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ON (ϕ, φ)-BEST PROXIMITY POINTS FOR PROXIMAL TYPE CONTRACTION MAPPINGS

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In this article, we introduce the notion of (ϕ, φ)-best proximity points and discuss the existence of such points for several types of proximal contraction mappings. As an application of this work, we obtain some new results related to (ϕ, φ)-fixed points of self mappings.

Keywords: ϕ-fixed points and (ϕ, φ)-fixed points and ϕ-best proximity points and (ϕ, φ)-best proximity points.

MSC2020: 47H10, 54H25

1. Introduction

Best proximity theory has developed extensively as a response to theoretical and applied issues which cannot be formulated by means of fixed point theory. In literature, the existence of best proximity points for different types of nonself maps are discussed in various directions. In [1], there is studied the existence of best proximity points for multivalued nonself mappings. In [2], best proximity results are stated with respect to Kakutani multimaps. Work [3] is dedicated to best proximity and equilibrium properties for a finite family of multimaps. Best proximity theorems for γ-controlled proximal contractions are presented in [4], while in [5] simulation functions are used as a main tool in obtaining proximity results. Prešic type operators are studied from this point of view in [6]. Papers [8] and [10] use proximal type contraction mappings in their findings. [9] is devoted to proximal cyclic contractions. In [11], a best proximity study is made in the setting of partially ordered metric spaces, in [20] best proximity results are provided in the setting of dualistic partial metric spaces, while [14] designed an iterative norm-convergent procedure for the determination of a best proximity point. [16] introduced the notion of α-proximal admissibility and studied best proximity points for mappings endowed with this property. In [17], existence properties of best proximity points are studied with respect to controlled proximal contractions for multimappings. In [18], simulation functions with adequate distances are used to state best proximity results. In [12] and [19] abstract spaces proved to be a suitable framework to study best proximity properties, while in [21] some auxiliary functions are used to define generalized contractions which led to best proximity results.

[15] extended the concept of fixed point by defining the notion of ϕ-fixed point. More specifically, a ϕ-fixed point is an element which is simultaneously a fixed point for a self

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mapping and also belongs to the set of zeros of a mapping with positive values. This concept was further extended in [7] to the so-called (ϕ, φ)-fixed points, by imposing the additional condition that the ϕ-fixed point has to be an element of the zeros of another mapping with positive values. In [13] there is defined the notion of ϕ-best proximity point for a nonself map, as an element which is a best proximity point for a nonself mapping, and also belongs to the set of the zeros of a function with positive values. In the present paper, we define two classes of proximal contractions by means of functions endowed with monotone type properties, continuity or other adequate properties. (ϕ, φ)-best proximity results are stated and proved with respect to mappings which fulfill axioms defined by the use of the proximal contractions. Examples and consequences of these results are also provided.

2. **Preliminaries**

In order to develop the new results of this work, we need the information within this section.

Along the paper, $\mathbb{N} = \{1, 2, ...\}$, and $\mathbb{R}_+ = [0, \infty)$.

Wardowski [22] introduced a class of functions $L\colon (0,\infty) \to \mathbb{R}$ which fulfill the axioms below, in order to define a type of generalized contraction.

(L_1) L is strictly increasing, that is, if $a_1 < a_2$ then $L(a_1) < L(a_2)$;

(L_2) for each sequence $\{c_n : c_n > 0\}$, we have $\lim_{n\to\infty} c_n = 0$ if and only if $\lim_{n\to\infty} L(c_n) = -\infty$;

(L_3) for each sequence $\{c_n : c_n > 0\}$ with $\lim_{n\to\infty} c_n = 0$, there exists $h \in (0, 1)$ such that $\lim_{n\to\infty} {c_n}^h L(c_n) = 0$.

This class will be denoted as $\mathfrak{L}$.

Example 1. 1. Consider $L\colon (0,\infty) \to \mathbb{R}$, $L(x) = \ln x$, which satisfies the axioms mentioned above, for any $h \in (0, 1)$.

2. Let $L\colon (0,\infty) \to \mathbb{R}$, $L(x) = -\frac{1}{\sqrt{x}}$. If we take $h \in \left(\frac{1}{2}, 1\right)$, it can be easily observed that $L \in \mathfrak{L}$.

Denote by $\mathfrak{M}$ the class of functions $L\colon {\mathbb{R}_+}^3 \setminus \{(0,0,0)\} \to \mathbb{R}$ which fulfill the below hypotheses

$(ML1 : a)$ $L(a_1, b_1, c_1) \le L(a_2, b_2, c_2)$ if and only if $a_1 + b_1 + c_1 \le a_2 + b_2 + c_2$;

$(ML1 : b)$ $L(a_1, b_1, c_1) < L(a_2, b_2, c_2)$ if and only if $a_1 + b_1 + c_1 < a_2 + b_2 + c_2$;

$(ML2)$ for each $\{c_n : c_n \ge 0\}, \{b_n : b_n \ge 0\}$ and $\{a_n : a_n \ge 0\}$ we have $\lim_{n\to\infty} c_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} L(c_n, b_n, a_n) = -\infty$;

$(ML3)$ for each $\{c_n : c_n \ge 0\}$, $\{b_n : b_n \ge 0\}$ and $\{a_n : a_n \ge 0\}$ with $\lim_{n\to\infty} c_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n = 0$, there exists $h \in (0, 1)$ so that $\lim_{n\to\infty} c_n^h L(c_n, b_n, a_n) = 0$.

Ali *et al.* [13] introduced conditions $(ML2)$ and $(ML3)$.

Example 2. As an element of the set $\mathfrak{M}$, we mention $L\colon {\mathbb{R}_+}^3 \setminus \{(0,0,0)\} \to \mathbb{R}$, $L(a, b, c) = \ln(a + b + c)$, fact which can be easily checked.

In the next section, we will represent (X, d_e) as a metric space, while P, and Q are non-void subsets of X. Furthermore,

$$D_e(P,Q) = \inf\{d_e(p,q) : \ p \in P, \ q \in Q\},$$

$$d_e(p,Q) = \inf\{d_e(p,q) : \ q \in Q\},$$

$$P_0 = \{p \in P : \ d_e(p,q) = D_e(P,Q) \text{ for some } q \in Q\},$$

$$Q_0 = \{q \in Q : \ d_e(p,q) = D_e(P,Q) \text{ for some } p \in P\}.$$

Also, recall that a point $p^* \in P$ is a best proximity point of $T\colon P \to Q$ if $d_e(p^*, Tp^*) = D_e(P, Q)$.

