PROJECTION METHODS WITH LINESEARCH TECHNIQUE FOR PSEUDOMONOTONE EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS

Li-Jun ZHU[1], Yonghong YAO[2], Mihai POSTOLACHE[3]

In this paper, we discuss pseudomonotone equilibrium problems and fixed point problems in real Hilbert spaces. With the help of linesearch technique, we propose a projection algorithm without any Lipschitz-type condition for solving equilibrium problems and fixed point problems. We show that the constructed algorithm converges strongly to a common element of the investigated problems.

Keywords: pseudomonotone equilibrium problem, fixed point problem, pseudo-contractive operators, projection.

MSC2010: 90C25, 47J25, 47J40, 65K10.

1. **Introduction**

Let H be a real Hilbert space. Let C be a nonempty closed and convex subset of H. Let $f : C \times C \to \mathbb{R}$ be a bifunction. Recall that the equilibrium problem (shortly, $EP(f, C)$) consists of finding a point $\tilde{x} \in C$ such that

$$f(\tilde{x}, x) \geq 0, \ \forall x \in C. \tag{1}$$

Denote the set of solutions of $EP(f, C)$ by $Sol(f, C)$.

The $EP(f, C)$ has attracted so much attention both in its theory and some relevant applications which can be refined from minimization problems, Nash equilibria ([2, 3, 14]), fixed point problems, variational inequalities ([11, 16]) and so on. Iterative algorithms for solving $EP(f, C)$ have investigated and further developed in many different forms such as the proximal point algorithms ([13]), the projection algorithms ([1]), the subgradient algorithms ([4, 8]) and the extragradient algorithms ([6, 19]).

[1] The Key Laboratory of Intelligent Information and Big Data Processing of NingXia Province, North Minzu University, Yinchuan 750021, China and Health Big Data Research Institute of North Minzu University, Yinchuan 750021, China, e-mail: zhulijun1995@sohu.com

[2] The Key Laboratory of Intelligent Information and Big Data Processing of NingXia Province, North Minzu University, Yinchuan 750021, China and Health Big Data Research Institute of North Minzu University, Yinchuan 750021, China and School of Mathematical Sciences, Tiangong University, Tianjin 300387, China, e-mail: yaoyonghong@aliyun.com or yyhtgu@hotmail.com

[3] Center for General Education, China Medical University, Taichung 40402, Taiwan and Department of Interior Design, Asia University, Taichung, Taiwan and Romanian Academy, Gh. Mihoc-C. Iacob Institute of Mathematical Statistics, and Applied Mathematics, Bucharest 050711, Romania and University "Politehnica" of Bucharest, Department of Mathematics and Informatics, Bucharest 060042, Romania, e-mail: mihai@mathem.pub.ro

In order to solve $EP(f,C)$, the bifunction f is always to be assumed to possess Lipschitz-type condition (L): there exists $\zeta_1 > 0, \zeta_2 > 0$ such that

$$f(u^\dagger, v^\dagger) + f(v^\dagger, w^\dagger) \geq f(u^\dagger, w^\dagger) - \zeta_1\|u^\dagger - v^\dagger\|^2 - \zeta_2\|v^\dagger - w^\dagger\|^2, \forall u^\dagger, v^\dagger, w^\dagger \in C,$$

and one of the following monotone properties (M1)-(M3):
(M1): strongly monotone if there exists $\zeta > 0$ such that

$$f(u^\dagger, v^\dagger) + f(v^\dagger, u^\dagger) \leq -\zeta\|u^\dagger - v^\dagger\|^2, \ \forall u^\dagger, v^\dagger \in C.$$

(M2): monotone if

$$f(u^\dagger, v^\dagger) + f(v^\dagger, u^\dagger) \leq 0, \ \forall u^\dagger, v^\dagger \in C.$$

(M3): pseudomonotone if

$$f(u^\dagger, v^\dagger) \geq 0 \text{ implies } f(v^\dagger, u^\dagger) \leq 0, \ \forall u^\dagger, v^\dagger \in C.$$

In [12], Mastroeni studied $EP(f,C)$ with f satisfying condition (L) and property (M1) by using auxiliary problem technique. Moudafi [13] investigated the proximal point algorithm for solving $EP(f,C)$ with f satisfying property (M2).

Note that the condition (L), in general, is not verified. Furthermore, even if the condition (L) holds, finding the constants ζ_1 and ζ_2 is not an easy work. In this respect, Nguyen, Strodiot and Nguyen [15] presented a hybrid method for solving $EP(f,C)$ without condition (L) and fixed point problems by using a linesearch procedure into the iterative step. Hung and Muu [7] extended the Tikhonov regularization method to the pseudomonotone equilibrium problem. Kazmi and Ali [9] studied the $EP(f,C)$ and a fixed point problem for an asymptotically quasi ψ-nonexpansivemapping. Kazmi and Yousuf [10] suggested an extragradient iterative method for finding a common solution to $EP(f,C)$ and fixed point problems of nonexpansive mappings. Yang and Liu [19] introduced and analyzed a subgradient extragradient algorithm for solving the pseudomonotone equilibrium problem and fixed point problems.

