INFINITELY MANY POSITIVE WEAK SOLUTIONS FOR A PERTURBED FOURTH-ORDER KIRCHHOFF-TYPE ON THE WHOLE SPACE

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This study aims to establish the existence of infinitely many weak solutions for a fourth-order Kirchhoff-type equation on an unbounded interval. The approach used in the persent study is based on variational methods and critical point theory.

Keywords: Fourth-order Kirchhoff-type equation, Weak solution, Critical point theory, Variational methods

1. Introduction

In the current research the following Kirchhoff type problem is considered:

$$\begin{cases} J(u) = \lambda\, \alpha(x) f(u(x)), \ a.e. \ x \in \mathbb{R}, \\ \\ u \in W^{2,2}(\mathbb{R}), \end{cases} \tag{1}$$

where λ is a positive parameter and

$$J(u) = u^{iv}(x) + K\left(\int_{\mathbb{R}}(q(x)|u'(x)|^2 + s(x)|u(x)|^2)dx\right)((-q(x)u'(x))' + s(x)u(x)),$$

in which $q, s \in L^{\infty}(\mathbb{R})$, with $q_0 = \operatorname{ess\,inf}_{\mathbb{R}} q > 0$ and $s_0 = \operatorname{ess\,inf}_{\mathbb{R}} s > 0$, $K : [0, +\infty[\to \mathbb{R}$ is a continuous function such that there exist positive numbers m_0 and m_1 with $m_0 \leq K(t) \leq m_1$ for all $t \geq 0$. Also $\alpha, f : \mathbb{R} \to \mathbb{R}$ are two functions such that $\alpha \in L^1(\mathbb{R})$, $\alpha(x) \geq 0$, for a.e. $x \in \mathbb{R}$, $\alpha \not\equiv 0$ and also $f : \mathbb{R} \to \mathbb{R}$ is a continuous and non-negative function.

This type of differential equation is a generalized classical D'Alembert's wave equation for the free vibration of an elastic beam proposed by Kirchhoff in [10]. In fact in problem (1), the function f represents the force that the foundation exerts on the beam and $K(\int_{\mathbb{R}}(q(x)|u'(x)|^2 + s(x)|u(x)|^2)dx)$ models the effects of the small changes in the length of the beam. Some important topics about modeling of Kirchhoff type strings and beams can be found in references [2, 10, 15] .

As we know, one of the most widely used differential equations is the fourth-order differential equation, which plays a key role in describing the large number of elastic deviations

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in beams and strings. Over the past two decades, many studies have been conducted on fourth-order differential equations and surprising results have been obtained. For example, readers might refer to the references cited at the end of the present study: [5, 8, 11, 13]. Also, to study the different types of Kirchhoff differential equations, they might refer to the cited references: [7, 9, 12, 14].

For example in [9], the authors considered the following fourth-order Kirchhoff-type problem:

$$\begin{cases} u^{iv} + K\left(\int_0^1(-A|u'(x)|^2 + B|u(x)|^2)dx\right)(Au'' + Bu) = \\ \lambda f(x,u) + \mu g(x,u) + h(u),\ x \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases} \tag{2}$$

where A and B are real constants, λ is a positive parameter, μ is a non-negative parameter, $K : [0, +\infty[\to \mathbb{R}$ is a continuous function such that there exist positive numbers m_0 and m_1 with $m_0 \leq K(t) \leq m_1$ for all $t \geq 0$, $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ are two L^2-Carathéodory functions and $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with the Lipschitz constant $L > 0$ and $h(0) = 0$.

The researchers of the present work examined the existence of at least three generalized solutions by using the variational methods and critical point theory.
It is to be noticed that if the interval is bounded, then there will not be much challenge to prove the existence of solution for the differential equation.

Therefore, since the operators which have been used to solve equations such as (1) on $\mathbb{R}$ are not compact, the study of such problems is very complicated.

In the present paper, using an infinitely many critical points theorem obtained in [4] , the existence of a sequence of weak solutions for the problem (1) is guaranteed.

