ON THE EXISTENCE OF POSITIVE SOLUTIONS FOR A CLASS OF INFINITE SEMIPOSITONE PROBLEMS

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We discuss the existence of a positive solution to the infinite semipositone problem

\[ -\Delta u = -au + bu^2 - du^3 - f(u) - \frac{c}{u^\alpha}, \quad x \in \Omega, \quad u = 0, \quad x \in \partial \Omega, \]

where \( \alpha \in (0, 1) \), \( a, b, d \) and \( c \) are positive constants, \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) with smooth boundary \( \partial \Omega \), \( \Delta \) is the Laplacian operator, and \( f : [0, \infty) \to \mathbb{R} \) is a nondecreasing continuous function such that \( f(u) \to \infty \) and \( f(u)/u \to 0 \) as \( u \to \infty \). We obtain our result via the method of sub- and supersolutions. We also extend our result to classes of infinite semipositone system and \( p \)-Laplacian problem.

Keywords: Positive solution; Infinite semipositone; Sub- and supersolutions
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1. Introduction

Consider the boundary value problem

\[
\begin{aligned}
-\Delta u &= -au + bu^2 - du^3 - f(u) - \frac{c}{u^\alpha}, & x \in \Omega, \\
u &= 0, & x \in \partial \Omega, 
\end{aligned}
\]

where \( \alpha \in (0, 1) \), \( a, b, d \) and \( c \) are positive constants, and \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) with smooth boundary \( \partial \Omega \), \( \Delta \) is the Laplacian operator, and \( f : [0, \infty) \to \mathbb{R} \) is a continuous function. We make the following assumptions:

(H1) \( f : [0, +\infty) \to \mathbb{R} \) is nondecreasing continuous functions such that

\[ \lim_{s \to +\infty} f(s) = \infty. \]

(H2) \( \lim_{s \to +\infty} \frac{f(s)}{s} = 0. \)

Note that (1.1) is as an infinite semipositone problems (\( \lim_{u \to 0} F(u) = -\infty \), where \( F(u) := -au + bu^2 - du^3 - f(u) - (c/u^\alpha) \)).

In [9], the authors have studied the case when \( F(u) := g(u) - (c/u^\alpha) \) where \( g \) is nonnegative and nondecreasing and \( \lim_{u \to \infty} g(u) = \infty \). The case \( g(u) := au - f(u) \)

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has been studied in [8], where $f(u) \geq au - M$ and $f(u) \leq Au^p$ on $[0, \infty)$ for some $M, A > 0, p > 1$ and this $g$ may have a falling zero. A simple example of this $g$ is $g(u) = u - u^p$, where $p > 1$. Note that this $g$ has a falling zero at $u = 1$, in fact $g$ is negative for $u > 1$. In this article, we consider the case when $g(u) := -au + bu^2 - du^3 - f(u)$ and we study more challenging infinite semipositone problem. A example of $f$ satisfying our hypotheses is $f(x) = x^p; 0 < p < 1$. Further, let $0, R_1$ and $R_2$ denote the zeros of $-au + bu^2 - du^3$ (such that $R_1 < R_2$), then $g(u) = -au + bu^2 - du^3 - u^p$ is negative for $u < R_1$ and $u > R_2$.

In recent years, there has been considerable progress on the study of semipositone problems ($F(0) < 0$ but finite)(see [2],[3],[6]). Many results have been obtained of infinite semipositone problems; see for example [7], [8], [9] and [10]. In [1], the authors establish the existence of a positive solution to $-\Delta u = -au + bu^2 - du^3$ with Dirichlet boundary conditions and the method employed in it uses the fact that $-\inf_{x \in [0,R_2]}(-au + bu^2 - du^3) < ar$, where $r$ is the first positive zero of $(-au + bu^2 - du^3)$. We will use in this paper this fact, too. The main tool used in this study is the method of sub- and supersolutions ([4]).

2. The main result

In this section, we shall establish our existence result via the method of sub-supersolution. A function $\psi$ is said to be a subsolution of (1.1) if it is in $C^2(\Omega) \cap C(\overline{\Omega})$ such that $\psi = 0$ on $\partial \Omega$ and

$$-\Delta \psi \leq -a\psi + b\psi^2 - d\psi^3 - f(\psi) - \frac{c}{\psi^\alpha} \quad \text{in } \Omega,$$

and $z$ is said supersolution of (1.1) if it is in $C^2(\Omega) \cap C(\overline{\Omega})$ such that $z = 0$ on $\partial \Omega$ and

$$-\Delta z \geq -az + bz^2 - dz^3 - f(z) - \frac{c}{z^\alpha} \quad \text{in } \Omega.$$

Then it is well known that if there exist a subsolution $\psi$ and supersolution $z$ such that $\psi \leq z$ in $\Omega$ then (1.1) has a solution $u$ such that $\psi \leq u \leq z$, see [4].

