n_2}|| < \frac{1}{2} \).
Inductively, we get sequences \((n_m)_{m \in \mathbb{N}}, (t_m)_{m \in \mathbb{N}}, (s_m)_{m \in \mathbb{N}}\) and \((z_m)_{m \in \mathbb{N}}\) with the property that \(t_m \to \infty, s_m \to \infty\) and

\[
||z_m - x_{n_m}|| < \frac{1}{m} \quad ||y - y_{n_m}|| < \frac{1}{m} \quad ||W(s_m, t_m)z_m - y_{n_m}|| < \frac{1}{m}
\]

hold for each \(m \in \mathbb{N}\). Now, \(||W(t_m, s_m)z_m - y|| \leq ||W(t_m, s_m)z_m - y_{n_m}|| + ||y_{n_m} - y|| \leq \frac{1}{m}\)

and therefore \(\lim_{m \to \infty} z_m = x\) and \(\lim_{m \to \infty} W(t_m, s_m)z_m = y\). So we proved \(y \in J_W(x)\).

**Theorem 5.2.** A two-parameter \(C_0\)-semigroup \(\{W(s, t)\}_{s, t \geq 0}\) on a Banach space \(X\) is tuple-hypercyclic if and only if \(J_W(x) = X\) for every \(x \in X\).

**Proof.** Let \(\{W(s, t)\}_{s, t \geq 0}\) be tuple-hypercyclic. Trivially \(HC(W) \subset A_W\). Since \(A_W\) is closed subset of \(X\) and \(HC(W)\) is dense in \(X\), we get \(A_W = X\). Conversely, suppose that \(J_W(x) = X\) for all \(x \in X\). It is enough to show that \(W\) is tuple-transitive. Let \(U\) and \(V\) be non empty open sets in \(X\) and \(x \in U\) be chosen. From \(J_W(x) = X\) there exists \((t, s) \in \mathbb{R}_+^2\) such that \(W(t, s)U \cap V \neq \emptyset\), which implies that \(W\) is tuple transitive. □

**REFERENCES**