2. **Preliminaries**

Let us recall some basic concepts that we will need later.
$W_0^{2,2}(\mathbb{R}) = H_0^2(\mathbb{R})$ denotes the closure of $C_0^\infty(\mathbb{R})$ in $W^{2,2}(\mathbb{R}) = H^2(\mathbb{R})$ and since $C_0^\infty(\mathbb{R})$ is dense in $W^{2,2}(\mathbb{R})$, we have $W_0^{2,2}(\mathbb{R}) = W^{2,2}(\mathbb{R})$, [[1], Corollary 3.19].
We denote by $\|.\|_t$ the usual norm on $L^t(\mathbb{R})$, for all $1 \leq t \leq +\infty$.
Also, the Sobolev space $W^{2,2}(\mathbb{R})$ is equipped with the norm

$$\| u \|_{W^{2,2}(\mathbb{R})} = \left(\int_{\mathbb{R}}(|u''(x)|^2 + |u'(x)|^2 + |u(x)|^2)dx\right)^{1/2},$$

for all $u \in W^{2,2}(\mathbb{R})$. On the other hand, we can consider $W^{2,2}(\mathbb{R})$ with the norm

$$\| u \| = \left(\int_{\mathbb{R}}(|u''(x)|^2 + q(x)|u'(x)|^2 + s(x)|u(x)|^2)dx\right)^{1/2},$$

for all $u \in W^{2,2}(\mathbb{R})$. According to

$$\begin{aligned} &(\min\{1, q_0, s_0\})^{\frac{1}{2}} \| u \|_{W^{2,2}(\mathbb{R})} \leq \| u \| \leq \\ &(\max\{1, \|q\|_\infty, \|s\|_\infty\})^{\frac{1}{2}} \| u \|_{W^{2,2}(\mathbb{R})}, \end{aligned} \tag{3}$$

the norm $\| \, . \, \|$ is equvalent to the $\| \, . \, \|_{W^{2,2}(\mathbb{R})}$ norm.
It is well known that $W^{2,2}(\mathbb{R})$ is continuously embedded in $L^\infty(\mathbb{R})$, [[6],Corollary 9.13]. Therefore there exists a constant $C_{q,s}$ (depending on the functions q and s) such that

$$\|u\|_\infty \leq \ C_{q,s} \| u \| \quad , \forall u \in W^{2,2}(\mathbb{R}).$$

In the following proposition, we obtain an approximation for this constant.

Proposition 2.1. *Let $u \in W^{2,2}(\mathbb{R})$; then*

$$\|u\|_\infty \leq \; C_{q,s} \parallel u \parallel \tag{4}$$

where $C_{q,s} = \left(\frac{1}{4\|q\|_\infty \|s\|_\infty}\right)^{\frac{1}{4}} \left(\frac{\max\{1,\|q\|_\infty,\|s\|_\infty\}}{\min\{1,q_0,s_0\}}\right)^{\frac{1}{2}}$.

Proof. According to inequality (3) and also by substituting $\|q\|_\infty$ and $\|s\|_\infty$ with $-A$ and B respectively in Proposition 2.1 from [8], the desired result will be obtained.

□

Let us define $F(\xi) = \int_0^\xi f(t)dt$ for all $\xi \in \mathbb{R}$ and $\widetilde{K}(\eta) = \int_0^\eta K(t)dt$ for all $\eta > 0$. Moreover we introduce the functional $I_\lambda : W^{2,2}(\mathbb{R}) \to \mathbb{R}$ associated with (1)

$$I_\lambda = \Phi - \lambda\Psi,$$

for every $u \in W^{2,2}(\mathbb{R})$, where

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}} |u''(x)|^2 dx + \frac{1}{2}\widetilde{K}\left(\int_{\mathbb{R}} (q(x)|u'(x)|^2 + s(x)|u(x)|^2)dx\right)dx \tag{5}$$

and

$$\Psi(u) = \int_{\mathbb{R}} \alpha(x)F(u(x))dx \tag{6}$$

for every $u \in W^{2,2}(\mathbb{R})$.
It should be noted that the assumptions on α and f guarantee that the functional Ψ is well defined (see [3], P.17). Also, Ψ is a Gâteaux differentiable sequentially weakly upper semicontinuous functional, whose Gâteaux derivative is given by

$$\Psi'(u)(v) = \int_{\mathbb{R}} \alpha(x)f(u(x))v(x)dx,$$

for every $v \in W^{2,2}(\mathbb{R})$.