Theorem 2.1. Let (H1) and (H2) hold, Then there exists positive constants $b_0 := b_0(a, d, \Omega)$ and $c_0 := c_0(a, b, d, \Omega)$ such that for $b \geq b_0$ and $c \leq c_0$, problem (1.1) has a positive solution.

Proof. Let $\lambda_1 > 0$ be the first eigenvalue of the operator $-\Delta$ with Dirichlet boundary condition and $\phi_1$ be the corresponding eigenfunction satisfying $\phi_1 > 0$ in $\Omega$ and $\frac{\partial \phi_1}{\partial \nu} < 0$ on $\partial \Omega$, where $\nu$ is outward normal vector on $\partial \Omega$ and $\|\phi_1\|_\infty = 1$, see [5]. Note that $\lambda_1$ and $\phi_1$ satisfy:

$$-\Delta \phi_1 = \lambda_1 \phi_1 \quad \text{in } \Omega,$$

$$\phi_1 = 0 \quad \text{on } \partial \Omega.$$

Let $\delta > 0$, $\mu > 0$, $m > 0$ be such that

$$\frac{2}{1 + \alpha} \{(1 - \frac{\alpha}{1 + \alpha})|\nabla \phi_1|^2 - \lambda_1 \phi_1^2\} \geq m \quad \text{in } \overline{\Omega}_\delta, \quad (2.1)$$

and $\phi_1 \in [\mu, 1]$ in $\Omega \setminus \overline{\Omega}_\delta$, where $\overline{\Omega}_\delta := \{x \in \Omega : d(x, \partial \Omega) \leq \delta\}$. This is possible since $|\nabla \phi_1| \neq 0$ on $\partial \Omega$ while $\phi_1 = 0$ on $\partial \Omega$.
Let \( b_0 > 2\sqrt{ab} \) and \( P(s) = -as + bs^2 - ds^3 \). Then the zeros of \( P(s) \) are
\[
0, R_1 = \frac{b-\sqrt{b^2-4ad}}{2d}, \quad \text{and} \quad R_2 = \frac{b+\sqrt{b^2-4ad}}{2d}.
\]
We note that \( P(s) \) can be factored as \( P(s) = -ds(s-R_1)(s-R_2) \). Let \( r = \frac{b-\sqrt{b^2-3ad}}{3d} \) denote the first positive zero of \( P'(s) \). Since \( P(s) \) is convex on \((0, \frac{b}{3d})\) and \( r < \frac{b}{3d} \), we have \( \rho := \inf_{s \in [0, R_2]} P(s) < a(\sqrt{b^2 - 3ad} / 3d) = a(\frac{b}{3d}) \) (see Fig 1). We note that
\[
\frac{\rho}{R_2} < \frac{a(b-\sqrt{b^2 - 3ad/3d})}{b+\sqrt{b^2 - 4ad/2d}} = \frac{2a^2d}{(b + \sqrt{b^2 - 4ad})(b + \sqrt{b^2 - 3ad})} \to 0 \text{ as } b \to \infty,
\]
\[
\frac{R_2}{R_1} = \frac{b + \sqrt{b^2 - 4ad}}{b - \sqrt{b^2 - 4ad}} = \frac{(b + \sqrt{b^2 - 4ad})^2}{4ad} \to \infty \text{ as } b \to \infty.
\]
Hence there exists \( b_0(1) := b_0(1)(a,d,\Omega) \) such that for every \( b > b_0(1) \) we have
\[
\frac{\rho}{R_2} < \frac{m}{6}, \tag{2.2}
\]
\[
[\frac{R_2}{2} \mu^{1/\alpha} , \frac{R_2}{2}] \subset (R_1, R_2) \text{ and } k_\mu := \inf_{s \in [\frac{R_2}{2} \mu^{1/\alpha}, \frac{R_2}{2}]} P(s) > 0. \]
Next we see that
\[
k_\mu \frac{R_2}{R_2} = \min \left\{ \frac{P\left(\frac{R_2}{2} \mu^{1/\alpha}\right)}{P\left(\frac{R_2}{2}\right)} \right\} = \min \left\{ \frac{d^2 R_2}{4} \mu^{1/\alpha}, \frac{d^2 R_2}{4} \mu^{1/\alpha} - R_1 \right\} \to \infty \text{ as } b \to \infty
\]
and hence there exists \( b_0(2) := b_0(2)(a,d,\Omega) \) such that for every \( b > b_0(2) \) we have
\[
\frac{k_\mu}{R_2} > \frac{2\lambda_1}{1 + \alpha}.
\]
Finally from (H1) and (H2), \( f(R_2) \to \infty \) and \( f(R_2/2)/(R_2/2) \to 0 \) as \( b \to \infty \). Thus there exists \( b_0^{(3)} := b_0^{(3)}(a, d, \Omega) \) such that for every \( b > b_0^{(3)} \) we have \( f(R_2) \geq 0 \) and

