COHOMOLOGICAL PROPERTIES AND ARENS REGULARITY OF BANACH ALGEBRAS

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In this paper, we study some cohomological properties of Banach algebras. For a Banach algebra A and a Banach A-bimodule B, we investigate the vanishing of the first Hochschild cohomology groups $H^1(A^n, B^m)$ and $H^1_{w^*}(A^n, B^m)$, where $0 \leq m, n \leq 3$. For amenable Banach algebra A, we show that there are Banach A-bimodules C, D and elements $\mathfrak{a}, \mathfrak{b} \in A^{**}$ such that

$$Z^1(A, C^*) = \{R_{D''(\mathfrak{a})}: D \in Z^1(A, C^*)\} = \{L_{D''(\mathfrak{b})}: D \in Z^1(A, D^*)\}.$$

where, for every $b \in B$, $L_b(a) = ba$ and $R_b(a) = ab$, for every $a \in A$. Moreover, under a condition, we show that if the second transpose of a continuous derivation from the Banach algebra A into A^* i.e., a continuous linear map from A^{**} into A^{***}, is a derivation, then A is Arens regular. Finally, we show that if A is a dual left strongly irregular Banach algebra such that its second dual is amenable, then A is reflexive.

Keywords: Arens regularity, topological centers, cohomological group, weakly amenable, Connes-amenability.

1. Introduction

A *derivation* from a Banach algebra A into a Banach A-bimodule B is a bounded linear mapping $D: A \longrightarrow B$ such that

$$D(ab) = aD(b) + D(a)b \qquad \text{for all} \qquad a, b \in A.$$

The space of continuous derivations from A into B is denoted by $Z^1(A, B)$. The easiest example of derivations is the *inner derivations*, which are given for each $b \in B$ by

$$\delta_b(a) = ab - ba \qquad \text{for all} \qquad a \in A.$$

The space of inner derivations from A into B is denoted by $B^1(A, B)$. The Banach algebra A is said to be *amenable*, if for every Banach A-bimodule B, all derivations from A into B^* are inner derivations, in the other word, $H^1(A, B^*) = Z^1(A, B^*)/B^1(A, B^*) = \{0\}$ and A is said to be *weakly amenable* if $H^1(A, A^*) = \{0\}$.

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The concept of amenability for a Banach algebra A, introduced by Johnson in 1972, see [15]. For a Banach A-bimodule B, the quotient space $H^1(A,B)$ of all continuous derivations from A into B modulo the subspace of inner derivations is called the first cohomology group of A with coefficients in B. Following [25] the Banach algebra A is called *super-amenable* if $H^1(A,B)=\{0\}$ for every Banach A-bimodule B (super-amenable Banach algebras are called contractible, too). It is clear that if A is super-amenable, then A is amenable.

In [17], Johnson, Kadison, and Ringrose introduced the notion of amenability for von Neumann algebras. The basic concepts, however, make sense for arbitrary dual Banach algebras. But is most commonly associated with Connes, see [4]. For this reason, this notion of amenability is called *Connes-amenability* (the origin of this name seems to be Helemskii, see [18]).

Let A be a Banach algebra. A Banach A-bimodule X is called dual if there is a closed submodule X_* of X^* such that $X=(X_*)^*$ (X_* is called the *predual* of X). A Banach algebra A is called dual if it is dual as a Banach A-bimodule.

Let A be a dual Banach algebra. A dual Banach A-bimodule X is called *normal* if, for every $x\in X$, the maps

$$A\longrightarrow X,\quad a\mapsto\begin{cases}a\cdot x\\ x\cdot a\end{cases}$$

are weak *-continuous (w^*-continuous). The dual Banach algebra A is called Connes-amenable if, for every dual Banach A-bimodule X, every w^*-continuous derivation $D\colon A\longrightarrow X$ is inner; or equivalently, $H^1_{w^*}(A,X)=\{0\}$ [25].

The second dual A^{**} of Banach algebra A endowed with the either *Arens multiplications* is a Banach algebra. The constructions of the two Arens multiplications in A^{**} lead us to the definition of *topological centers* for A^{**} with respect to both Arens multiplications. The topological centers of Banach algebras, module actions and applications of them were introduced and discussed in [1, 19, 21]. To state our results, we need to fix some notations and recall some definitions.