Furthermore, denote by BP_T the set of the best proximity points of a mapping $T\colon P \to Q$, P, $Q \subseteq X$. If $\varphi\colon X \to [0, \infty)$, let Ze_φ be the set of the zeros of φ, that is $Ze_\varphi = \{x \in X : \varphi(x) = 0\}$. If we consider also $\psi\colon X \to [0, \infty)$, denote by $Ze_{(\varphi,\psi)} = \{x \in X : \varphi(x) = 0, \psi(x) = 0\}$. In this context, an element $p \in P$ is called (ϕ, φ)-best proximity point of T if $p \in BP_T \cap Ze_{(\phi,\varphi)}$.

3. Existence of (ϕ, φ)-best proximity points by type I proximal contractions

In this section, we study the existence of (ϕ, φ)-best proximity points of $T\colon P \to Q$ by considering type I proximal contractions based on the family $\mathfrak{L}$ and the following family $\mathfrak{K}$.

Let $W\colon [0, \infty)^3 \to [0, \infty)$ be a mapping which satisfies the next axioms

(K1) $W(c, b, a) = 0$ if and only if $c = b = a = 0$;

(K2) W is continuous;

(K3) $c \leq W(c, b, a)$.

Example 3. It may be observed that $W\colon [0, \infty)^3 \to [0, \infty)$, $W(c, b, a) = c + b + a$ is a mapping which belongs to the set $\mathfrak{K}$.

Throughout this section, P and Q are non-void subsets of X, (X, d_e) is a metric space, and ϕ, $\varphi\colon P \to [0, \infty)$ are lower semi continuous functions.

Definition 3.1. *A mapping $T\colon P \to Q$ is called $L^I_{(\phi,\varphi)}$-proximal contraction if there exist the functions $\alpha\colon P \times P \to [0, \infty)$, $L \in \mathfrak{L}$, $W \in \mathfrak{K}$ and a constant $\kappa > 0$ such that for all τ_1, τ_2, γ_1, $\gamma_2 \in P$, with $\alpha(\tau_1, \tau_2) \geq 1$ and $d_e(\gamma_1, T\tau_1) = D_e(P, Q) = d_e(\gamma_2, T\tau_2)$, we get $\alpha(\gamma_1, \gamma_2) \geq 1$, and*

$$(1) \qquad \kappa + L(W(d_e(\gamma_1, \gamma_2), \phi(\gamma_1), \varphi(\gamma_2))) \leq L(W(d_e(\tau_1, \tau_2), \phi(\tau_1), \varphi(\tau_2))),$$

whenever $\min\{W(d_e(\gamma_1, \gamma_2), \phi(\gamma_1), \varphi(\gamma_2)), W(d_e(\tau_1, \tau_2), \phi(\tau_1), \varphi(\tau_2))\} > 0$.

By using this definition, a property regarding the existence of a (ϕ, φ)-best proximity point of such a mapping can be stated, as follows.

Theorem 1. *Let P and Q be non-void subsets of X, and (X, d_e) be a complete metric space. Consider that P_0 is closed with respect to d_e and a $L^I_{(\phi,\varphi)}$-proximal contraction mapping $T\colon P \to Q$, which fulfills the next conditions:*

(i) *$T(P_0) \subseteq Q_0$;*

(ii) *there are τ_1, $\tau_2 \in P_0$, so that $\alpha(\tau_1, \tau_2) \geq 1$ and $d_e(\tau_2, T\tau_1) = D_e(P, Q)$;*

(iii) *every sequence $\{\tau_n\} \subseteq P_0$ with $\tau_n \to \tau$ and $\alpha(\tau_n, \tau_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, necessarily satisfies the inequality $\alpha(\tau_n, \tau) \geq 1$, for all $n \in \mathbb{N}$.*

Then T has a (ϕ, φ)-best proximity point.

Proof. Hypothesis (ii) ensures the existence of $\tau_1, \tau_2 \in P_0$ with $\alpha(\tau_1, \tau_2) \geq 1$ and $d_e(\tau_2, T\tau_1) = D_e(P, Q)$. Without loss of generality, we may assume that $\tau_1 \neq \tau_2$. Keeping in mind the fact that $T\tau_2 \in Q_0$, there can be found $\tau_3 \in P_0$ with $d_e(\tau_3, T\tau_2) = D_e(P, Q)$. We may consider once again that $\tau_2 \neq \tau_3$. By using inequality (1), since $\min\{W(d_e(\tau_2, \tau_3), \phi(\tau_2), \varphi(\tau_3)), W(d_e(\tau_1, \tau_2), \phi(\tau_1), \varphi(\tau_2))\} > 0$, we get

$$(2) \qquad \kappa + L(W(d_e(\tau_2, \tau_3), \phi(\tau_2), \varphi(\tau_3))) \leq L(W(d_e(\tau_1, \tau_2), \phi(\tau_1), \varphi(\tau_2))),$$

and $\alpha(\tau_2, \tau_3) \geq 1$.

The above arguments allow us to state that $\tau_2, \tau_3 \in P_0$, with $\alpha(\tau_2, \tau_3) \geq 1$ and $d_e(\tau_3, T\tau_2) = D_e(P, Q)$. By hypothesis (i), $T\tau_3 \in Q_0$, therefore there can be found an element $\tau_4 \in P_0$ endowed with the property that $d_e(\tau_4, T\tau_3) = D_e(P, Q)$. We may again consider $\tau_3 \neq \tau_4$; then relation (1) compels

$$\kappa + L(W(d_e(\tau_3, \tau_4), \phi(\tau_3), \varphi(\tau_4))) \leq L(W(d_e(\tau_2, \tau_3), \phi(\tau_2), \varphi(\tau_3))) \tag{3}$$

and $\alpha(\tau_3, \tau_4) \geq 1$. By combining inequalities (2) and (3), it follows that

$$L(W(d_e(\tau_3, \tau_4), \phi(\tau_3), \varphi(\tau_4))) \leq L(W(d_e(\tau_1, \tau_2), \phi(\tau_1), \varphi(\tau_2))) - 2\kappa.$$

Iteratively, there can be obtained a sequence $\{\tau_i\}$ in P_0, so that $\{T\tau_i\}$ in Q_0, $\alpha(\tau_i, \tau_{i+1}) \geq 1$, $d_e(\tau_{i+1}, T\tau_i) = D_e(P, Q)$ and, for any $i \in \mathbb{N}$,

$$L(W(d_e(\tau_i, \tau_{i+1}), \phi(\tau_i), \varphi(\tau_{i+1}))) \leq L(W(d_e(\tau_1, \tau_2), \phi(\tau_1), \varphi(\tau_2))) - (i-1)\kappa. \tag{4}$$