On the other hand, according to the inequality

$$\frac{\min\{1, m_0\}}{2}\|u\|^2 \leq \Phi(u) \leq \frac{\max\{1, m_1\}}{2}\|u\|^2 \tag{7}$$

Φ is coercive and convex. By standard arguments, one has that Φ is Gâteaux differentiable and sequentially weakly lower semi-continuous, and its Gâteaux derivative is the functional $\Phi'(u) \in (W^{2,2}(\mathbb{R}))^*$ given by

$$\Phi'(u)(v) = \lim_{\theta\to 0}\frac{\Phi(u+\theta\, v) - \Phi(u)}{\theta} = \frac{d}{d\theta}[\Phi(u+\theta\, v)]\Big|_{\theta=0} = \int_{\mathbb{R}} u''(x)v''(x)dx +$$
$$K\Big(\int_{\mathbb{R}} (q(x)|u'(x)|^2 + s(x)|u(x)|^2)dx\Big) \times \int_{\mathbb{R}} \big(q(x)u'(x)v'(x) + s(x)u(x)v(x)\big)dx$$

for every $v \in W^{2,2}(\mathbb{R})$.

Definition 2.1. *Fixing the real parameter λ, a function $u \in W^{2,2}(\mathbb{R})$ is said to be a weak solution of* (1) *if*

$$\int_{\mathbb{R}} u''(x)v''(x)dx + K\left(\int_{\mathbb{R}} (q(x)|u'(x)|^2 + s(x)|u(x)|^2)dx\right) \times$$
$$\int_{\mathbb{R}} \big(q(x)u'(x)v'(x) + s(x)u(x)v(x)\big)dx - \lambda\int_{\mathbb{R}} \alpha(x)f(u(x))v(x)dx = 0$$

for every $v \in W^{2,2}(\mathbb{R})$*.*

Hence, the critical points of I_λ are exactly the weak solutions of (1).

Lemma 2.1. *Suppose* $f : \mathbb{R} \to \mathbb{R}$ *is a non-negative continuous function. If* $u_0 \not\equiv 0$ *is a weak solution for problem* (1) *then* u_0 *is non-negative.*

Proof. Let $v(x) = \bar{u}_0(x) = \max\{-u_0(x), 0\}$ and we assume that $\Theta = \{x \in \mathbb{R} : u_0(x) \leq 0\}$. Then we have

$$\int_{\Theta\cup\Theta^c} u_0''(x)\bar{u}_0''(x)dx + K\left(\int_{\mathbb{R}}(q(x)|u_0'(x)|^2 + s(x)|u_0(x)|^2)dx\right) \times$$
$$\int_{\Theta\cup\Theta^c}\big(q(x)u_0'(x)\bar{u}_0'(x) + s(x)u_0(x)\bar{u}_0(x)\big)dx = \int_{\mathbb{R}} \lambda\alpha(x)f(u_0(x))\bar{u}_0(x)dx,$$

that is

$$\int_{\Theta}\big(-|u_0''(x)|^2\big)dx + m_0\int_{\Theta}\big(-q(x)|u_0'(x)|^2 - s(x)|u_0(x)|^2\big)dx \geq 0,$$

hence $\min\{1, m_0\}\|\bar{u}_0\| = 0$ which means that $u_0 \geq 0$ and the proof is complete. □

Our main tool to investigate the existence of infinitely many solutions for the problem (1) is the classical Ricceri's variational principle which we now recall.