Assume that T is an operator from normed linear space X into normed linear space Y. T^* is the adjoint of T from Y^* into X^*. We say that T^* is $weak^*-weak^*$ continuous, if for each $\{y'_\alpha\}\subseteq Y^*$, $y'_\alpha\xrightarrow{w^*}y'$ in Y^* implies $T^*y'_\alpha\xrightarrow{w^*}T^*y'$ in X^*.

Let X,Y,Z be normed linear spaces and $m\colon X\times Y\to Z$ be a bounded bilinear mapping. Arens in [1] offers two natural extensions m^{***} and m^{t***t} of m from $X^{**}\times Y^{**}$ into Z^{**}, for more information see [9, 19, 21].

The mapping m^{***} is the unique extension of m such that $x''\to m^{***}(x'',y'')$ from X^{**} into Z^{**} is $weak^*-weak^*$ continuous for every $y''\in Y^{**}$, but the mapping $y''\to m^{***}(x'',y'')$ is not in general $weak^*-weak^*$ continuous from Y^{**} into Z^{**} unless $x''\in X$. Hence *the first topological center* of m may be defined as follows

$$Z_1(m) = \{x'' \in X^{**}:\ y'' \to m^{***}(x'', y'') \text{ isweak}^* - \text{weak}^* \text{ continuous}\}.$$
Now, let $m^t: Y \times X \to Z$ be the transpose of m defined by $m^t(y,x) = m(x,y)$ for every $x \in X$ and $y \in Y$. Then m^t is a continuous bilinear map from $Y \times X$ to Z, and so it may be extended as above to $m^{t***}: Y^{**} \times X^{**} \to Z^{**}$. The mapping $m^{t***t}: X^{**} \times Y^{**} \to Z^{**}$ in general is not equal to m^{***}, see [1]. If $m^{***} = m^{t***t}$, then m is called Arens regular. The mapping $y'' \to m^{t***t}(x'', y'')$ is $weak^* - weak^*$ continuous for every $x'' \in X^{**}$, but the mapping $x'' \to m^{t***t}(x'', y'')$ from X^{**} into Z^{**} is not in general weak *-weak * continuous for every $y'' \in Y^{**}$. So we define *the second topological center* of m as
$$Z_2(m) = \{y'' \in Y^{**}:\ x'' \to m^{t***t}(x'', y'') \text{ isweak}^* - \text{weak}^* \text{ continuous}\}.$$
It is clear that m is *Arens regular* if and only if $Z_1(m) = X^{**}$ or $Z_2(m) = Y^{**}$. Arens regularity of m is equivalent to the following
$$\lim_i \lim_j \langle z', m(x_i, y_j)\rangle = \lim_j \lim_i \langle z', m(x_i, y_j)\rangle,$$
whenever both limits exist for all bounded sequences $(x_i)_i \subseteq X$, $(y_i)_i \subseteq Y$ and $z' \in Z^*$, see [7].