Applying the limit $i \to \infty$ in relation (4), we obtain that $\lim_{i\to\infty} L(W(d_e(\tau_i, \tau_{i+1}), \phi(\tau_i), \varphi(\tau_{i+1}))) = -\infty$. By $(L2)$ and the previous equality, it follows $\lim_{i\to\infty} W(d_e(\tau_i, \tau_{i+1}), \phi(\tau_i), \varphi(\tau_{i+1})) = 0$. Thus, by using $(K1)$ and $(K2)$, we conclude that $\lim_{i\to\infty} d_e(\tau_i, \tau_{i+1}) = 0$, $\lim_{i\to\infty} \phi(\tau_i) = 0$ and $\lim_{i\to\infty} \varphi(\tau_i) = 0$. Denote by $d_{e_i} = d_e(\tau_i, \tau_{i+1})$, $\phi_i = \phi(\tau_i)$, $\varphi_i = \varphi(\tau_i)$. By using hypothesis $(L3)$, there exists $h \in (0, 1)$ so that

$$\lim_{i\to\infty} W(d_{e_i}, \phi_i, \varphi_{i+1})^h L(W(d_{e_i}, \phi_i, \varphi_{i+1})) = 0.$$

By taking $W_i = W(d_{e_i}, \phi_i, \varphi_{i+1})$, we have $\lim_{i\to\infty} W_i^h L(W_i) = 0$. Taking advantage of relation (4), we get

$$W_i^h L(W_i) - W_i^h L(W_1) \leq -W_i^h (i-1)\kappa \leq 0, \text{ for all } i \in \mathbb{N}.$$

These relations lead to $\lim_{i\to\infty}(i-1)W_i^h = 0$, which ensures us that there is $i_1 \in \mathbb{N}$, $i_1 > 1$, for which $(i-1)W_i^h \leq 1$, for each $i \geq i_1$. Hence, we obtain

$$W_i \leq \frac{1}{(i-1)^{1/h}}, \text{ for all } i \geq i_1. \tag{5}$$

By using axiom (K3) and inequality (5), we get

$$d_{e_i} \leq W_i \leq \frac{1}{(i-1)^{1/h}}, \text{ for all } i \geq i_1. \tag{6}$$

Next, we prove that $\{\tau_i\}$ is a Cauchy sequence in P_0. Consider $i, j \in \mathbb{N}$ with $j > i > i_1$. By considering the triangle inequality and inequality (6), we obtain

$$\begin{aligned} d_e(\tau_i, \tau_j) &\leq d_e(\tau_i, \tau_{i+1}) + d_e(\tau_{i+1}, \tau_{i+2}) + \cdots + d_e(\tau_{j-1}, \tau_j) \\ &= \sum_{m=i}^{j-1} d_{e_m} \leq \sum_{m=i}^{\infty} d_{e_m} \leq \sum_{m=i-1}^{\infty} \frac{1}{m^{1/h}}. \end{aligned}$$

The convergence of the series $\sum_{m=1}^{\infty} \frac{1}{m^{1/h}}$ implies $\lim_{i,j\to\infty} d_e(\tau_i, \tau_j) = 0$, that is, $\{\tau_i\}$ is a Cauchy sequence in P_0. Since P_0 is closed, we have $\tau^* \in P_0$, $\tau_i \to \tau^*$. Hypothesis (iii) yields $\alpha(\tau_i, \tau^*) \geq 1$, $i \in \mathbb{N}$. As $T\tau^* \in Q_0$, there is $\nu^* \in P_0$ such that $d_e(\nu^*, T\tau^*) = D_e(P, Q)$. Hence, we have obtained $\alpha(\tau_i, \tau^*) \geq 1$, $d_e(\tau_{i+1}, T\tau_i) = D_e(P, Q)$ and $d_e(\nu^*, T\tau^*) = D_e(P, Q)$. Without loss of generality, we may presume that $\tau_i \neq \nu^*$ and $\tau_i \neq \tau^*$ for all $i \in \mathbb{N}$. From (1), it follows that

$$\kappa + L(W(d_e(\tau_{i+1}, \nu^*), \phi(\tau_{i+1}), \varphi(\nu^*))) \leq L(W(d_e(\tau_i, \tau^*), \phi(\tau_i), \varphi(\tau^*))), i \in \mathbb{N},$$

which leads to

$$W(d_e(\tau_{i+1}, \nu^*), \phi(\tau_{i+1}), \varphi(\nu^*)) < W(d_e(\tau_i, \tau^*), \phi(\tau_i), \varphi(\tau^*)).$$

Taking the limit $i \to \infty$ and using the continuity of W we get

$$W(d_e(\tau^*, \nu^*), 0, \varphi(\nu^*)) \leq W(d_e(\tau^*, \tau^*), 0, \varphi(\tau^*)). \tag{7}$$

Since ϕ, φ are lower semi continuous functions, $\tau_i \to \tau^*$ and $\lim_{i\to\infty} \phi(\tau_i) = \lim_{i\to\infty} \varphi(\tau_i) = 0$, we get $\phi(\tau^*) = \varphi(\tau^*) = 0$. Hence, inequality (7) compels $W(d_e(\tau^*, \nu^*), 0, \varphi(\nu^*)) \leq 0$. By using condition (K1), the last inequality implies $d_e(\tau^*, \nu^*) = 0$, that is, $\tau^* = \nu^*$, and $\varphi(\nu^*) = 0$. Thus, $d_e(\tau^*, T\tau^*) = D_e(P, Q)$ and $\phi(\tau^*) = \varphi(\tau^*) = 0$; τ^* is a (ϕ, φ)-best proximity point of T. □

Remark 1. Theorem 1 can be further generalized by modifying the properties of the proximal contraction mapping $T: P \to Q$ as follows. A mapping $T: P \to Q$ is called $L^I_{(\phi,\varphi)}$-weakly proximal contraction if there exist the functions $\alpha: P \times P \to [0, \infty)$, $L \in \mathfrak{L}$, $W \in \mathfrak{K}$ and the constants $\kappa > 0$, $r_1 \geq 0$, $r_2 \geq 0$ such that for all τ_1, τ_2, γ_1, $\gamma_2 \in P$, satisfying $\alpha(\tau_1, \tau_2) \geq 1$ and $d_e(\gamma_1, T\tau_1) = D_e(P, Q) = d_e(\gamma_2, T\tau_2)$, we get $\alpha(\gamma_1, \gamma_2) \geq 1$ and

$$\kappa + L(W(d_e(\gamma_1, \gamma_2), \phi(\gamma_1), \varphi(\gamma_2))) \leq L(W(d_e(\tau_1, \tau_2), \phi(\tau_1), \varphi(\tau_2)) + |A|),$$

whenever $\min\{W(d_e(\gamma_1, \gamma_2), \phi(\gamma_1), \varphi(\gamma_2)), W(d_e(\tau_1, \tau_2), \phi(\tau_1), \varphi(\tau_2))\} > 0$. We have denoted by

$$\begin{aligned} A &= r_1(W(d_e(\tau_1, \gamma_1), \phi(\tau_1), \varphi(\gamma_1)) - W(d_e(\tau_1, \tau_2), \phi(\tau_1), \varphi(\tau_2))) \\ &\quad + r_2(W(d_e(\gamma_1, \tau_2), \phi(\tau_1), \varphi(\gamma_1)) - W(0, \phi(\tau_1), \varphi(\gamma_1))). \end{aligned}$$

In order to develop further our theory, we introduce the concept of graphic $L^I_{(\phi,\varphi)}$-proximal contraction mapping.