\[
f\left(\frac{R_2}{2}\phi_1^{1/\alpha}\right) \leq f\left(\frac{R_2}{2}\right) \leq \min\left\{\lambda_1, \frac{m}{3}\right\}\left(\frac{R_2}{2}\right). \tag{2.3}\]

For a given \( a, d > 0 \), define \( b_0 := \max\{b_0^{(1)}, b_0^{(2)}, b_0^{(3)}\} \) and \( c_0 := c_0(a, b, d, \Omega) := \min \left\{ \frac{m}{3} (\frac{R_2}{2})^{1+\alpha}, (\frac{R_2}{2})^{\alpha/2} \right\} \), and let \( b \geq b_0 \) and \( c \leq c_0 \). We will show that \( \psi := R\phi_1^{2/1+\alpha} \) is a subsolution of (1.1), where \( R := \frac{R_2}{2} \).

We first note that

\[
\nabla \psi = R\left(\frac{2}{1+\alpha}\right)\phi_1^{\frac{1-\alpha}{1+\alpha}}\nabla \phi_1
\]

and

\[
-\Delta \psi = -R\left(\frac{2}{1+\alpha}\right)\left\{\phi_1^{\frac{1-\alpha}{1+\alpha}} \Delta \phi_1 + \frac{1-\alpha}{1+\alpha} \phi_1^{\frac{1-\alpha}{1+\alpha}} \left|\nabla \phi_1\right|^2 \right\}
= R\left(\frac{2}{1+\alpha}\right)\frac{1}{\left(\phi_1^{\frac{1-\alpha}{1+\alpha}}\right)^\alpha} \left\{\lambda_1 \phi_1^2 - \left(\frac{1-\alpha}{1+\alpha}\right)\left|\nabla \phi_1\right|^2 \right\}.
\]

Next for \( x \in \Omega_\delta \) since \( \frac{1}{\left(\phi_1^{\frac{1-\alpha}{1+\alpha}}\right)^\alpha} \geq 1 \), from (2.1),(2.2),(2.3) and \( c \leq c_0 \) we have

\[
-\Delta \psi = R\left(\frac{2}{1+\alpha}\right)\frac{1}{\left(\phi_1^{\frac{1-\alpha}{1+\alpha}}\right)^\alpha} \left\{\lambda_1 \phi_1^2 - \left(\frac{1-\alpha}{1+\alpha}\right)\left|\nabla \phi_1\right|^2 \right\}
\leq -mR\frac{1}{\left(\phi_1^{\frac{1-\alpha}{1+\alpha}}\right)^\alpha}
= -mR \frac{1}{3(\phi_1^{\frac{1-\alpha}{1+\alpha}})^\alpha} - mR \frac{1}{3(\phi_1^{\frac{1-\alpha}{1+\alpha}})^\alpha} - mR \frac{1}{3(\phi_1^{\frac{1-\alpha}{1+\alpha}})^\alpha}
\leq -mR - mR \frac{3}{3} - mR \frac{3}{3(\phi_1^{\frac{1-\alpha}{1+\alpha}})^\alpha}
\leq -\rho - f(R\phi_1^{\frac{2}{1+\alpha}}) - mR^{1+\alpha}/3(\phi_1^{\frac{2}{1+\alpha}})^\alpha
\leq -a\psi + b\psi^2 - d\psi^3 - f(\psi) - \frac{c}{\psi^\alpha}. \tag{2.4}
\]
Also for \( x \in \Omega \setminus \overline{\Omega}_\delta \), since \( 0 < \mu \leq \phi \), from (2.3) and \( c \leq c_0 \),