Motivated and inspired by the above work, the purpose of this paper is to further investigate $EP(f,C)$ and fixed point problems. We devote to solve the pseudomonotone equilibrium problems and fixed point problems of pseudocontractive operators. We propose an iterative algorithm based on the projected method and hybrid method with linesearch technique for finding a common solution of the equilibrium problems and fixed point problems. We prove the strong convergence of the proposed algorithm.

2. **Preliminaries**

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Recall that the distance between a point x and C is defined as $dist(x, C) = \inf\{\|x - a\| : a \in C\}$. Especially, if $C = \{u \in H : \langle w, u - v \rangle \leq 0\}$ for $w, v \in H$ with $w \neq 0$, then

$$dist(u^\dagger, C) = \begin{cases} \dfrac{|\langle w, u^\dagger - v \rangle|}{\|w\|}, & if \ u^\dagger \notin C, \\ 0, & if \ u^\dagger \in C. \end{cases}$$

An operator $S : C \to C$ is said to be pseudocontractive if

$$\|Su - Su^\dagger\|^2 \le \|u - u^\dagger\|^2 + \|(I-S)u - (I-S)u^\dagger\|^2, \ \forall u, u^\dagger \in C.$$

$S : C \to C$ is called κ-Lipschitz if $\|Su - Su^\dagger\| \le \kappa \|u - u^\dagger\|, \ \forall u, u^\dagger \in C$. If $\kappa = 1$, then S is called nonexpansive. If $\kappa < 1$, then S is called κ-contractive.

Recall that the metric projection P_C is an orthographic projection from H onto C and satisfies $\|\hat{u} - P_C[\hat{u}]\| \le \|u^\dagger - \hat{u}\|, \ \forall u^\dagger \in C$.

In the sequel, we use the following symbols.

- $z_n \rightharpoonup z^\dagger$ means the weak convergence of z_n to $z^\dagger$ as $n \to \infty$.
- $z_n \to p^\dagger$ stands for the strong convergence of z_n to $z^\dagger$ as $n \to \infty$.
- $Fix(S)$ denotes the fixed point set of S.

Let $g : C \to (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function. Recall that the subdifferential ∂g of g is defined, for each $u \in C$, by

$$\partial g(u) := \{v^\dagger \in H : g(u^\dagger) - g(u) \ge \langle v^\dagger, u^\dagger - u\rangle, \forall u^\dagger \in C\}. \tag{2}$$

It is known that $u^\dagger$ solves the following minimization problem

$$\min_{u \in C}\{g(u)\}$$

if and only if

$$0 \in \partial g(u^\dagger) + N_C(u^\dagger), \tag{3}$$

where $N_C(u^\dagger)$ means the normal cone of C at $u^\dagger$ defined by

$$N_C(u^\dagger) = \{\omega \in H : \langle \omega, u - u^\dagger\rangle \le 0, \forall u \in C\}. \tag{4}$$

Proposition 2.1 ([12])**.** *Let $f : C \times C \to \mathbb{R}$ be a bifunction with $f(\tilde{z}, \tilde{z}) = 0, \forall \tilde{z} \in C$. Then the following conclusions are equivalent*

(i) $z^\dagger \in Sol(f, C)$;

(ii) $z^\dagger$ *solves the following minimization problem*

$$\min_{z \in C} f(z^\dagger, z).$$

Next, we consider the following auxiliary equilibrium problem which consists of finding a point $z^\dagger \in C$ with the property

$$f(z^\dagger, z) + \tfrac{1}{2\vartheta}\|z^\dagger - z\|^2 \ge 0, \forall z \in C, \text{ where } \vartheta > 0.$$

By Proposition 2.1, $z^\dagger \in Sol(f, C)$ implies that $z^\dagger$ also solves the following minimization problem

$$\min_{z \in C}\{f(z^\dagger, z) + \frac{1}{2\vartheta}\|z^\dagger - z\|^2\}. \tag{5}$$

Let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the following assumptions:

(f1): $f(u^\dagger, u^\dagger) = 0, \forall u^\dagger \in C$;

(f2): f is jointly sequently weakly continuous on $D \times D$, where D is an open convex set containing C (recall that f is called jointly sequently weakly continuous on $D \times D$, if $x^n \rightharpoonup x^\dagger$ and $y^n \rightharpoonup y^\dagger$, then $f(x^n, y^n) \to f(x^\dagger, y^\dagger)$);

(f3): $f(u^\dagger, \cdot)$ is convex and subdifferentiable for all $u^\dagger \in C$;

(f4): f is pseudomonotone.

Lemma 2.1 ([12, 17])**.** *Let $f : C \times C \to \mathbb{R}$ be a bifunction. If f satisfies assumptions (f1)-(f4), then $u^\dagger \in Sol(f, C)$ if and only if $u^\dagger$ solves the problem* (5)*.*

Lemma 2.2 ([18])**.** *Assume that the bi-function $f : C\times C \to \mathbb{R}$ satisfies assumptions (f1)-(f4). For given two points $\bar{u}, \bar{v} \in C$ and two sequences $\{u_n\} \subset C$ and $\{v_n\} \subset C$, if $u_n \rightharpoonup \bar{u}$ and $v_n \rightharpoonup \bar{v}$, respectively, then, for any $\epsilon > 0$, there exist $\delta > 0$ and $N_\epsilon \in \mathbb{N}$ verifying $\partial f(v_n, \cdot)(u_n) \subset \partial f(\bar{v}, \cdot)(\bar{u}) + \frac{\epsilon}{\delta} B$ for every $n \geq N_\epsilon$, where $B := \{b \in H | \|b\| \leq 1\}$.*