The mapping m is left strongly Arens irregular if $Z_1(m) = X$ and m is right strongly Arens irregular if $Z_2(m) = Y$. *The first Arens product* is defined as follows in three steps. For a, b in A, f in A^* and m, n in A^{**}, the elements $f.a$, $m.f$ of A^* and $m.n$ of A^{**} are defined as follows:
$$\langle f.a, b\rangle = \langle f, ab\rangle, \quad \langle m.f, a\rangle = \langle m, f.a\rangle, \quad \langle m.n, f\rangle = \langle m, n.f\rangle.$$
The second Arens product is defined as follows. For a, b in A, f in A^* and m, n in A^{**}, the elements $a \diamond f$, $f \diamond m$ of A^* and $m \diamond n$ of A^{**} are defined by the equalities
$$\langle a \diamond f, b\rangle = \langle f, ba\rangle, \quad \langle f \diamond m, a\rangle = \langle m, a \diamond f\rangle, \quad \langle m \diamond n, f\rangle = \langle n, f \diamond m\rangle.$$
The Arens regularity of a normed algebra A is defined to be the Arens regularity of its algebra multiplication when considered as a bilinear mapping $m: A \times A \to A$. Let B be a Banach A-bimodule, and let
$$\pi_\ell: A \times B \longrightarrow B \quad \text{and} \quad \pi_r: B \times A \longrightarrow B,$$
be the right and left module actions of A on B. By above notation, the transpose of π_r denoted by $\pi_r^t: A \times B \to B$. Then
$$\pi_\ell^*: B^* \times A \longrightarrow B^* \quad \text{and} \quad \pi_r^{t*t}: A \times B^* \longrightarrow B^*.$$
Thus B^* is a left Banach A-module and a right Banach A-module with respect to the module actions π_r^{t*t} and π_ℓ^*, respectively. The second dual B^{**} is a Banach A^{**}-bimodule with the following module actions
$$\pi_\ell^{***}: A^{**} \times B^{**} \longrightarrow B^{**} \quad \text{and} \quad \pi_r^{***}: B^{**} \times A^{**} \longrightarrow B^{**},$$
where A^{**} is considered as a Banach algebra with respect to the first Arens product. Similarly, B^{**} is a Banach A^{**}-bimodule with the module actions
$$\pi_\ell^{t***t}: A^{**} \times B^{**} \longrightarrow B^{**} \quad \text{and} \quad \pi_r^{t***t}: B^{**} \times A^{**} \longrightarrow B^{**},$$
where A^{**} is considered as a Banach algebra with respect to the second Arens product. In this way we write $Z(\pi_\ell) = Z_{B^{**}}(A^{**})$ and $Z(\pi_r) = Z_{A^{**}}(B^{**})$.

Let B be a Banach A-bimodule. Then we say that B factors on the left (right) with respect to A, if $B = BA$ ($B = AB$). Thus B factors on both sides, if $B = BA = AB$.

2. Weak *-weak * continuous derivations

Let B be a Banach A-bimodule. In this section, we study the cohomological properties of Banach algebra A whenever every derivation in $Z^1(A^{**}, B^*)$ is weak *-weak * continuous.

Theorem 2.1. *Let B be a Banach A-bimodule and let every derivation $D: A^{**} \longrightarrow B^*$ is weak *-weak * continuous. If $Z^{\ell}_{B^{**}}(A^{**}) = A^{**}$ and $H^1(A, B^*) = \{0\}$, then $H^1(A^{**}, B^*) = \{0\}$.*

Proof. Let $D: A^{**} \longrightarrow B^*$ be a derivation. Then $D|_A: A \to B^*$ is a derivation. Since $H^1(A, B^*) = \{0\}$, there exists $b' \in B^*$ such that $D|_A = \delta_{b'}$. Suppose that $a'' \in A^{**}$ and $(a_\alpha)_\alpha \subseteq A$ such that $a_\alpha \overset{w^*}{\longrightarrow} a''$ in A^{**}. Then

$$\begin{aligned} D(a'') &= w^* - \lim_\alpha D|_A(a_\alpha) \\ &= w^* - \lim_\alpha \delta_{b'}(a_\alpha) \\ &= w^* - \lim_\alpha (a_\alpha b' - b' a_\alpha) \\ &= a'' b' - b' a''. \end{aligned}$$

We now show that $b' a'' \in B^*$. Assume that $(b''_\beta)_\beta \in B^{**}$ such that $b'' = w^* - \lim_\beta b''_\beta$. Since $Z^{\ell}_{B^{**}}(A^{**}) = A^{**}$, we have

$$\langle b'a'', b''_\beta \rangle = \langle a''.b''_\beta, b' \rangle \to \langle a''.b'', b' \rangle = \langle b'a'', b'' \rangle.$$

Thus, $b'a'' \in (B^{**}, weak^*)^* = B^*$, and so $H^1(A^{**}, B^*) = \{0\}$.

Corollary 2.1. *Let A be an Arens regular Banach algebra and let every derivation $D: A^{**} \to A^*$ is weak *-weak * continuous. If A is weakly amenable, then $H^1(A^{**}, A^*) = \{0\}$.*

By the following result, we show that weak amenability of the Banach algebra A is essential in vanishing of $H^1(A^{**}, A^*)$.