\[
-\Delta \psi = R\left(\frac{2}{1+\alpha}\right)\frac{1}{(\phi^\alpha_1)^\alpha} \{ \lambda_1 \phi_1^2 - \left(\frac{1-\alpha}{1+\alpha}\right) |\nabla \phi_1|^2 \}
\]

\[
\leq R\left(\frac{2}{1+\alpha}\right) \lambda_1 \phi_1^2
\]

\[
\leq R\left(\frac{2}{1+\alpha}\right) \lambda_1
\]

\[
= 2\left[R\left(\frac{2}{1+\alpha}\right) \lambda_1\right] - R\left(\frac{2}{1+\alpha}\right) \lambda_1
\]

\[
\leq \frac{4\lambda_1}{1+\alpha} \{ R - R\lambda_1 \}
\]

\[
\leq k\mu - \frac{c}{(R\phi_1^2)^\alpha} - f(R\phi_1^2)^\alpha
\]

\[
\leq -a\psi + b\psi^2 - d\psi^3 - f(\psi) - \frac{c}{\psi^\alpha}. \quad (2.5)
\]

According to (2.4) and (2.5), we can conclude that \( \psi \) is a subsolution of (1.1). We also show that \( z := R^2 \) is a supersolution, by noting that

\[
-\Delta z = 0 \geq -f(z) - \frac{c}{z^\alpha} = -az + bz^2 - dz^3 - f(z) - \frac{c}{z^\alpha}.
\]

Further \( z \geq \psi \). Thus, (1.1) has a positive solution. This completes the proof of Theorem 2.1. \( \Box \)

3. Extension of (1.1) to system (3.1)

In this section, we consider the extension of (1.1) to the following system:

\[
\begin{cases}
-\Delta u = -a_1 u + b_1 u^2 - d_1 u^3 - f_1(u) - \frac{c_1}{u^\alpha}, & x \in \Omega,

-\Delta v = -a_2 v + b_2 v^2 - d_2 v^3 - f_2(v) - \frac{c_2}{v^\alpha}, & x \in \Omega,

u = 0 = v, & x \in \partial \Omega,
\end{cases}
\]

(3.1)

where \( \alpha \in (0,1) \), \( a_1, a_2, b_1, b_2, d_1, d_2, c_1 \) and \( c_2 \) are positive constants, \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) with smooth boundary \( \partial \Omega \), and \( f_i : [0, \infty) \to \mathbb{R} \) is a continuous function for \( i = 1, 2 \). We make the following assumptions:

(H3) \( f_i : [0, +\infty) \to \mathbb{R} \) is nondecreasing continuous functions such that

\[
\lim_{s \to +\infty} f_i(s) = \infty \text{ for } i = 1, 2.
\]

(H4) \( \lim_{s \to +\infty} \frac{f_i(s)}{s} = 0 \) for \( i = 1, 2 \).

We prove the following result by finding sub-super solutions to infinite semipositone system (3.1).

**Theorem 3.1.** Let (H3) and (H4) hold, Then there exists positive constants \( b_0^* := b_0^*(a_1, a_2, d_1, d_2, \Omega) \) and \( c_0^* := c_0^*(a_1, a_2, b_1, b_2, d_1, d_2, \Omega) \) such that for \( \min\{b_1, b_2\} \geq b_0^* \) and \( \max\{c_1, c_2\} \leq c_0^* \), problem (3.1) has a positive solution.
Let $(R_1^{(i)}, R_2^{(i)}, \rho^{(i)}, k_\mu^{(i)})$, $P_i(s) := -a_is + b_is^2 - d_is^3$ for $i = 1, 2$ be given, as in section 2. By the same argument as in section 2, there exists $b_0 := b_0(a_1, a_2, d_1, d_2, \Omega)$ such that for $\min\{b_1, b_2\} > b_0$ we have

$$\frac{\rho^{(i)}}{R_2^{(i)}} < \frac{m}{6}, \quad \frac{k_\mu^{(i)}}{R_2^{(i)}} > \frac{2\lambda_1}{1 + \alpha},$$

and $f_i(R_2^{(i)} \phi_1 \frac{1}{2}) \leq \min \left\{ \lambda_1, \frac{m}{3} \right\} \left( R_2^{(i)} \frac{1}{2} \right)$ for $i = 1, 2$. Define