Definition 3.2. *A mapping $T: P \to Q$ is called a graphic $L^I_{(\phi,\varphi)}$-proximal contraction if there are the functions $\alpha: P \times P \to [0, \infty)$, $L \in \mathfrak{L}$, $W \in \mathfrak{K}$ and a constant $\kappa > 0$ such that for all $\tau_1, \gamma_1, \gamma_2 \in P$ with $d_e(\gamma_1, T\tau_1) = D_e(P, Q) = d_e(\gamma_2, T\gamma_1)$ and $\alpha(\tau_1, \gamma_1) \geq 1$, we get $\alpha(\gamma_1, \gamma_2) \geq 1$, and*

$$\kappa + L(W(d_e(\gamma_1, \gamma_2), \phi(\gamma_1), \varphi(\gamma_2))) \leq L(W(d_e(\tau_1, \gamma_1), \phi(\tau_1), \varphi(\gamma_1))),$$

whenever $\min\{W(d_e(\gamma_1, \gamma_2), \phi(\gamma_1), \varphi(\gamma_2)), W(d_e(\tau_1, \gamma_1), \phi(\tau_1), \varphi(\gamma_1))\} > 0$.

With regard to such proximal contractions, we are in a position to state and prove an existence result.

Theorem 2. *Let P and Q be non-void subsets of X, and (X, d_e) be a complete metric space. Consider that P_0 is a closed subset of X. Let $T: P \to Q$ be a graphic $L^I_{(\phi,\varphi)}$-proximal contraction mapping which satisfy these conditions:*

(i) $T(P_0) \subseteq Q_0$;

(ii) *there are* $\tau_1, \tau_2 \in P_0$ *with* $\alpha(\tau_1, \tau_2) \geq 1$ *and* $d_e(\tau_2, T\tau_1) = D_e(P, Q)$;

(iii) $\mathrm{Graph}(T_\alpha) = \{(\tau, \gamma) : \tau, \gamma \in P_0$ *with* $\alpha(\tau, \gamma) \geq 1$ *and* $d_e(\gamma, T\tau) = D_e(P, Q)\}$ *is closed.*

Then T has a (ϕ, φ)-best proximity point.

Proof. Following the procedure of the proof of Theorem 1, here we obtain a Cauchy sequence $\{\tau_i\}$ in P_0, so that that $\alpha(\tau_i, \tau_{i+1}) \geq 1$, $d_e(\tau_{i+1}, T\tau_i) = D_e(P, Q)$, for all $i \in \mathbb{N}$ and $\lim_{i\to\infty} \phi(\tau_i) = \lim_{i\to\infty} \varphi(\tau_{i+1}) = 0$. Furthermore, $\tau^* \in P_0$, and $\tau_i \to \tau^*$. The closedness of $\mathrm{Graph}(T_\alpha)$ compels $(\tau^*, \tau^*) \in \mathrm{Graph}(T_\alpha)$. Hence, $d_e(\tau^*, T\tau^*) = D_e(P, Q)$. By the lower semi continuity of the functions ϕ and φ, we get $\phi(\tau^*) = \varphi(\tau^*) = 0$. Thus, τ^* becomes a (ϕ, φ)-best proximity point of T. □

3.1. Existence of (ϕ,φ)-best proximity points by type II proximal contractions

In this section, we will discuss type II (ϕ,φ)-proximal contractions based on the family $\mathfrak{M}$. Again, throughout this subsection P and Q are non-void subsets X, (X,d_e) is a metric space, and $\phi,\varphi\colon P\to[0,\infty)$ are lower semi continuous functions.

Definition 3.3. *A mapping $T\colon P\to Q$ is called $L^{II}_{(\phi,\varphi)}$-proximal contraction if there exist functions $\alpha\colon P\times P\to[0,\infty)$, $L\in\mathfrak{M}$, and a constant $\kappa>0$ such that for all $\tau_1,\tau_2,\gamma_1,\gamma_2\in P$ with $\alpha(\tau_1,\tau_2)\geq 1$ and $d_e(\gamma_1,T\tau_1)=D_e(P,Q)=d_e(\gamma_2,T\tau_2)$, we get $\alpha(\gamma_1,\gamma_2)\geq 1$ and*

$$\kappa+L(d_e(\gamma_1,\gamma_2),\phi(\gamma_1),\varphi(\gamma_2))\leq L(d_e(\tau_1,\tau_2),\phi(\tau_1),\varphi(\tau_2)) \tag{8}$$

whenever $\min\{d_e(\gamma_1,\gamma_2)+\phi(\gamma_1)+\varphi(\gamma_2),d_e(\tau_1,\tau_2)+\phi(\tau_1)+\varphi(\tau_2)\}>0$.

Theorem 3. *Let P and Q be non-void subsets X, and (X,d_e) be a complete metric space. Consider that P_0 is closed, and $T\colon P\to Q$ a $L^{II}_{(\phi,\varphi)}$-proximal contraction mapping which fulfills these conditions*

(i) *$T(P_0)\subseteq Q_0$;*

(ii) *there are $\tau_1,\tau_2\in P_0$ with $\alpha(\tau_1,\tau_2)\geq 1$ and $d_e(\tau_2,T\tau_1)=D_e(P,Q)$;*

(iii) *every $\{\tau_n\}\subseteq P_0$ with $\tau_n\to\tau$ and $\alpha(\tau_n,\tau_{n+1})\geq 1$ for all $n\in\mathbb{N}$, implies $\alpha(\tau_n,\tau)\geq 1$, for all $n\in\mathbb{N}$.*

Then T has a (ϕ,φ)-best proximity point.

Proof. By hypothesis (ii), there are τ_1, $\tau_2\in P_0$ so that $\alpha(\tau_1,\tau_2)\geq 1$ and $d_e(\tau_2,T\tau_1)=D_e(P,Q)$. Without loss of generality, we may assume that $\tau_1\neq\tau_2$. As $T\tau_2\in Q_0$, we have $\tau_3\in P_0$, $d_e(\tau_3,T\tau_2)=D_e(P,Q)$; consider again $\tau_2\neq\tau_3$. By using the contractive condition (8), we get

$$\kappa+L(d_e(\tau_2,\tau_3),\phi(\tau_2),\varphi(\tau_3))\leq L(d_e(\tau_1,\tau_2),\phi(\tau_1),\varphi(\tau_2)) \tag{9}$$

and $\alpha(\tau_2,\tau_3)\geq 1$. This compels that τ_2 and τ_3 are elements of P_0, for which $\alpha(\tau_2,\tau_3)\geq 1$ and $d_e(\tau_3,T\tau_2)=D_e(P,Q)$.