Lemma 2.3 ([5])**.** *Let $f : C \times C \to \mathbb{R}$ be a bi-function satisfying assumptions (f1)-(f4). Let $\{\vartheta_n\}_{n=0}^{\infty}$ be a real number sequence satisfying $\vartheta_n \in [\underline{\vartheta}, \overline{\vartheta}] \subset (0, 1]$. For a given bounded sequence $\{z_n\}$ in C, y_n solves the following strongly convex program*

$$\min_{z^\dagger \in C} \left\{ f(z_n, z^\dagger) + \frac{1}{2\vartheta_n} \|z_n - z^\dagger\|^2 \right\}.$$

Then $\{y_n\}$ is bounded.

Lemma 2.4 ([15])**.** *For all $x, x^\dagger \in H$ and $\forall \varsigma \in [0, 1]$, the following equality holds*

$$\|\varsigma x + (1 - \varsigma)x^\dagger\|^2 = \varsigma\|x\|^2 + (1 - \varsigma)\|x^\dagger\|^2 - \varsigma(1 - \varsigma)\|x - x^\dagger\|^2.$$

Lemma 2.5. *Let the operator $S : C \to C$ be κ-Lipschitz and pseudocontractive. For any $\tilde{x} \in C$ and $x^\dagger \in Fix(S)$, we have*

$$\|x^\dagger - S[(1 - \vartheta)\tilde{x} + \vartheta S\tilde{x}]\|^2 \leq \|\tilde{x} - x^\dagger\|^2 + (1 - \vartheta)\|\tilde{x} - S[(1 - \vartheta)\tilde{x} + \vartheta S\tilde{x}]\|^2,$$

where $\vartheta \in (0, \frac{1}{\sqrt{1+\kappa^2}+1})$.

Lemma 2.6 ([20])**.** *Let $S : C \to C$ be a continuous pseudocontractive operator. Then S is demi-closedness.*

Lemma 2.7 ([20])**.** *Suppose $\{\varpi_n\} \subset [0, \infty)$, $\{\nu_n\} \subset (0, 1)$, and $\{\varrho_n\}$ are three real number sequences satisfying*
(i) $\varpi_{n+1} \leq (1 - \nu_n)\varpi_n + \varrho_n, \forall n \geq 1$;
(ii) $\sum_{n=1}^{\infty} \nu_n = \infty$;
(iii) $\limsup_{n\to\infty} \frac{\varrho_n}{\nu_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\varrho_n| < \infty$.
Then $\lim_{n\to\infty} \varpi_n = 0$.

Lemma 2.8 ([4])**.** *Let $\{w_n\}$ be a sequence of real numbers. Assume there exists at least a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $w_{n_k} \leq w_{n_k+1}$ for all $k \geq 0$. For every $n \geq N_0$, define an integer sequence $\{\tau(n)\}$ as*

$$\tau(n) = \max\{i \leq n : w_{n_i} < w_{n_i+1}\}.$$

Then $\tau(n) \to \infty$ as $n \to \infty$ and for all $n \geq N_0$, we have $\max\{w_{\tau(n)}, w_n\} \leq w_{\tau(n)+1}$.

3. Main results

In this section, we first give some assumptions and conditions. Let H be a real Hilbert space. Let C be a nonempty closed convex subset of H and D be a given open set which contains C. Let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the assumptions (f1)-(f4) in Section 2. Let $S : C \to C$ be a κ-Lipschitz pseudocontractive operator. Let the operator $\psi : C \to C$ be ρ-contractive. Let $\sigma \in (0, 1)$ be a constant.

Assume that the real number sequences $\{\varsigma_n\}$, $\{\vartheta_n\}$, $\{\tau_n\}$, $\{\epsilon_n\}$, $\{\gamma_n\}$ and $\{\mu_n\}$ satisfy the following conditions:

(C1) $0 < \underline{\varsigma} < \varsigma_n < \overline{\varsigma} < \vartheta_n < \overline{\vartheta} < \frac{1}{\sqrt{1+\kappa^2}+1} (\forall n \geq 0)$;
(C2) $0 < \underline{\tau} \leq \tau_n < \tau_1 < \tau_2 < \epsilon_n \leq \overline{\tau} < \infty$;
(C3) $\gamma_n \in [0, 1]$, $\lim_{n\to\infty} \gamma_n = 0$ and $\sum_{n=0}^{\infty} \gamma_n = \infty$;
(C4) $0 < \liminf_{n\to\infty} \mu_n \leq \limsup_{n\to\infty} \mu_n < 1$.
Next, we construct an iterative algorithm for finding a common solution of the equilibrium problem $EP(f, C)$ and fixed point problem of S.