Theorem 2.1 ([4], Theorem 2.1)**.** *Let* X *be a reflexive real Banach space, let* $\Phi, \Psi : X \to \mathbb{R}$ *be two Gâteaux differentiable functionals such that* Φ *is sequentially weakly lower semicontinuous, strongly continuous, and coercive and* Ψ *is sequentially weakly upper semicontinuous. For every* $r > \inf_X \Phi$*, let us put*

$$\varphi(r) := \inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\sup_{v\in\Phi^{-1}(]-\infty,r[)}\Psi(v) - \Psi(u)}{r - \Phi(u)}$$

and

$$\gamma := \liminf_{r\to+\infty}\varphi(r), \qquad \delta := \liminf_{r\to(\inf_X \Phi)^+}\varphi(r).$$

Then, one has

(a) *for every* $r > \inf_X \Phi$ *and every* $\lambda \in]0, \frac{1}{\varphi(r)}[$*, the restriction of the functional* $I_\lambda = \Phi - \lambda\Psi$ *to* $\Phi^{-1}(]-\infty, r[)$ *admits a global minimum, which is a critical point (local minimum) of* I_λ *in* X*.*

(b) *If* $\gamma < +\infty$ *then, for each* $\lambda \in]0, \frac{1}{\gamma}[$*, the following alternative holds: either*

(b_1) I_λ *possesses a global minimum,*

or

(b_2) *there is a sequence* $\{u_n\}$ *of critical points (local minima) of* I_λ *such that*

$$\lim_{n\to+\infty}\Phi(u_n) = +\infty.$$

(c) *If* $\delta < +\infty$ *then, for each* $\lambda \in]0, \frac{1}{\delta}[$*, the following alternative holds: either*

(c_1) *there is a global minimum of* Φ *which is a local minimum of* I_λ*,*

or

(c_2) *there is a sequence of pairwise distinct critical points (local minima) of* I_λ*, with* $\lim_{n\to+\infty}\Phi(u_n) = \inf_X \Phi$*, which weakly converges to a global minimum of* Φ*.*

3. Main results

Put

$$\tau := \frac{540\ \min\{1, m_0\}}{86111\,(\max\{1, m_1\})(\max\{1, \|q\|_\infty, \|s\|_\infty\})C_{q,s}{}^2}\,, \tag{8}$$

$$A := \|\alpha\|_1 \liminf_{\xi\to+\infty} \frac{F(\xi)}{\xi^2}, \tag{9}$$

and

$$B := \limsup_{\xi\to+\infty} \frac{F(\xi)\int_{\frac{3}{8}}^{\frac{5}{8}} \alpha(x)dx}{\xi^2}. \tag{10}$$

Qur main result is the following.

Theorem 3.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and non-negative function, and assume that*
(i) $F(t) \geq 0$ for every $t \in]0, \frac{3}{8}[\cup]\frac{5}{8}, 1[$ where $F(\xi) = \int_0^\xi f(t)dt$ for all $\xi \in \mathbb{R}$,
(ii) $A < \tau B$, where τ, A and B are given by (8) , (9) and (10) respectively .
Then for every

$$\lambda \in \left]\frac{86111}{1080}\frac{(\max\{1, m_1\})(\max\{1, \|q\|_\infty, \|s\|_\infty\})}{B}, \frac{\min\{1, m_0\}}{2A\, C_{q,s}{}^2}\right[$$

the problem (1) admits a sequence of positive weak solutions which is unbounded in $W^{2,2}(\mathbb{R})$.

Proof. Our aim is to apply Theorem 2.1, part (b) with $X = W^{2,2}(\mathbb{R})$. For this purpose fix λ as in our conclusion and Φ, Ψ are the functionals introduced in section 2. In the previous section, we showed that Φ and Ψ have the necessary conditions in Theorem 2.1. Our assumptions on f guarantee that $F \in C^1(\mathbb{R})$ and $F'(t) = f(t) \geq 0$ for all $t \in \mathbb{R}$, so F is non-decreasing. Hence for every $\xi > 0$ we have

$$\sup_{|t|\leq\xi} F(t) = F(\xi).$$

Now, we claim that there are many critical points for the functional I_λ in $W^{2,2}(\mathbb{R})$. For our goal, let $\{\xi_n\}$ be a sequence of positive numbers such that $\lim_{n\to\infty} \xi_n = +\infty$ and

$$\lim_{n\to\infty} \frac{\int_{\mathbb{R}} \alpha(x) \sup_{|t|\leq\xi_n} F(t)dx}{\xi_n^2} = \|\alpha\|_1 \lim_{n\to\infty} \frac{F(\xi_n)}{\xi_n^2} = A.$$

Set $r_n := \frac{\min\{1,m_0\}}{2}\left(\frac{\xi_n}{C_{q,s}}\right)^2$, for each $n \in \mathbb{N}$.