By continuing this process, from $T\tau_3\in Q_0$ there can be obtained that $\tau_4\in P_0$, $d_e(\tau_4,T\tau_3)=D_e(P,Q)$. Without any loss we can assume once again $\tau_3\neq\tau_4$. Therefore, inequality (8) implies

$$\kappa+L(d_e(\tau_3,\tau_4),\phi(\tau_3),\varphi(\tau_4))\leq L(d_e(\tau_2,\tau_3),\phi(\tau_2),\varphi(\tau_3)) \tag{10}$$

and $\alpha(\tau_3,\tau_4)\geq 1$. Having in mind inequalities (9) and (10), it follows that

$$L(d_e(\tau_3,\tau_4),\phi(\tau_3),\varphi(\tau_4))\leq L(d_e(\tau_1,\tau_2),\phi(\tau_1),\varphi(\tau_2))-2\kappa.$$

Iteratively, a sequence $\{\tau_n\}$ in P_0 can be obtained so that and $\{T\tau_n\}$ in Q_0, $\alpha(\tau_n,\tau_{n+1})\geq 1$, $d_e(\tau_{n+1},T\tau_n)=D_e(P,Q)$ and, for any $i\in\mathbb{N}$

$$L(d_e(\tau_i,\tau_{i+1}),\phi(\tau_i),\varphi(\tau_{i+1}))\leq L(d_e(\tau_1,\tau_2),\phi(\tau_1),\varphi(\tau_2))-(i-1)\kappa. \tag{11}$$

Applying the limit to infinity in inequality (11), we obtain that $\lim_{i\to\infty}L(d_e(\tau_i,\tau_{i+1}),\phi(\tau_i),\varphi(\tau_{i+1}))=-\infty$. Combining this equality with $(ML2)$, we get $\lim_{i\to\infty}d_e(\tau_i,\tau_i)=0$, $\lim_{i\to\infty}\phi(\tau_i)=0$ and $\lim_{i\to\infty}\varphi(\tau_{i+1})=0$. Denote by $d_{e_i}=d_e(\tau_i,\tau_{i+1})$, $\phi_i=\phi(\tau_i)$, $\varphi_i=\varphi(\tau_i)$. By using $(ML3)$, there exists $j\in(0,1)$, so that

$$\lim_{i\to\infty}d_{e_i}^j L(d_{e_i},\phi_i,\varphi_{i+1})=0.$$

Putting $L_i = L(d_{e_i}, \phi_i, \varphi_{i+1})$ in the above, it follows that $\lim_{i\to\infty} d_{e_i}^j L_i = 0$. Relation (11) leads us to

$$d_{e_i}^j L_i - d_{e_i}^j L_1 \leq -d_{e_i}^j (i-1)\kappa \leq 0, \text{ for all } i \in \mathbb{N}.$$

This yields that $\lim_{i\to\infty} d_{e_i}^j (i-1) = 0$. Thus, there is $i_1 \in \mathbb{N}$, $i_1 > 1$, with $d_{e_i}^j (i-1) \leq 1$, for all $i \geq i_1$. Hence, we get

$$d_{e_i} \leq \frac{1}{(i-1)^{1/j}}, \text{ for all } i \geq i_1.$$

By following the proof of Theorem 1 and using the above inequality one can check that $\{\tau_i\}$ is a Cauchy sequence in P_0 and $\tau_i \to \tau^* \in P_0$. As $\lim_{i\to\infty} \phi(\tau_i) = \lim_{i\to\infty} \varphi(\tau_i) = 0$ and $\tau_i \to \tau^*$, by the lower semi continuity ϕ, φ, we get $\phi(\tau^*) = \varphi(\tau^*) = 0$. Moreover, the convergence of $\{\tau_n\}$ to τ^* and hypothesis (iii) imply that $\alpha(\tau_i, \tau^*) \geq 1$, for all $i \in \mathbb{N}$, since $\alpha(\tau_i, \tau_{i+1}) \geq 1$, for all $i \in \mathbb{N}$. As $T\tau^* \in Q_0$, there is $\nu^* \in P_0$, so that $d_e(\nu^*, T\tau^*) = D_e(P,Q)$. Thus, we have $\alpha(\tau_i, \tau^*) \geq 1$, for all $i \in \mathbb{N}$, $d_e(\tau_{i+1}, T\tau_i) = D_e(P,Q)$ and $d_e(\nu^*, T\tau^*) = D_e(P,Q)$.

Without loss of generality, we can assume that $\tau_i \neq \nu^*$, and $\tau_i \neq \tau^*$, for all $i \in \mathbb{N}$.

Having in mind relation (8), we get

$$\kappa + L(d_e(\tau_{i+1}, \nu^*), \phi(\tau_{i+1}), \varphi(\nu^*)) \leq L(d_e(\tau_i, \tau^*), \phi(\tau_i), \varphi(\tau^*)), \text{ for all } i \in \mathbb{N},$$

which compels

$$L(d_e(\tau_{i+1}, \nu^*), \phi(\tau_{i+1}), \varphi(\nu^*)) < L(d_e(\tau_i, \tau^*), \phi(\tau_i), \varphi(\tau^*)), \text{ for all } i \in \mathbb{N}.$$

Taking advantage of property $(ML1:b)$ and the above inequality, it follows that

$$d_e(\tau_{i+1}, \nu^*) + \phi(\tau_{i+1}) + \varphi(\nu^*) < d_e(\tau_i, \tau^*) + \phi(\tau_i).$$

By letting $i \to \infty$ in the above inequalities, we get $d_e(\tau^*, \nu^*) + \varphi(v^*) \leq 0$, thus, $\tau^* = \nu^*$. We have proved that $d_e(\tau^*, T\tau^*) = D_e(P,Q)$ and $\phi(\tau^*) = \varphi(\tau^*) = 0$. Therefore, τ^* is (ϕ, φ)-best proximity point of T.