$c_0^\ast := c_0^\ast(a_1, a_2, b_1, b_2, d_1, d_2, \Omega)$

$$:= \min \left\{ \frac{m}{3} \left( \frac{R_2^{(1)}}{2} \right)^\alpha, \frac{m}{3} \left( \frac{R_2^{(2)}}{2} \right)^\alpha, \left( \frac{R_2^{(1)}}{2} \right)^\alpha \mu^{2\alpha/1 + \alpha}(k_\mu^{(1)}) - \frac{2\lambda_1}{1 + \alpha} R_2^{(1)}, \right\}$$

and $(\psi_1, \psi_2) := (R_2 \phi_1 2/1 + \alpha, R_2 \phi_1 2/1 + \alpha)$, where $R_2 = R_2^{(1)}/2$. Let $\min\{b_1, b_2\} > b_0^\ast$ and $\max\{c_1, c_1\} \leq c_0^\ast$, then for $x \in \overline{\Omega}_\delta$ we have

$$-\Delta \psi_1 = R_2^{(1)} \left( \frac{2}{1 + \alpha} \right) \left( \frac{1}{\phi_1 2/1 + \alpha} \right) \left\{ \lambda_1 \phi_1^2 - \left( \frac{1 - \alpha}{1 + \alpha} \right) |\nabla \phi_1|^2 \right\}$$

$$\leq -mR_2^{(1)} \frac{1}{(\phi_1 2/1 + \alpha)} - \frac{mR_2^{(1)}}{3} - \frac{mR_2^{(1)}}{3(\phi_1 2/1 + \alpha)}$$

$$\leq -\rho_1 - f(R_2^{(1)} \phi_1 \frac{2}{1 + \alpha}) - \frac{mR_2^{(1)}[R_2^{(2)}]^{\alpha}/3}{(R_2^{(2)} \phi_1 \frac{2}{1 + \alpha})^\alpha}$$

$$\leq -a_1 \psi_1 + b_1 \psi_1^2 - d_1 \psi_1^3 - f(\psi_1) - \frac{c_1}{\psi_2^\ast}.$$
Similarly
\[-\Delta \psi_2 \leq -a_2 \psi_2 + b_2 \psi_2^2 - d_2 \psi_2^3 - f(\psi_2) - \frac{c_2}{\psi_1}, \quad x \in \Omega.\]
Thus the \((\psi_1, \psi_2)\) is a subsolution of (3.1). It is obvious that \((z_1, z_2) := (R_2^{(1)}, R_2^{(2)})\) is a supersolution of (3.1), such that \((z_1, z_2) \geq (\psi_1, \psi_2)\). Thus Theorem 3.1 is proven. \(\square\)

4. Extension of (1.1) to problem (4.1)

In this section, we consider the extension of (1.1) to the following problem:
\[
\begin{aligned}
-\Delta_p u &= -au + bu^2 - du^3 - f(u) - \frac{c}{u^\alpha}, \quad x \in \Omega, \\
u &= 0, \quad x \in \partial \Omega,
\end{aligned}
\]
where \(\Delta_p z = \text{div}(|\nabla z|^{p-2} \nabla z)\), \(p > 1\), \(\alpha \in (0, 1)\), \(a, b, d\) and \(c\) are positive constants, \(\Omega\) is a bounded domain in \(\mathbb{R}^N\) with smooth boundary \(\partial \Omega\), and \(f : [0, \infty) \to \mathbb{R}\) is a continuous function. Then we have the following result.

**Theorem 4.1.** Let (H1) and (H2) hold, Then there exists positive constants \(b_0^{**} := b^{**}_0(a, d, \Omega)\) and \(c_0^{**} := c^{**}_0(a, b, d, \Omega)\) such that for \(b \geq b^{**}_0\) and \(c \leq c^{**}_0\), problem (4.1) has a positive solution.