Proposition 2.1. *Let A be a Banach algebra such that is an ideal in A^{**}. If A is not weakly amenable, then $H^1(A^{**}, A^*) \neq \{0\}$.*

Proof. Let $d: A \longrightarrow A^*$ be a derivation and $\pi: A^{**} \longrightarrow A$ be a bounded homomorphism. Now; define $D := d \circ \pi: A^{**} \longrightarrow A^*$. Clearly, D is a bounded derivation which it is not inner. This shows that $H^1(A^{**}, A^*) \neq \{0\}$.

Example 2.1. (i) *Let K be a compact metric space, d be a metric on K and $\alpha \in$* $(0,1]$. *The Lipschitz algebra $Lip_\alpha K$ is the space of complex-valued functions f on K such that*

$$p_\alpha(f) = sup\left\{\frac{|f(x) - f(y)|}{d(x,y)^\alpha} : x, y \in K, x \neq y\right\}$$

is finite. A subspace of $Lip_\alpha K$ *that contains* $f \in Lip_\alpha K$ *such that*

$$\frac{|f(x) - f(y)|}{d(x,y)^\alpha} \to 0 \quad as \quad d(x,y) \to 0$$

is denoted by $lip_\alpha K$. *Let* $\alpha \in (0, \frac{1}{2})$. *Then by [6, Theorem 4.4.34] or [2, Theorem 3.8],* $lip_\alpha K$ *is Arens regular and by [2, Theorem 3.10] it is weakly amenable. Then by Corollary 2,* $H^1((lip_\alpha K)^{**}, (lip_\alpha K)^*) = \{0\}$.

(ii) *Let* ω *be a weight sequence on* $\mathbb{Z}$ *such that*

$$sup\left\{\frac{\omega(m+n)}{\omega(m)\omega(n)}\left(\frac{1+|n|}{1+|m+n|}\right) : m, n \in \mathbb{Z}\right\}$$

is finite. The Beurling algebra $\ell^1(\mathbb{Z}, \omega)$ *is not weakly amenable [2, Theorem 2.3]. Then by Proposition 2, we have*

$$H^1(\ell^1(\mathbb{Z}, \omega)^{**}, \ell^\infty(\mathbb{Z}, \omega)) \neq \{0\}.$$

Let B be a dual Banach algebra, with predual X and suppose that

$$X^\perp = \{x''' : x'''|_X = 0 \text{ where } x''' \in X^{***}\} = \{b'' : b''|_X = 0 \text{ where } b'' \in B^{**}\}.$$

Then the canonical projection $P: X^{***} \longrightarrow X^*$ gives a continuous linear map $P: B^{**} \longrightarrow B$. Thus, we can write the following equality

$$B^{**} = X^{***} = X^* \oplus \ker P = B \oplus X^\perp,$$

as a direct sum of Banach A-bimodules.

Theorem 2.2. *Let* B *be a Banach* A*-bimodule such that every derivation from* A^{**} *into* B *is weak* **-weak continuous and* $A^{**}B, BA^{**} \subseteq B$.

(i) *If* $H^1(A,B) = 0$, *then* $H^1(A^{**}, B) = \{0\}$.

(ii) *Suppose that* A *has a left bounded approximate identity (=LBAI), has a predual* X *and* $AB^*, B^*A \subseteq X$. *If* $H^1(A,B) = 0$, *then*

$$H^1(A^{**}, B^{**}) = \{0\}.$$

Proof. (i) Proof is similar to the proof of Theorem 2.