The proof has been completed. □

Remark 2. A generalization of Theorem 3 can be obtained by using the following modified form of $L^{II}_{(\phi,\varphi)}$-proximal contractions. A mapping $T\colon P \to Q$ is called a $L^{II}_{(\phi,\varphi)}$-weakly proximal contraction, if there can be found the functions $\alpha\colon P \times P \to \mathbb{R}_+$, $L \in \mathfrak{M}$ and the constants $\kappa > 0$, $r_1 \geq 0$, $r_2 \geq 0$ such that for all $\tau_1, \tau_2, \gamma_1, \gamma_2 \in P$ with $\alpha(\tau_1, \tau_2) \geq 1$ and $d_e(\gamma_1, T\tau_1) = D_e(P,Q) = d_e(\gamma_2, T\tau_2)$, we get $\alpha(\gamma_1, \gamma_2) \geq 1$ and

$$\kappa + L(d_e(\gamma_1, \gamma_2), \phi(\gamma_1), \varphi(\gamma_2)) \leq L(d_e(\tau_1, \tau_2) + |A|, \phi(\tau_1), \varphi(\tau_2) + |B|)$$

whenever $\min\{d_e(\gamma_1, \gamma_2) + \phi(\gamma_1) + \varphi(\gamma_2), d_e(\tau_1, \tau_2) + \phi(\tau_1) + \varphi(\tau_2)\} > 0$, where $A = r_1\{d_e(\tau_1, \gamma_1) - d_e(\tau_1, \tau_2)\} + r_2 d_e(\gamma_1, \tau_2)$ and $B = \varphi(\gamma_1) - \varphi(\tau_2)$.

We continue by introducing the notion of graphic $L^{II}_{(\phi,\varphi)}$-proximal contraction mappings.

Definition 3.4. *A mapping $T\colon P \to Q$ is called graphic $L^{II}_{(\phi,\varphi)}$-proximal contraction, if there are functions $\alpha\colon P \times P \to [0,\infty)$, $L \in \mathfrak{M}$, and a constant $\kappa > 0$ such that for all $\tau_1, \gamma_1, \gamma_2 \in P$ with $d_e(\gamma_1, T\tau_1) = D_e(P,Q) = d_e(\gamma_2, T\gamma_1)$ and $\alpha(\tau_1, \gamma_1) \geq 1$, we get $\alpha(\gamma_1, \gamma_2) \geq 1$, and*

$$\kappa + L(d_e(\gamma_1, \gamma_2), \phi(\gamma_1), \varphi(\gamma_2)) \leq L(d_e(\tau_1, \gamma_1), \phi(\tau_1), \varphi(\gamma_1))$$

whenever $\min\{d_e(\gamma_1, \gamma_2) + \phi(\gamma_1) + \varphi(\gamma_2), d_e(\tau_1, \gamma_1) + \phi(\tau_1) + \varphi(\gamma_1)\} > 0$.

This definition enables us to develop a result on the existence of a (ϕ, φ)-best proximity point.

Theorem 4. *Let P and Q be non-void subsets X, (X, d_e) a complete metric space, and P_0 be closed. Consider $T\colon P \to Q$ be a graphic $L^{II}_{(\phi,\varphi)}$-proximal contraction mapping if the next axioms are fulfilled*

(i) $T(P_0) \subseteq Q_0$;

(ii) *there are* $\tau_1, \tau_2 \in P_0$ *with* $\alpha(\tau_1, \tau_2) \geq 1$ *and* $d_e(\tau_2, T\tau_1) = D_e(P, Q)$;

(iii) $\mathrm{Graph}(T_\alpha) = \{(\tau, \gamma) : \tau, \gamma \in P_0$ *with* $\alpha(\tau, \gamma) \geq 1$ *and* $d_e(\gamma, T\tau) = D_e(P, Q)\}$ *is closed.*

Then T has a (ϕ, φ)-best proximity point.

Proof. By using the definition of a graphic $L^{II}_{(\phi,\varphi)}$-proximal contraction and the hypotheses of the theorem, we obtain a Cauchy sequence $\{\tau_i\}$ in P_0 with $\alpha(\tau_i, \tau_{i+1}) \geq 1$, $d_e(\tau_{i+1}, T\tau_i) = D_e(P, Q)$ for all $i \in \mathbb{N}$ and $\lim_{i\to\infty} \phi(\tau_i) = \lim_{i\to\infty} \varphi(\tau_{i+1}) = 0$. Moreover, $\tau^* \in P_0$ with $\tau_i \to \tau^*$. Thus, by the closedness of $\mathrm{Graph}(T_\alpha)$, we have $(\tau^*, \tau^*) \in \mathrm{Graph}(T_\alpha)$. Hence, $d_e(\tau^*, T\tau^*) = D_e(P, Q)$. By the lower semi continuity of ϕ and φ, we get $\phi(\tau^*) = \varphi(\tau^*) = 0$. Therefore, τ^* becomes a (ϕ, φ)-best proximity point of T. □

4. Consequence and Example

The following result is obtained from Theorem 1 (or from Theorem 3) by taking $L(x) = \ln x$ and $W(c, b, a) = c + b + a$ (or by taking $L(c, b, a) = \ln(c + b + a)$).

Corollary 1. *Let P and Q be non-void subsets of complete metric space (X, d_e). Consider that P_0 is closed. Let $\alpha\colon P \times P \to [0, \infty)$, ϕ, $\varphi\colon P \to [0, \infty)$ and $T\colon P \to Q$ be mappings such that for all $\tau_1, \tau_2, \gamma_1, \gamma_2 \in P$ with $\alpha(\tau_1, \tau_2) \geq 1$ and $d_e(\gamma_1, T\tau_1) = D_e(P, Q) = d_e(\gamma_2, T\tau_2)$, we get*

$$\alpha(\gamma_1, \gamma_2) \geq 1 \text{ and } d_e(\gamma_1, \gamma_2) + \phi(\gamma_1) + \varphi(\gamma_2) \leq \kappa(d_e(\tau_1, \tau_2) + \phi(\tau_1) + \varphi(\tau_2))$$

where $0 < \kappa < 1$. Also assume that the below conditions are fulfilled

(1) $T(P_0) \subseteq Q_0$;

(2) *there are* $\tau_1, \tau_2 \in P_0$ *with* $\alpha(\tau_1, \tau_2) \geq 1$ *and* $d_e(\tau_2, T\tau_1) = D_e(P, Q)$;

(3) *every* $\{\tau_n\} \subseteq P_0$ *with* $\tau_n \to \tau$ *and* $\alpha(\tau_n, \tau_{n+1}) \geq 1$ *for all* $n \in \mathbb{N}$, *satisfies* $\alpha(\tau_n, \tau) \geq 1$, *for all* $n \in \mathbb{N}$;

(4) $\phi, \varphi\colon P \to [0, \infty)$ *are lower semi continuous functions.*

Then T has a (ϕ, φ)-best proximity point.