Algorithm 3.1. *Step 0. For given initial value $x_0 \in H$, set $n = 0$.*

Step 1. Assume that $\{x_n\}$ has been given. Compute

$$z_n = (1 - \varsigma_n)x_n + \varsigma_n S[(1 - \vartheta_n)x_n + \vartheta_n Sx_n]. \tag{6}$$

Step 2. Compute

$$y_n = \arg\min_{z^\dagger \in C} \left\{ f(z_n, z^\dagger) + \frac{1}{2\tau_n}\|z_n - z^\dagger\|^2 \right\}. \tag{7}$$

Criterion: if $y_n = z_n$, then set $u_n = z_n$ and jump to Step 4; otherwise, continuous to the next Step 3.

Step 3. Find the smallest positive integer m verifying

$$f(u_{n,m}, y_n) + \frac{1}{2\epsilon_n}\|y_n - z_n\|^2 \leq 0, \tag{8}$$

where

$$u_{n,m} = (1 - \sigma^m)z_n + \sigma^m y_n, \tag{9}$$

and consequently, write $\sigma^m = \sigma_n$ and $u_{n,m} = u_n$.

Step 4. Construct

$$C_n = \{u^\dagger \in C : f(u_n, u^\dagger) \leq 0\} \tag{10}$$

and compute

$$x_{n+1} = \gamma_n \psi(x_n) + (1 - \gamma_n)[\mu_n z_n + (1 - \mu_n)P_{C_n}(z_n)]. \tag{11}$$

Step 5. Set $n := n + 1$ and return to Step 1.

Remark 3.1. *The inequality* (8) *is well-defined, i.e., there exists a positive integer m such that* (8) *holds.*

In fact, if (8) is invalid, m must violate the inequality (8). Thus, for every $m \in \mathbb{N}$, we get

$$f(u_{n,m}, y_n) + \frac{1}{2\epsilon_n}\|y_n - z_n\|^2 > 0. \tag{12}$$

Since $u_{n,m} = (1 - \sigma^m)z_n + \sigma^m y_n$ and $\sigma \in (0, 1)$, $u_{n,m} \to z_n$ as $m \to \infty$. Thanks to the condition (f2), we deduce that $f(u_{n,m}, y_n) \to f(z_n, y_n)$. This together with (12) implies that

$$f(z_n, y_n) + \frac{1}{2\epsilon_n}\|y_n - z_n\|^2 \geq 0. \tag{13}$$

By the definition of y_n, we have

$$f(z_n, y_n) + \frac{1}{2\tau_n}\|z_n - y_n\|^2 \leq f(z_n, z_n) + \frac{1}{2\tau_n}\|z_n - z_n\|^2 = 0. \tag{14}$$

In terms of (13) and (14), we obtain $0 < (\frac{1}{2\tau_n} - \frac{1}{2\epsilon_n})\|y_n - z_n\|^2 \leq 0$. This leads a contradiction. Hence, the search rule (8) is well-defined.

Proposition 3.1. *We have the following statements:*
(i) *If $y_n = z_n$, then $z_n \in Sol(f, C)$;*
(ii) *$Sol(f, C) \subset C_n$;*
(iii) *If $y_n \neq z_n$, then $z_n \notin C_n$ and $f(u_n, z_n) > 0$.*

Proof. (i) If $y_n = z_n$, then $0 \in \partial f(z_n, \cdot)(z_n) + N_C(z_n)$. Hence, $\langle \xi_n, z - z_n\rangle \geq 0, \forall z \in C$ where $\xi_n \in \partial f(z_n, \cdot)(z_n)$. By the subdifferentiable inequality of $f(z_n, \cdot)$, we have $f(z_n, z) - f(z_n, z_n) \geq \langle \xi_n, z - z_n\rangle$ for all $z \in C$. Therefore, $f(z_n, z) \geq 0 (\forall z \in C)$ which implies that $z_n \in Sol(f, C)$.

(ii) Pick up any $q \in Sol(f, C)$. Then, we have $f(q, q^\dagger) \geq 0$ for all $q^\dagger \in C$. By the pseudomonotonicity (f4) of f, we get $f(q^\dagger, q) \leq 0$ for all $q^\dagger \in C$. Note that $u_n \in C$. Then, $f(u_n, q) \leq 0$ which implies that $q \in C_n$. Therefore, $Sol(f, C) \subset C_n$.

(iii) By (9), (f1) and (f3), we have

$$\begin{aligned} 0 = f(u_n, u_n) &= f(u_n, (1 - \sigma_n)z_n + \sigma_n y_n) \\ &\leq (1 - \sigma_n)f(u_n, z_n) + \sigma_n f(u_n, y_n) \\ &\leq (1 - \sigma_n)f(u_n, z_n) - \frac{\sigma_n}{2\epsilon_n}\|y_n - z_n\|^2 \text{ (by (8))} \\ &< (1 - \sigma_n)f(u_n, z_n), \end{aligned}$$

thus, $f(u_n, z_n) > 0$ and so $z_n \notin C_n$ by (10). □

Next, we show the convergence of Algorithm 3.1.