(ii) Set $B^{**} = B \oplus X^\perp$. Then we have

$$H^1(A^{**}, B^{**}) = H^1(A^{**}, B) \oplus H^1(A^{**}, X^\perp).$$

Since $H^1(A,B) = \{0\}$, by (i), $H^1(A^{**}, B) = \{0\}$. Now let $\widetilde{D} \in Z^1(A^{**}, X^\perp)$ and we take $D = \widetilde{D}|_A$. It is clear that $D \in Z^1(A^{**}, X^\perp)$. Assume that $a'', x'' \in A^{**}$ and $(a_\alpha)_\alpha, (x_\beta)_\beta \subseteq A$ such that $a_\alpha \xrightarrow{w^*} a''$ and $x_\beta \xrightarrow{w^*} x''$ on A^{**}. Since $AB^*, B^*A \subseteq X$, for every $b' \in B^*$, by using the weak *-weak continuity of $\widetilde{D}$, we have

$$\langle \widetilde{D}(a'' \diamond x''), b' \rangle = \lim_\beta \lim_\alpha \langle D(a_\alpha x_\beta), b' \rangle$$

$$= \lim_\beta \lim_\alpha \langle (D(a_\alpha) x_\beta + a_\alpha D(x_\beta)), b' \rangle$$

$$= \lim_{\beta}\lim_{\alpha}\langle D(a_\alpha)x_\beta, b'\rangle + \lim_{\beta}\lim_{\alpha}\langle a_\alpha D(x_\beta), b'\rangle$$
$$= \lim_{\beta}\lim_{\alpha}\langle D(a_\alpha), x_\beta b'\rangle + \lim_{\beta}\lim_{\alpha}\langle D(x_\beta)), b' a_\alpha\rangle$$
$$= 0.$$

Since A has a LBAI, A^{**} has a left unit e'' with respect to the second Arens product [6, Proposition 2.9.16]. Then $D(x'') = D(e'' \diamond x'') = 0$, and so $D = 0$.

Example 2.2. (i) *Assume that G is a compact group. Then we know that $L^1(G)$ is $M(G)$-bimodule and $L^1(G)$ is an ideal in the second dual of $M(G)$, $M(G)^{**}$. By [20, Corollary 1.2], we have $H^1(L^1(G), M(G)) = \{0\}$. Then by Theorem 2, every weak *-weak continuous derivation from $L^1(G)^{**}$ into $M(G)$ is inner.*

(ii) *We know that c_0 is a C^*-algebra and every C^*-algebra is weakly amenable, so c_0 is weakly amenable. Then by Theorem 2, every weak *- weak continuous derivation from ℓ^∞ into ℓ^1 is inner.*

Theorem 2.3. *Let B be a Banach A-bimodule and A has a $LBAI$. Suppose that $AB^{**}, B^{**}A \subseteq B$ and every derivation from A^{**} into B^* is weak *-weak * continuous. If $H^1(A, B^*) = \{0\}$, then $H^1(A^{**}, B^{***}) = \{0\}$.*

Proof. Take $B^{***} = B^* \oplus B^\perp$, where $B^\perp = \{b''' \in B^{***}: b'''|_B = 0\}$. Then we have

$$H^1(A^{**}, B^{***}) = H^1(A^{**}, B^*) \oplus H^1(A^{**}, B^\perp).$$

Since $H^1(A, B^*) = \{0\}$, similar to Theorem 2(i), we have $H^1(A^{**}, B^*) = \{0\}$. It suffices to show that $H^1(A^{**}, B^\perp) = 0$. Let $(e_\alpha)_\alpha \subseteq A$ be a LBAI for A such that $e_\alpha \overset{w^*}{\rightarrow} e''$ in A^{**} where e'' is a left unit for A^{**} with respect to the second Arens product. Let $a'' \in A^{**}$ and suppose that $(a_\beta)_\beta \subseteq A$ such that $a_\beta \overset{w^*}{\rightarrow} a''$ in A^{**}. Let $D \in Z^1(A^{**}, B^\perp)$. Then for every $b'' \in B^{**}$, by weak *-weak * continuity of D, we have

$$\langle D(a''), b''\rangle = \langle D(e'' \diamond a''), b''\rangle$$
$$= \lim_{\beta}\lim_{\alpha}\langle (D(e_\alpha a_\beta), b''\rangle$$
$$= \lim_{\beta}\lim_{\alpha}\langle (D(e_\alpha)a_\beta + e_\alpha D(a_\beta)), b''\rangle$$
$$= \lim_{\beta}\lim_{\alpha}\langle D(e_\alpha)a_\beta, b''\rangle + \lim_{\beta}\lim_{\alpha}\langle e_\alpha D(a_\beta), b''\rangle$$
$$= \lim_{\beta}\lim_{\alpha}\langle D(e_\alpha), a_\beta b''\rangle + \lim_{\beta}\lim_{\alpha}\langle D(a_\beta), b'' e_\alpha\rangle$$
$$= 0.$$

It follows that $D = 0$, and so the result holds.