For every $u \in W^{2,2}(\mathbb{R})$ by using (4) we have

$$\Phi^{-1}(]-\infty, r_n[) = \{u \in W^{2,2}(\mathbb{R});\ \Phi(u) < r_n\} \subseteq$$
$$\left\{u \in W^{2,2}(\mathbb{R});\ \frac{\min\{1, m_0\}}{2}\|u\|^2 < \frac{\min\{1, m_0\}}{2}\left(\frac{\xi_n}{C_{q,s}}\right)^2\right\} =$$
$$\{u \in X;\ C_{q,s}\|u\| < \xi_n\} \subseteq \left\{u \in W^{2,2}(\mathbb{R});\ \ \|u\|_\infty \leq \xi_n\right\}.$$

Now, since $0 \in \Phi^{-1}(]-\infty, r_n[)$ then the following inequalities are obtained:

$$\varphi(r_n) = \inf_{u\in\Phi^{-1}(]-\infty,r_n[)} \frac{\sup_{v\in\Phi^{-1}(]-\infty,r_n[)} \Psi(v) - \Psi(u)}{r_n - \Phi(u)} \leq \frac{\sup_{v\in\Phi^{-1}(]-\infty,r_n[)} \Psi(v)}{r_n}$$

$$\leq \frac{\int_{\mathbb{R}} \alpha(x) \sup_{|t|\leq \xi_n} F(t)dx}{r_n} = \frac{\|\alpha\|_1 F(\xi_n)}{\frac{\min\{1,m_0\}}{2}\left(\frac{\xi_n}{C_{q,s}}\right)^2}$$

$$= \frac{2\,{C_{q,s}}^2}{\min\{1,m_0\}}\,\|\alpha\|_1\,\frac{F(\xi_n)}{{\xi_n}^2},$$

for every $n \in \mathbb{N}$. Hence, it follows that

$$\gamma \leq \liminf_{n\to\infty} \Phi(r_n) \leq \frac{2\,{C_{q,s}}^2}{\min\{1,m_0\}} A < +\infty,$$

because condition (ii) shows $A < +\infty$. Now, we will prove that the functional I_λ is unbounded from below. to this end, let $\{d_n\}$ be a sequence of positive numbers such that $\lim_{n\to\infty} d_n = +\infty$ and

$$\lim_{n\to+\infty} \frac{F(d_n)\int_{\frac{3}{8}}^{\frac{5}{8}} \alpha(x)dx}{{d_n}^2} = B. \tag{11}$$

Let $\{v_n\}$ be a sequence in $W^{2,2}(\mathbb{R})$ which is defined by

$$v_n(x) := \begin{cases} -\frac{64\,d_n}{9}(x^2 - \frac{3}{4}x) & \text{if } x \in [0, \frac{3}{8}], \\ d_n & \text{if } x \in]\frac{3}{8}, \frac{5}{8}], \\ -\frac{64\,d_n}{9}(x^2 - \frac{5}{4}x + \frac{1}{4}) & \text{if } x \in]\frac{5}{8}, 1], \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

One can compute that

$$\| v_n \|_{W^{2,2}(\mathbb{R})}{}^2 = \frac{86111}{540}\,{d_n}^2,$$

and the following can be inferred from (3) and (7)

$$(\min\{1,m_0\})(\min\{1,q_0,s_0\})\frac{86111}{1080}\,{d_n}^2 \leq \Phi(v_n) \leq$$

$$(\max\{1,m_1\})(\max\{1,\|q\|_\infty,\|s\|_\infty\})\frac{86111}{1080}\,{d_n}^2. \tag{13}$$