Theorem 3.1. *Suppose that $\Omega := Sol(f, C) \cap Fix(S) \neq \emptyset$. Then the sequence $\{x_n\}$ generated by (11) converges strongly to $p = P_\Omega \psi(p)$.*

Proof. Set $v_n = \mu_n z_n + (1 - \mu_n)P_{C_n}(z_n)$ for all $n \geq 0$. It follows that

$$\begin{aligned} \|v_n - p\| &= \|\mu_n(z_n - p) + (1 - \mu_n)(P_{C_n}(z_n) - P_{C_n}(p))\| \\ &\leq \mu_n\|z_n - p\| + (1 - \mu_n)\|z_n - p\| \\ &= \|z_n - p\|. \end{aligned} \tag{15}$$

From (6) and Lemmas 2.4 and 2.5, we deduce

$$\begin{aligned} \|z_n - p\|^2 &= \|(1 - \varsigma_n)(x_n - p) + \varsigma_n(S[(1 - \vartheta_n)x_n + \vartheta_n Sx_n] - p)\|^2 \\ &= (1 - \varsigma_n)\|x_n - p\|^2 - \varsigma_n(1 - \varsigma_n)\|S[(1 - \vartheta_n)x_n + \vartheta_n Sx_n] - x_n\|^2 \\ &\quad + \varsigma_n\|S[(1 - \vartheta_n)x_n + \vartheta_n Sx_n] - p\|^2 \\ &\leq (1 - \varsigma_n)\|x_n - p\|^2 - \varsigma_n(1 - \varsigma_n)\|S[(1 - \vartheta_n)x_n + \vartheta_n Sx_n] - x_n\|^2 \\ &\quad + \varsigma_n(\|x_n - p\|^2 + (1 - \vartheta_n)\|x_n - S[(1 - \vartheta_n)x_n + \vartheta_n Sx_n]\|^2) \\ &= \|x_n - p\|^2 - \varsigma_n(\vartheta_n - \varsigma_n)\|x_n - S[(1 - \vartheta_n)x_n + \vartheta_n Sx_n]\|^2 \\ &\leq \|x_n - p\|^2. \end{aligned} \tag{16}$$

By (6), (15) and (16), we obtain

$$\begin{aligned}\|x_{n+1}-p\| &= \|\gamma_n(\psi(x_n)-p)+(1-\gamma_n)(v_n-p)\| \\ &\le \gamma_n\|\psi(x_n)-p\|+(1-\gamma_n)\|v_n-p\| \\ &\le \gamma_n\|\psi(x_n)-\psi(p)\|+\gamma_n\|\psi(p)-p\|+(1-\gamma_n)\|z_n-p\| \\ &\le \gamma_n\rho\|x_n-p\|+\gamma_n\|\psi(p)-p\|+(1-\gamma_n)\|x_n-p\| \\ &= [1-(1-\rho)\gamma_n]\|x_n-p\|+\gamma_n\|\psi(p)-p\|.\end{aligned} \tag{17}$$

By induction, we can conclude that $\|x_{n+1}-p\| \le \max\{\|x_0-p\|, \|\psi(p)-p\|/(1-\rho)\}$. Thus, the sequence $\{x_n\}$ is bounded. Consequently, the sequences $\{v_n\}$ and $\{z_n\}$ are all bounded. According to Lemma 2.3, we deduce that $\{y_n\}$ is bounded. Hence, $\{u_n\}$ is also bounded.

For each $w_n \in \partial f(u_n,\cdot)(u_n)$, define

$$Q_n = \{u^\dagger \in C : \langle w_n, u^\dagger - u_n\rangle \le 0\}. \tag{18}$$

Thanks to Lemma 2.2, $\{w_n\}$ is bounded. Hence, there exists a positive constant M such that $\|w_n\| \le M$ for all n. By the subdifferentiable inequality of $f(u_n,\cdot)$ at u_n, we obtain

$$f(u_n,u^\dagger) = f(u_n,u^\dagger) - f(u_n,u_n) \ge \langle w_n, u^\dagger - u_n\rangle, \forall u^\dagger \in C, \tag{19}$$

which implies that $C_n \subset Q_n$.

Replacing $u^\dagger$ by y_n in (19), we get

$$f(u_n,y_n) \ge \langle w_n, y_n - u_n\rangle.$$

This together with (8) implies that

$$\langle w_n, u_n - y_n\rangle \ge \frac{1}{2\epsilon_n}\|y_n - z_n\|^2. \tag{20}$$

Note that $z_n - u_n = \frac{\sigma_n}{1-\sigma_n}(u_n - y_n)$. It follows from (20) that

$$\langle w_n, z_n - u_n\rangle \ge \frac{\sigma_n}{2\epsilon_n(1-\sigma_n)}\|y_n - z_n\|^2. \tag{21}$$

Since $z_n \notin C_n$ and $C_n \subset Q_n$, we get

$$\begin{aligned}\|z_n - P_{C_n}(z_n)\| &= dist(z_n, C_n) \\ &\ge dist(z_n, Q_n) \\ &= \frac{|\langle w_n, z_n - u_n\rangle|}{\|w_n\|} \\ &\ge \frac{\sigma_n}{2\tau M}\|y_n - z_n\|^2.\end{aligned} \tag{22}$$

By Lemma 2.4, we have

$$\begin{aligned}\|v_n - p\|^2 &= \|\mu_n(z_n - p) + (1-\mu_n)(P_{C_n}(z_n) - p)\|^2 \\ &= \mu_n\|z_n - p\|^2 + (1-\mu_n)\|P_{C_n}(z_n) - p\|^2 \\ &\quad - \mu_n(1-\mu_n)\|z_n - P_{C_n}(z_n)\|^2 \\ &\le \|z_n - p\|^2 - \mu_n(1-\mu_n)\|z_n - P_{C_n}(z_n)\|^2.\end{aligned} \tag{23}$$