Example 4. Let $X = \mathbb{R}^2$ endowed with the metric $d_e((\tau_1, \tau_2), (\overline{\tau_1}, \overline{\tau_2})) = |\tau_1 - \overline{\tau_1}| + |\tau_2 - \overline{\tau_2}|$. Let $P = \{(\tau, 0) : \tau \in \mathbb{R}\}$ and $Q = \{(\tau, 1) : \tau \in \mathbb{R}\}$. Define

$$\alpha\colon P \times P \to [0, \infty), \quad \alpha((\tau, 0), (\overline{\tau}, 0)) = \begin{cases} 1, & \text{if } \tau, \overline{\tau} \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Consider the mappings

$$T\colon P \to Q, \quad T(\tau, 0) = \begin{cases} \left(\frac{\tau}{2+2\tau}, 1\right), & \text{if } \tau \geq 0; \\ (\tau^3, 1), & \text{if } \tau < 0 \end{cases}$$

and

$$\phi, \varphi\colon P \to [0, \infty), \quad \phi((\tau, 0)) = \left|\frac{\tau}{2}\right|, \ \varphi((\tau, 0)) = |\tau|.$$

First let us observe that $D_e(P, Q) = 1$, $P_0 = P$, and $Q_0 = Q$.

Obviously, $T(P_0) \subseteq Q_0$.

Also we can consider $(\tau_1,0)=(0,0)$, and $(\tau_2,0)=(0,1)$ and notice that

$$d_e((0,0),T(0,0))=d((0,0),(0,1))=1=D_e(P,Q).$$

Furthermore, it is clear that for any sequence $\{(\tau_n,0)\}\subseteq P_0$, for which $(\tau_n,0)\to(\tau,0)$ and $\alpha((\tau_n,0),(\tau_{n+1},0))\geq 1$, $n\in\mathbb{N}$, the inequality $\alpha(\tau_n,\tau)\geq 1$, $n\in\mathbb{N}$ holds true.

Suppose now we take $(\tau_1,0),(\tau_2,0)\in P$, so that $\alpha((\tau_1,0),(\tau_2,0))=1$; that is τ_1, $\tau_2\geq 0$. Moreover, consider that

$$d_e((\gamma_1,0),T(\tau_1,0))=D_e(P,Q),\qquad d_e((\gamma_2,0),T(\tau_2,0))=D_e(P,Q),$$

which compels

$$\gamma_1=\frac{\tau_1}{2+2\tau_1},\qquad \gamma_1=\frac{\tau_2}{2+2\tau_2}.$$

Furthermore, it follows that

$$\begin{aligned}
d_e((\gamma_1,0),(\gamma_2,0)) &= d_e\Big(\Big(\frac{\tau_1}{2+2\tau_1},0\Big),\Big(\frac{\tau_2}{2+2\tau_2},0\Big)\Big)+\phi\Big(\Big(\frac{\tau_1}{2+2\tau_1},0\Big)\Big)+\varphi\Big(\Big(\frac{\tau_2}{2+2\tau_2},0\Big)\Big)\\
&=\left|\frac{\tau_1}{2+2\tau_1}-\frac{\tau_2}{2+2\tau_2}\right|+\left|\frac{\tau_1}{4+4\tau_1}\right|+\left|\frac{\tau_2}{2+2\tau_2}\right|\\
&\leq\frac{1}{2}\left(|\tau_1-\tau_2|+\left|\frac{\tau_1}{2}\right|+|\tau_2|\right)\\
&=\frac{1}{2}\left(d_e((\tau_1,0),(\tau_2,0)),\phi(\tau_1),\varphi(\tau_2)\right).
\end{aligned}$$

The remaining conditions of Corollary 1 are also satisfied. Thus, T has a (ϕ,φ)-best proximity point.

5. **Application in Fixed Point Theory**

In this section, we will discuss results which ensure the existence of (ϕ,φ)-fixed points of self-mappings $T\colon P\to P$. These properties can be considered as applications of the above stated results in fixed point theory. They are obtained by considering $P=Q$ in the second section.

Theorem 5. *Let (P,d_e) be a complete metric space, and $T\colon P\to P$ be a mapping for which there exist $\alpha\colon P\times P\to[0,\infty)$, $\phi,\varphi\colon P\to[0,\infty)$, $L\in\mathfrak{L}$, $W\in\mathfrak{K}$ and a constant $\kappa>0$ such that for all $\tau_1,\tau_2\in P$ with $\alpha(\tau_1,\tau_2)\geq 1$, the next relations hold*

$$\begin{aligned}
&\alpha(T\tau_1,T\tau_2)\geq 1 \text{ and}\\
&\kappa+L(W(d_e(T\tau_1,T\tau_2),\phi(T\tau_1),\varphi(T\tau_2)))\leq L(W(d_e(\tau_1,\tau_2),\phi(\tau_1),\varphi(\tau_2))),
\end{aligned}$$

whenever $\min\{W(d_e(T\tau_1,T\tau_2),\phi(T\tau_1),\varphi(T\tau_2))),W(d_e(\tau_1,\tau_2),\phi(\tau_1),\varphi(\tau_2))\}>0$. *Also, consider that the below hypotheses hold true*

(1) *there is a point $\tau_1\in P$ so that $\alpha(\tau_1,T\tau_1)\geq 1$;*
(2) *every $\{\tau_n\}\subseteq P$ with $\tau_n\to\tau$ and $\alpha(\tau_n,\tau_{n+1})\geq 1$, for all $n\in\mathbb{N}$, fulfills the inequality $\alpha(\tau_n,\tau)\geq 1$, for all $n\in\mathbb{N}$;*
(3) *$\phi,\varphi\colon P\to[0,\infty)$ are lower semi continuous.*

Then T has a (ϕ,φ)-fixed point in P.

Remark 3. Consider a mapping $T\colon P\to P$ for which there can be found $\alpha\colon P\times P\to[0,\infty)$, ϕ, $\varphi\colon P\to[0,\infty)$, $L\in\mathfrak{L}$, $W\in\mathfrak{K}$ and constants $\kappa>0$, $r_1\geq 0$, $r_2\geq 0$ such that for all $\tau_1,\tau_2\in P$ with $\alpha(\tau_1,\tau_2)\geq 1$, we get $\alpha(T\tau_1,T\tau_2)\geq 1$ and

$$\kappa+L(W(d_e(T\tau_1,T\tau_2),\phi(T\tau_1),\varphi(T\tau_2)))\leq L(W(d_e(\tau_1,\tau_2),\phi(\tau_1),\varphi(\tau_2))+|A|)$$

whenever $\min\{W(d_e(T\tau_1, T\tau_2), \phi(T\tau_1), \varphi(T\tau_2)), W(d_e(\tau_1, \tau_2), \phi(\tau_1), \varphi(\tau_2))\} > 0$, where

$$\begin{aligned} A &= r_1(W(d_e(\tau_1, T\tau_1), \phi(\tau_1), \varphi(T\tau_1)) - W(d_e(\tau_1, \tau_2), \phi(\tau_1), \varphi(\tau_2))) \\ &\quad + r_2(W(d_e(T\tau_1, \tau_2), \phi(\tau_1), \varphi(T\tau_1)) - W(0, \phi(\tau_1), \varphi(T\tau_1))). \end{aligned}$$

Note that if the above mapping T exists on the complete metric space (P, d_e) along with the axioms (1) - (3) of Theorem 5, then T must possess a (ϕ, φ)-fixed point in P.