Also, by using condition (i), we infer

$$\int_{\mathbb{R}} \alpha(x)F(v_n(x))dx \geq \int_{\frac{3}{8}}^{\frac{5}{8}} \alpha(x)F(d_n)dx = F(d_n)\int_{\frac{3}{8}}^{\frac{5}{8}} \alpha(x)dx,$$

for every $n \in \mathbb{N}$. Therefore, we have

$$I_\lambda(v_n) = \Phi(v_n) - \lambda\,\Psi(v_n)$$

$$\leq (\max\{1,m_1\})(\max\{1,\|q\|_\infty,\|s\|_\infty\})\frac{86111}{1080}\,{d_n}^2 - \lambda F(d_n)\int_{\frac{3}{8}}^{\frac{5}{8}} dx,$$

for every $n \in \mathbb{N}$. If $B < +\infty$, let

$$\epsilon \in \left]\frac{(\max\{1,m_1\})(\max\{1,\|q\|_\infty,\|s\|_\infty\})}{\lambda B}\frac{86111}{1080}, 1\right[.$$

By (11) there is N_ϵ such that

$$F(d_n)\int_{\frac{3}{8}}^{\frac{5}{8}} \alpha(x)dx > \epsilon\,B\,{d_n}^2, \qquad (\forall n > N_\epsilon).$$

Consequently, one has

$$\begin{aligned} I_\lambda(v_n) &\le (\max\{1,m_1\})(\max\{1,\|q\|_\infty,\|s\|_\infty\})\frac{86111}{1080}\,{d_n}^2 - \lambda\epsilon B{d_n}^2 \\ &= {d_n}^2\left((\max\{1,m_1\})(\max\{1,\|q\|_\infty,\|s\|_\infty\})\frac{86111}{1080} - \lambda\epsilon B\right), \end{aligned}$$

for every $n > N_\epsilon$. Thus, it follows that

$$\lim_{n\to\infty} I_\lambda(v_n) = -\infty.$$

If $B = +\infty$, then consider

$$M > \frac{(\max\{1,m_1\})(\max\{1,\|q\|_\infty,\|s\|_\infty\})}{\lambda}\frac{86111}{1080}.$$

By (11) there is $N(M)$ such that

$$F(d_n)\int_{\frac{3}{8}}^{\frac{5}{8}} \alpha(x)dx > M{d_n}^2, \qquad (\forall n > N(M)).$$

So, we have

$$\begin{aligned} I_\lambda(v_n) &\le (\max\{1,m_1\})(\max\{1,\|q\|_\infty,\|s\|_\infty\})\frac{86111}{1080}\,{d_n}^2 - \lambda M{d_n}^2 \\ &= {d_n}^2\left((\max\{1,m_1\})(\max\{1,\|q\|_\infty,\|s\|_\infty\})\frac{86111}{1080} - \lambda M\right), \end{aligned}$$

for every $n > N(M)$. Taking into account the choice of M, also in this case, one has

$$\lim_{n\to\infty} I_\lambda(v_n) = -\infty.$$

Therefore according to Theorem 2.1, the functional I_λ admits an unbounded sequence $\{u_n\} \subset W^{2,2}(\mathbb{R})$ of critical points. Finally, since the weak solutions of the problem (1) are exactly the solutions of the equation $I'_\lambda(u) = 0$, Theorem 2.1 and Lemma 2.1 guarantee the conclusion.

□

Remark 3.1. *We note that assumption (ii) in Theorem 3.1 could be replaced by the following more general hypotheses:*

(iii) there exists two sequences $\{a_n\}$ and $\{b_n\}$ such that

$$0 \le a_n < \frac{1}{C_{q,s}}\Big(\frac{540\,\min\{1,m_0\}}{86111\,(\max\{1,m_1\})(\max\{1,\|q\|_\infty,\|s\|_\infty\})}\Big)^{\frac{1}{2}} b_n$$

for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} b_n = +\infty$ such that

$$\lim_{n\to\infty} \frac{F(b_n)\|\alpha\|_1 - F(a_n)\int_{\frac{3}{8}}^{\frac{5}{8}} \alpha(x)dx}{\frac{\min\{1,m_0\}}{2}\Big(\frac{b_n}{C_{q,s}}\Big)^2 - (\max\{1,m_1\})(\max\{1,\|q\|_\infty,\|s\|_\infty\})\frac{86111}{1080}\,{a_n}^2} <$$

$$\frac{1080}{86111\,(\max\{1,m_1\})(\max\{1,\|q\|_\infty,\|s\|_\infty\})}\limsup_{\xi\to+\infty}\frac{F(\xi)\int_{\frac{3}{8}}^{\frac{5}{8}} \alpha(x)dx}{\xi^2}.$$