**Proof.** We shall establish Theorem 4.1 by constructing positive sub-super solutions to equation (4.1). Let \(\lambda_1\) be the first eigenvalue of the problem
\[-\Delta_p \phi_1 = \lambda_1 \phi_1^{p-1}, \quad x \in \Omega, \quad \phi_1 = 0, \quad x \in \partial \Omega,
\]
where \(\phi_1\) denote the corresponding eigenfunction, satisfying \(\phi_1 > 0\) in \(\Omega\) and \(|\nabla \phi_1| > 0\) on \(\partial \Omega\), see [5]. Without loss of generality, we let \(\|\phi_1\|_{\infty} = 1\). Let \(\delta > 0\), \(\mu > 0\), \(m > 0\) be such that
\[
\left(\frac{p}{p-1+\alpha}\right)^{p-1} \left\{ \frac{(1-\alpha)(p-1)}{p-1+\alpha} |\nabla \phi_1|^p - \lambda_1 \phi_1^p \right\} \geq m \quad \text{in} \quad \overline{\Omega}_\delta,
\]
and \(\phi_1 \in [\mu, 1]\) in \(\Omega \setminus \overline{\Omega}_\delta\), where \(\overline{\Omega}_\delta := \{x \in \Omega : d(x, \partial \Omega) \leq \delta\}\). This is possible since \(|\nabla \phi_1| \neq 0\) on \(\partial \Omega\) while \(\phi_1 = 0\) on \(\partial \Omega\). Also let \(R_1, R_2\) be as in section 2 and \(b_0^{**}\) be such that for every \(b > b_0^{**}\)
\[
\frac{\rho}{R_2^{p-1}} < \frac{m}{6}, \quad \frac{k_\mu}{R_2^{p-1}} > \left(\frac{\lambda_1}{2}\right) \left(\frac{p}{p-1+\alpha}\right)^{p-1},
\]
and
\[
f\left([R_2^{p-1}\phi_1]^{\frac{p}{p-1+\alpha}}\right) \leq \min \left\{ \lambda_1, \frac{m}{3} \right\} \left(\frac{R_2}{2}\right)^{p-1}.
\]
Define
\[
c_0^{**} := c^{**}_0(a, b, d, \Omega)
\]
\[:= \min \left\{ \left(\frac{m}{3}\right) \left(\frac{R_2}{2}\right)^{(p-1)(1+\alpha)}, \left(\frac{R_2}{2}\right)^{\alpha(p-1)} \mu^{\frac{1-\alpha}{p-1+\alpha}} [k_\mu - R_2 \lambda_1 (\frac{p}{p-1+\alpha})^{p-1}] \right\},
\]
and \(\psi := R \phi_1^{\frac{p}{p-1+\alpha}}\). Then
\[
\nabla \psi = R \left(\frac{p}{p-1+\alpha}\right)^{\frac{1-\alpha}{p-1+\alpha}} \nabla \phi_1.
\]
\[ \Delta_p \psi = \text{div}(\lvert \nabla \psi \rvert^{p-2} \nabla \psi) \]
\[ = R^{p-1} \left( \frac{p}{p-1 + \alpha} \right)^{p-1} \text{div} \left( \phi_1^{\frac{(1-\alpha)(p-1)}{p-1 + \alpha}} \nabla \phi_1 \nabla \phi_1 \right) \]
\[ = R^{p-1} \left( \frac{p}{p-1 + \alpha} \right)^{p-1} \left\{ \nabla \phi_1^{\frac{(1-\alpha)(p-1)}{p-1 + \alpha}} \nabla \phi_1 \nabla \phi_1 + \phi_1^{\frac{(1-\alpha)(p-1)}{p-1 + \alpha}} \Delta_p \phi_1 \right\} \]
\[ = R^{p-1} \left( \frac{p}{p-1 + \alpha} \right)^{p-1} \left\{ \frac{1 - \alpha}{p - 1 + \alpha} \phi_1^{\frac{-\alpha}{p-1 + \alpha}} \nabla \phi_1^{p - \lambda_1 \phi_1} - \phi_1^{\frac{p-p-1}{p-1 + \alpha}} \right\} \]
\[ = R^{p-1} \left( \frac{p}{p-1 + \alpha} \right)^{p-1} \left\{ \frac{1 - \alpha}{p - 1 + \alpha} \phi_1^{\frac{p}{p-1 + \alpha}} \nabla \phi_1^{p - \lambda_1 \phi_1} \right\}. \]

By the same argument as in the proof of theorem 2.1, we can show that \( \psi \) is a subsolution of (4.1) for \( b \geq b_0^{**} \) and \( c \leq c_0^{**} \). Next, it is easy to check that \( z := R_2 \) is a supersolution of (4.1) with \( z \geq \psi \). Hence (4.1) has a positive solution and the proof is complete.

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