Theorem 6. *Let (P, d_e) be a complete metric space and $T\colon P \to P$ be a mapping for which there are $\alpha\colon P \times P \to [0, \infty)$, ϕ, $\varphi\colon P \to [0, \infty)$, $L \in \mathfrak{L}$, $W \in \mathfrak{K}$ and a constant $\kappa > 0$ such that for all $\tau_1 \in P$ with $\alpha(\tau_1, T\tau_1) \geq 1$, we get*

$$\begin{aligned} &\alpha(T\tau_1, T^2\tau_1) \geq 1 \text{ and} \\ &\kappa + L(W(d_e(T\tau_1, T^2\tau_1), \phi(T\tau_1), \varphi(T^2\tau_1))) \leq L(W(d_e(\tau_1, T\tau_1), \phi(\tau_1), \varphi(T\tau_1))), \end{aligned}$$

whenever $\min\{W(d_e(T\tau_1, T^2\tau_1), \phi(T\tau_1), \varphi(T^2\tau_1)), W(d_e(\tau_1, T\tau_1), \phi(\tau_1), \varphi(T\tau_1))\} > 0$. Also, consider that the below hypotheses hold

1. *there is a point $\tau_1 \in P$ with $\alpha(\tau_1, T\tau_1) \geq 1$;*
2. $\mathrm{Graph}(T_\alpha) = \{(\tau, T\tau) : \tau \in P$ *with* $\alpha(\tau, T\tau) \geq 1\}$ *is closed;*
3. *$\phi, \varphi\colon P \to [0, \infty)$ are lower semi continuous.*

Then T has a (ϕ, φ)-best proximity point.

Theorem 7. *Let (P, d_e) be a complete metric space. Let $T\colon P \to P$ be a mapping for which there are $\alpha\colon P \times P \to [0, \infty)$, ϕ, $\varphi\colon P \to [0, \infty)$, $L \in \mathfrak{M}$ and a constant $\kappa > 0$ such that for all $\tau_1, \tau_2 \in P$ with $\alpha(\tau_1, \tau_2) \geq 1$, we get $\alpha(T\tau_1, T\tau_2) \geq 1$, and*

$$\kappa + L(d_e(T\tau_1, T\tau_2), \phi(T\tau_1), \varphi(T\tau_2)) \leq L(d_e(\tau_1, \tau_2), \phi(\tau_1), \varphi(\tau_2))$$

whenever $\min\{d_e(T\tau_1, T\tau_2) + \phi(T\tau_1) + \varphi(T\tau_2), d_e(\tau_1, \tau_2) + \phi(\tau_1) + \varphi(\tau_2)\} > 0$. Also, consider the given below hypotheses hold

1. *there is a point $\tau_1 \in P$ so that $\alpha(\tau_1, T\tau_1) \geq 1$;*
2. *every $\{\tau_n\} \subseteq P$ with $\tau_n \to \tau$ and $\alpha(\tau_n, \tau_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, checks the inequality $\alpha(\tau_n, \tau) \geq 1$, for all $n \in \mathbb{N}$;*
3. *$\phi, \varphi\colon P \to [0, \infty)$ are lower semi continuous.*

Then T has a (ϕ, φ)-fixed point in P.

Remark 4. Consider a map $T\colon P \to P$ for which there can be found $\alpha\colon P \times P \to [0, \infty)$, ϕ , $\varphi\colon P \to [0, \infty)$, $L \in \mathfrak{M}$ and constants $\kappa > 0$, $r_1 \geq 0$, $r_2 \geq 0$ such that for all $\tau_1, \tau_2 \in P$ with $\alpha(\tau_1, \tau_2) \geq 1$, we get $\alpha(T\tau_1, T\tau_2) \geq 1$ and

$$\kappa + L(d_e(T\tau_1, T\tau_2), \phi(T\tau_1), \varphi(T\tau_2)) \leq L(d_e(\tau_1, \tau_2) + |A|, \phi(\tau_1), \varphi(\tau_2) + |B|)$$

whenever $\min\{d_e(T\tau_1, T\tau_2) + \phi(T\tau_1) + \varphi(T\tau_2), d_e(\tau_1, \tau_2) + \phi(\tau_1) + \varphi(\tau_2)\} > 0$, where $A = r_1\{d_e(\tau_1, T\tau_1) - d_e(\tau_1, \tau_2)\} + r_2 d_e(T\tau_1, \tau_2)$ and $B = \varphi(T\tau_1) - \varphi(\tau_2)$.
Note that if the above mapping T exists on a complete metric space (P, d_e) along with the axioms (1) - (3) of Theorem 7, then T certainly has a (ϕ, φ)-fixed point in P.

Theorem 8. *Let (P, d_e) be a complete metric space, and $T\colon P \to P$ be a mapping for which there are $\alpha\colon P \times P \to [0, \infty)$, ϕ, $\varphi\colon P \to [0, \infty)$, $L \in \mathfrak{M}$ and a constant $\kappa > 0$ such that for all $\tau_1 \in P$ with $\alpha(\tau_1, T\tau_1) \geq 1$, we get $\alpha(T\tau_1, T^2\tau_1) \geq 1$ and*

$$\kappa + L(d_e(T\tau_1, T^2\tau_1), \phi(T\tau_1), \varphi(T^2\tau_1)) \leq L(d_e(\tau_1, T\tau_1), \phi(\tau_1), \varphi(T\tau_1))$$

whenever $\min\{d_e(T\tau_1, T^2\tau_1) + \phi(T\tau_1) + \varphi(T^2\tau_1), d_e(\tau_1, T\tau_1) + \phi(\tau_1) + \varphi(T\tau_1)\} > 0$. Also, consider that the below hypotheses hold

(1) *there is a point* $\tau_1 \in P$ *with* $\alpha(\tau_1, T\tau_1) \geq 1$*;*
(2) $\mathrm{Graph}(T_\alpha) = \{(\tau, T\tau) : \tau \in P \text{ with } \alpha(\tau, T\tau) \geq 1\}$ *is closed;*
(3) $\phi, \varphi\colon P \to [0, \infty)$ *are lower semi continuous.*

Then T *has a* (ϕ, φ)*-best proximity point.*

6. **Conclusions**

In this work, we have introduced two classes of proximal contractions defined by using functions with suitable properties related to different types of monotone, continuity, or convergence. The (ϕ, φ)-best proximity points were also introduced and their existence with regard to proximal mappings defined here is obtained. Fixed point results, examples and other consequences are formulated as corollaries of these theorems.

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