It is clear that from (iii) by choosing $a_n = 0$ *for all* $n \in \mathbb{N}$, *(ii) will be obtained. Let* $r_n := \frac{\min\{1,m_0\}}{2}\left(\frac{b_n}{C_{q,s}}\right)^2$ *for every* $n \in \mathbb{N}$. *Now we have*

$$\varphi(r_n) \le \frac{\sup_{v\in\Phi^{-1}(]-\infty,r_n[)} \int_{\mathbb{R}} \alpha(x)F(v(x))dx - \int_{\mathbb{R}} \alpha(x)F(v_n(x))dx}{r_n - \Phi(v_n)}$$

$$\le \frac{F(b_n)\|\alpha\|_1 - F(a_n)\int_{\frac{3}{8}}^{\frac{5}{8}} \alpha(x)dx}{\frac{\min\{1,m_0\}}{2}\left(\frac{b_n}{C_{q,s}}\right)^2 - (\max\{1, m_1\})(\max\{1, \|q\|_\infty, \|s\|_\infty\})\frac{86111}{1080}\, {a_n}^2}$$

where

$$v_n(x) := \begin{cases} -\dfrac{64\, a_n}{9}(x^2 - \dfrac{3}{4}x) & \text{if } x \in [0, \frac{3}{8}], \\ a_n & \text{if } x \in]\frac{3}{8}, \frac{5}{8}], \\ -\frac{64\, a_n}{9}(x^2 - \frac{5}{4}x + \frac{1}{4}) & \text{if } x \in]\frac{5}{8}, 1], \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

The continuation of our argument is similar to the process of proving Theorem 3.1 and so for every

$$\lambda \in \left] \frac{86111}{1080} \frac{(\max\{1, m_1\})(\max\{1, \|q\|_\infty, \|s\|_\infty\})}{\limsup_{\xi\to+\infty} \frac{F(\xi)\int_{\frac{3}{8}}^{\frac{5}{8}} \alpha(x)dx}{\xi^2}}, \right.$$

$$\left. \frac{1}{\frac{F(b_n)\|\alpha\|_1 - F(a_n)\int_{\frac{3}{8}}^{\frac{5}{8}} \alpha(x)dx}{\frac{\min\{1,m_0\}}{2}\left(\frac{b_n}{C_{q,s}}\right)^2 - (\max\{1, m_1\})(\max\{1, \|q\|_\infty, \|s\|_\infty\})\frac{86111}{1080}\, {a_n}^2}} \right[$$

the problem (1) *admits a sequence of positive weak solutions which is unbounded in* $W^{2,2}(\mathbb{R})$.

Please note the following example to illustrate Theorem 3.1.

Example 3.1. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be the function defined as*

$$f(t) = \begin{cases} 0 & \text{if } t \in \mathbb{R} \setminus \bigcup_{n\in\mathbb{N}} [n!n, (n+1)!] \\ \\ 4[(n+1)^2 - 1] \min\{t - n!n, (n+1)! - t\} & \text{if } t \in [n!n, (n+1)!], \end{cases}$$

and also put $a_n = n!$ *and* $b_n = n!n$ *for each* $n \in \mathbb{N}$. *Then we have (see* [5]*)*

$$\liminf_{\xi\to+\infty} \frac{F(\xi)}{\xi^2} = 0$$

and

$$\limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^2} = 1.$$

Let $\alpha : \mathbb{R} \to \mathbb{R}$ *be the function defined as follows*

$$\alpha(x) = \begin{cases} 80 & \textit{if } x \in [\frac{3}{8}, \frac{5}{8}] \\ \\ 0 & \textit{otherwise}, \end{cases}$$

and so one has

$$A = \|\alpha\|_1 \liminf_{\xi \to +\infty} \frac{F(\xi)}{\xi^2} = 0$$

and

$$B = \limsup_{\xi \to +\infty} \frac{F(\xi) \int_{\frac{3}{8}}^{\frac{5}{8}} \alpha(x)dx}{\xi^2} = 20.$$