Combining (16) with (23) to derive

$$\begin{aligned}\|v_n - p\|^2 &\le \|x_n - p\|^2 - \varsigma_n(\vartheta_n - \varsigma_n)\|x_n - S[(1-\vartheta_n)x_n + \vartheta_n Sx_n]\|^2 \\ &\quad - \mu_n(1-\mu_n)\|z_n - P_{C_n}(z_n)\|^2.\end{aligned} \tag{24}$$

By virtue of (17) and (24), we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \gamma_n\|\psi(x_n) - p\|^2 + (1-\gamma_n)\|v_n - p\|^2 \\ &\le \gamma_n\|\psi(x_n) - p\|^2 + \|x_n - p\|^2 - \mu_n(1-\mu_n)\|z_n - P_{C_n}(z_n)\|^2 \\ &\quad - \varsigma_n(\vartheta_n - \varsigma_n)\|x_n - S[(1-\vartheta_n)x_n + \vartheta_n Sx_n]\|^2.\end{aligned} \tag{25}$$

Regarding the convergence of $\{\|x_n - p\|\}_{n\ge 0}$, there are two possible cases. Case 1: there exists n_0 such that $\|x_{n+1} - p\| \le \|x_n - p\|$ when $n \ge n_0$. Case 2: there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $\|x_{n_i} - p\| < \|x_{n_i+1} - p\|, \forall i \ge 1$.

For case 1, we conclude that $\lim_{n\to\infty}\|x_n - p\|$ exists. In the light of (25), we have

$$\begin{aligned}&\varsigma_n(\vartheta_n - \varsigma_n)\|x_n - S[(1-\vartheta_n)x_n + \vartheta_n Sx_n]\|^2 + \mu_n(1-\mu_n)\|z_n - P_{C_n}(z_n)\|^2 \\ &\le \gamma_n\|\psi(x_n) - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \to 0\end{aligned} \tag{26}$$

which implies that

$$\lim_{n\to\infty}\|z_n - P_{C_n}(z_n)\| = 0 \tag{27}$$

and

$$\lim_{n\to\infty}\|x_n - S[(1-\vartheta_n)x_n + \vartheta_n Sx_n]\| = 0. \tag{28}$$

By (6) and (28), we deduce

$$\lim_{n\to\infty}\|z_n - x_n\| = \lim_{n\to\infty}\varsigma_n\|x_n - S[(1-\vartheta_n)x_n + \vartheta_n Sx_n]\| = 0. \tag{29}$$

Since S is κ-Lipschitz continuous, we deduce

$$\begin{aligned}\|x_n - Sx_n\| &\le \|x_n - S[(1-\vartheta_n)x_n + \vartheta_n Sx_n]\| \\ &\quad + \|S[(1-\vartheta_n)x_n + \vartheta_n Sx_n] - Sx_n\| \\ &\le \|x_n - S[(1-\vartheta_n)x_n + \vartheta_n Sx_n]\| + \kappa\vartheta_n\|x_n - Sx_n\|.\end{aligned}$$

This means that

$$\|x_n - Sx_n\| \le \frac{1}{1-\kappa\vartheta_n}\|x_n - S[(1-\vartheta_n)x_n + \vartheta_n Sx_n]\|.$$

This together with (28) implies that

$$\lim_{n\to\infty}\|x_n - Sx_n\| = 0. \tag{30}$$

Since y_n solves strongly convex program $\min_{z^\dagger\in C}\left\{f(z_n, z^\dagger) + \frac{1}{2\tau_n}\|z_n - z^\dagger\|^2\right\}$, there exists $\zeta_n \in \partial f(z_n,\cdot)(y_n)$ such that

$$\langle \zeta_n, v - y_n\rangle + \frac{1}{\tau_n}\langle y_n - z_n, v - y_n\rangle \ge 0, \forall v \in C. \tag{31}$$

Using the subdifferentiable inequality of $f(z_n,\cdot)$ at y_n, we have

$$f(z_n, v) - f(z_n, y_n) \ge \langle \zeta_n, v - y_n\rangle, \forall v \in C. \tag{32}$$

By (31) and (32), we deduce

$$f(z_n, v) - f(z_n, y_n) + \frac{1}{\tau_n}\langle y_n - z_n, v - y_n\rangle \geq 0, \forall v \in C. \tag{33}$$

It follows that

$$f(z_n, v) - f(z_n, y_n) + \frac{1}{\underline{\tau}}\|y_n - z_n\|\|v - y_n\| \geq 0, \forall v \in C. \tag{34}$$

Thanks to (22) and (27), we get

$$\lim_{n\to\infty} \sigma_n \|y_n - z_n\|^2 = 0. \tag{35}$$

Since the sequence $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup q$ as $j \to \infty$ and

$$\limsup_{n\to\infty}\langle \psi(p) - p, x_{n+1} - p\rangle = \lim_{j\to\infty}\langle \psi(p) - p, x_{n_j+1} - p\rangle.$$

By (29), $z_{n_j} \rightharpoonup q$ as $j \to \infty$. Next, we show $q \in \Omega$. Firstly, using Lemma 2.6 and (30), we conclude that $q \in Fix(S)$. Next, we prove $q \in Sol(f, C)$ by considering two cases.