Therefore, the condition (ii) of Theorem 3.1 is satisfied and so according to the Theorem 3.1 for any

$$\lambda > \frac{86111}{1080} \frac{(\max\{1, m_1\})(\max\{1, \|q\|_\infty, \|s\|_\infty\})}{B} \approx 12$$

the problem

$$u^{iv}(x) + (2 + \tanh(\int_{\mathbb{R}} (|u'(x)|^2 + |u(x)|^2)dx))(-u''(x) + u(x)) = \lambda\, \alpha(x) f(u(x))$$

admits a sequence of positive weak solutions which is unbounded in $W^{2,2}(\mathbb{R})$.

Remark 3.2. *In Theorem 3.1, the conclusion (c) can be used instead of (b). The deformation of A and B by replacing $\xi \to +\infty$ with $\xi \to 0^+$, will be as follows:*

$$A = \|\alpha\|_1 \liminf_{\xi \to 0^+} \frac{F(\xi)}{\xi^2}$$

and

$$B = \limsup_{\xi \to 0^+} \frac{F(\xi) \int_{\frac{3}{8}}^{\frac{5}{8}} \alpha(x)dx}{\xi^2}.$$

In this case, will be achieved a sequence of pairwise distinct positive weak solutions to the problem (1) which converges uniformly to zero.

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REFERENCES

[1] *R. A Adams*, Sobolev spaces, Academic Press, 1975.

[2] *A. Arosio*, A geometrical nonlinear correction to the Timoshenko beam equation, Nonlinear Anal., **47** (2001) 729-740.

[3] *G. Bonanno, G. Barletta, D. O'Regan*, A variational approach to multiplicity results for boundary-value problems on the real line, Proceedings of the Royal Society of Edinburgh., **145** (2015) 13-29.

[4] *G. Bonanno and G. Molica Bisci* , Infinitely many solutions for a boundary value problem with discontinuous nonlinearities, Bound. Value Probl. 2009, Article ID 670675 (2009).

[5] *G. Bonanno, B. Di Bella*, Infinitely many solutions for a fourth-order elastic beam equations, Nonl. Differ. Equ. Appl., **18** (2011) 357-368.

[6] *H. Brezis*, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer, 2011.

[7] *X. He, W. Zou*, Infinitely many positive solutions for Kirchhoff-type problems, Nonlinear Anal., **70** (2009) 1407-1414.

[8] *M.R.Heidari Tavani*, Existence results for fourth-order elastic beam equations on the real line, Dynamic Systems and Applications., **27** (2018) 149-163.

[9] M.R.Heidari Tavani, G. Afrouzi, S. Heidarkhani, Multiplicity results for perturbed fourth-order Kirchhoff-type problems, Opuscula Mathematica. 37 (2017) 755-772.

[10] *Kirchhoff*, Vorlesungen uber mathematische Physik: Mechanik, Teubner, Leipzig (1883).

[11] *X.L. Liu,W.T. Li*, Existence and multiplicity of solutions for fourth-order boundary values problems with parameters, J.Math. Anal. Appl., **327** (2007) 362-375.

[12] *T.F. Ma*, Positive solutions for a nonlinear Kirchhoff type beam equation, Applied Mathematics Letters., **18** (2005) 479-482

[13] *X . Wang*, Infinitely many solutions for a fourth-order differential equation on a nonlinear elastic foundation. Bound. Value Probl,. 2013, **258** (2013).

[14] *F. Wang, M. Avci, Y. An*, Existence of solutions for fourth order elliptic equations of Kirchhoff-type, J. Math. Anal. Appl., **409** (2014) 140-146.

[15] *S. Woinowsky-Krieger*, The effect of axial force on the vibration of hinged bars, J. Appl. Mech., **17** (1950), 35-36.