(i) If $\liminf_{j\to\infty} \sigma_{n_j} > 0$, then from (35) we have

$$\lim_{j\to\infty} \|y_{n_j} - z_{n_j}\| = 0. \tag{36}$$

Because of $z_{n_j} \rightharpoonup q(j \to \infty)$, we conclude that $y_{n_j} \rightharpoonup q$ as $j \to \infty$ by (36). In (34), replacing n by n_j and letting $j \to \infty$, we have

$$f(q, v) \geq 0, \forall v \in C.$$

This means that $q \in Sol(f, C)$.

(ii) If $\liminf_{j\to\infty} \sigma_{n_j} = 0$, then there exists a subsequence $\{\sigma_{n_{j_i}}\}$ of $\{\sigma_{n_j}\}$ such that $\lim_{i\to\infty} \sigma_{n_{j_i}} = 0$.

Since $\{y_{n_{j_i}}\}$ is bounded, without loss of generality, we may assume that $y_{n_{j_i}} \rightharpoonup q^\dagger \in C$ as $i \to \infty$. Since $y_{n_{j_i}}$ solves (7), for $\forall y^\dagger \in C$, we have

$$f(z_{n_{j_i}}, y_{n_{j_i}}) + \frac{1}{2\tau_{n_{j_i}}}\|y_{n_{j_i}} - z_{n_{j_i}}\|^2 \leq f(z_{n_{j_i}}, y^\dagger) + \frac{1}{2\tau_{n_{j_i}}}\|z_{n_{j_i}} - y^\dagger\|^2. \tag{37}$$

Since τ_n is bounded, we may assume, without loss of generality, that $\lim_{i\to\infty} \tau_{n_{j_i}} = \rho^\dagger \leq \tau_1$. Letting $i \to \infty$ in (37), we deduce together with (f2) that

$$f(q, q^\dagger) + \frac{1}{2\rho^\dagger}\|q - q^\dagger\|^2 \leq f(q, y^\dagger) + \frac{1}{2\rho^\dagger}\|q - y^\dagger\|^2, \quad \forall y^\dagger \in C. \tag{38}$$

Setting $y^\dagger = q$ in (38), we get

$$f(q, q^\dagger) + \frac{1}{2\rho^\dagger}\|q - q^\dagger\|^2 \leq 0. \tag{39}$$

On the other hand, m is the smallest positive integer satisfying (8), so we have

$$f(u_{n_{j_i},m-1}, y_{n_{j_i}}) > -\frac{1}{2\tau_2}\|y_{n_{j_i}} - z_{n_{j_i}}\|^2. \tag{40}$$

Note that $u_{n_{j_i},m-1} = (1-\sigma_{n_{j_i}-1})z_{n_{j_i}} + \sigma_{n_{j_i}-1}y_{n_{j_i}} \rightharpoonup q$. Letting $i \to \infty$ in (40), we obtain

$$f(q, q^\dagger) \geq -\frac{1}{2\tau_2}\|q - q^\dagger\|^2. \tag{41}$$

Taking into account (39) and (41), we deduce $0 \leq (\frac{1}{2\rho^\dagger} - \frac{1}{2\tau_2})\|q - q^\dagger\|^2 \leq 0$ which implies that $q^\dagger = q$. Therefore, from (38), we have

$$f(q, y^\dagger) + \frac{1}{2\rho^\dagger}\|q - y^\dagger\|^2 \geq 0, \quad \forall y^\dagger \in C.$$

According to Lemma 2.1, $q \in Sol(f, C)$. Hence,

$$\limsup_{n\to\infty}\langle \psi(p) - p, x_{n+1} - p\rangle = \lim_{j\to\infty}\langle \psi(p) - p, x_{n_j+1} - p\rangle = \langle \psi(p) - p, q - p\rangle \leq 0$$

because of $p = P_\Omega\psi(p)$.

From (11), we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq (1-\gamma_n)^2\|v_n - p\|^2 + 2\gamma_n\langle \psi(x_n) - \psi(p), x_{n+1} - p\rangle \\ &\quad + 2\gamma_n\langle \psi(p) - p, x_{n+1} - p\rangle \\ &\leq (1-\gamma_n)^2\|x_n - p\|^2 + 2\gamma_n\rho\|x_n - p\|\|x_{n+1} - p\| \\ &\quad + 2\gamma_n\langle \psi(p) - p, x_{n+1} - p\rangle \\ &\leq (1-\gamma_n)^2\|x_n - p\|^2 + \gamma_n\rho\|x_n - p\|^2 + \gamma_n\rho\|x_{n+1} - p\|^2 \\ &\quad + 2\gamma_n\langle \psi(p) - p, x_{n+1} - p\rangle.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq [1 - \frac{2(1-\rho)\gamma_n}{1-\gamma_n\rho}]\|x_n - p\|^2 + \frac{\gamma_n^2}{1-\gamma_n\rho}\|x_n - p\|^2 \\ &\quad + \frac{2\gamma_n}{1-\gamma_n\rho}\langle \psi(p) - p, x_{n+1} - p\rangle \\ &\leq [1 - \frac{2(1-\rho)\gamma_n}{1-\gamma_n\rho}]\|x_n - p\|^2 + \frac{\gamma_n^2}{1-\gamma_n\rho}M_0 \\ &\quad + \frac{2\gamma_n}{1-\gamma_n\rho}\langle \psi(p) - p, x_{n+1} - p\rangle.\end{aligned} \tag{42}$$

where M_0 is a constant such that $\sup_n\{\|x_n - p\|^2\} \leq M_0$.

Combining (42) and Lemma 2.7, we deduce that $x_n \to p$.

For Case 2, by Lemma 2.8, there exists a nondecreasing sequence m_k verifying $m_k \to \infty$,

$$\max\{\|x_k - p\|, \|x_{m_k} - p\|\} \leq \|x_{m_k+1} - p\|, \forall k \geq 1.$$

From (26), we also have

$$\begin{aligned}&\varsigma_{m_k}(\vartheta_{m_k} - \varsigma_{m_k})\|x_{m_k} - S[(1-\vartheta_{m_k})x_{m_k} + \vartheta_{m_k}Sx_{m_k}]\|^2 \\ &+ \mu_{m_k}(1-\mu_{m_k})\|z_{m_k} - P_{C_{m_k}}(z_{m_k})\|^2 \\ &\leq \gamma_{m_k}\|\psi(x_{m_k}) - p\|^2 + \|x_{m_k} - p\|^2 - \|x_{m_k+1} - p\|^2 \\ &\leq \gamma_{m_k}\|\psi(x_{m_k}) - p\|^2 \to 0.\end{aligned}$$

which implies that

$$\lim_{k\to\infty} \|z_{m_k} - P_{C_{m_k}}(z_{m_k})\| = 0 \text{ and } \lim_{k\to\infty} \|x_{m_k} - Sx_{m_k}\| = 0.$$

Consequently, by the similar argument as that in Case 1, we can obtain

$$\limsup_{n\to\infty} \langle \psi(p) - p, x_{m_k+1} - p\rangle \leq 0. \tag{43}$$

By (42), we derive

$$\begin{aligned}
\|x_{m_k+1} - p\|^2 &\leq [1 - \frac{2(1-\rho)\gamma_{m_k}}{1-\gamma_{m_k}\rho}]\|x_{m_k} - p\|^2 + \frac{\gamma_{m_k}^2}{1-\gamma_{m_k}\rho}M_0 \\
&\quad + \frac{2\gamma_{m_k}}{1-\gamma_{m_k}\rho}\langle \psi(p) - p, x_{m_k+1} - p\rangle \\
&\leq [1 - \frac{2(1-\rho)\gamma_{m_k}}{1-\gamma_{m_k}\rho}]\|x_{m_k+1} - p\|^2 + \frac{\gamma_{m_k}^2}{1-\gamma_{m_k}\rho}M_0 \\
&\quad + \frac{2\gamma_{m_k}}{1-\gamma_{m_k}\rho}\langle \psi(p) - p, x_{m_k+1} - p\rangle.
\end{aligned}$$

It follows that

$$\|x_{m_k+1} - p\|^2 \leq \frac{\gamma_{m_k}}{2(1-\rho)}M_0 + \frac{1}{1-\rho}\langle \psi(p) - p, x_{m_k+1} - p\rangle.$$

This together with (43) implies that $\|x_{m_k+1} - p\|$ as $k \to \infty$. Hence, $\|x_k - p\| \to 0$ as $k \to \infty$. This completes the proof. □

Algorithm 3.2. *Step 0. For given initial value $x_0 \in H$, set $n = 0$.*

Step 1. Assume that $\{x_n\}$ has been given. Compute

$$y_n = \arg\min_{z^\dagger \in C} \left\{ f(x_n, x^\dagger) + \frac{1}{2\tau_n}\|x_n - x^\dagger\|^2 \right\}.$$

Criterion: if $y_n = x_n$, then set $u_n = x_n$ and jump to Step 3; otherwise, continuous to the next Step 2.

Step 2. Find the smallest positive integer m verifying

$$f(u_{n,m}, y_n) + \frac{1}{2\epsilon_n}\|y_n - x_n\|^2 \leq 0,$$

where

$$u_{n,m} = (1 - \sigma^m)x_n + \sigma^m y_n,$$

and consequently, write $\sigma^m = \sigma_n$ and $u_{n,m} = u_n$.

Step 3. Construct

$$C_n = \{u^\dagger \in C : f(u_n, u^\dagger) \leq 0\}$$

and compute

$$x_{n+1} = \gamma_n\psi(x_n) + (1-\gamma_n)[\mu_n x_n + (1-\mu_n)P_{C_n}(x_n)].$$

Step 4. Set $n := n + 1$ and return to Step 1.

Theorem 3.2. *Suppose that $Sol(f, C) \neq \emptyset$. Then the sequence $\{x_n\}$ generated by Algorithm 3.2 converges strongly to $p^\dagger = P_{Sol(f,C)}\psi(p^\dagger)$.*

4. **Acknowledgments**

Li-Jun Zhu was supported by the National Natural Science Foundation of China (11861003) , the Natural Science Foundation of Ningxia province(NZ17015) and (NXYLXK2017B09). Yonghong Yao was supported in part by the grant TD13-5033.

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