It is known that neither the weak amenability of A implies that of A^{**}, nor the weak amenability of A^{**} implies that of A. The question "*when the weak* *amenability of* A^{**} *implies that of* A*?*" is investigated in many works; see [3, 7, 10, 11, 12] for more details. We now by Theorem 2 consider the converse of the above question, i.e., "*under which conditions the weak amenability of* A *implies that of* A^{**}*?*", as follows:

Corollary 2.2. *Assume that* A *is a Banach algebra with* $LBAI$ *such that it is two-sided ideal in* A^{**} *and every derivation* $D: A^{**} \rightarrow A^{***}$ *is weak* * *- weak* * *continuous. If* A *is weakly amenable, then* A^{**} *is weakly amenable.*

Example 2.3. *Assume that* G *is a locally compact group. We know that* $L^1(G)$ *is weakly amenable Banach algebra, see [16]. Then by Corollary 2, every weak* **-weak* * *continuous derivation from* $L^1(G)^{**}$ *into* $L^1(G)^{***}$ *is inner.*

Theorem 2.4. *Let* A *be an amenable and Arens regular Banach algebra. If for any normal Banach* A*-bimodule* B *with predual* X*, we have* $AB^*, B^*A \subseteq X$*, then* $H^1_{w^*}(A^{**}, B^{**}) = \{0\}$.

Proof. If the Banach algebra A is amenable and Arens regular, then A^{**} is Connes-amenable and the converse holds whenever A is an ideal in A^{**}, too [25, Theorem 4.4.8]. Thus $H^1_{w^*}(A^{**}, B) = \{0\}$ and by the argument before Theorem 2, we have $B^{**} = B \oplus X^{\perp}$. These imply that $H^1_{w^*}(A^{**}, B^{**}) = H^1_{w^*}(A^{**}, X^{\perp})$. It is known that every amenable Banach algebra possesses a BAI, so by a similar argument in the proof of Theorem 2(ii), we obtain that $H^1_{w^*}(A^{**}, X^{\perp}) = \{0\}$.

Proposition 2.2. *Suppose that* A *is an amenable Banach algebra. If for every Banach* A*-bimodule* B*, we have* $AB^{**}, B^{**}A \subseteq B$*, then*

$$H^1_{w^*}(A^{**}, B^{***}) = \{0\}.$$

Proof. By applying a similar argument in the proof of Theorem 2(ii), we obtain the desire.

Corollary 2.3. *Assume that* A *is a weakly amenable Banach algebra with a LBAI. If* A *is an ideal in* A^{**}*, it follows that*

$$H^1_{w^*}(A^{**}, A^{***}) = \{0\}.$$

Example 2.4. *Assume that* G *is a compact group. It is known that* $L^1(G)$ *has a BAI and is a two-sided ideal in* $L^1(G)^{**}$*. We know that* $L^1(G)$ *is weakly amenable, hence by Corollary 2,*

$$H^1_{w^*}(L^{\infty}(G)^*, L^{\infty}(G)^{**}) = \{0\}.$$

Proposition 2.3. *Let* A *be a Banach algebra such that* A *is an ideal in* A^{**} *and* A^* *factors. Then* A *is amenable if and only if* A^{**} *is Connes-amenable.*

Proof. By [3, Corollary 2.8](i), A is Arens regular. Then by [24, Theorem 4.4], the proof completes.

A Banach space A is called weakly sequentially complete if every weakly Cauchy sequence in A has a weak limit in A.

Theorem 2.5. *Let* A *be an Arens regular dual Banach algebra such that* A^* *is weakly sequentially complete (WSC). If* $H^1_{w^*}(A^{**}, A^{***}) = \{0\}$*, then* $H^1_{w^*}(A, A^*) = \{0\}$.

Proof. Let $D: A \longrightarrow A^*$ be a w^*-continuous derivation. Since A^* is WSC, every derivation $D: A \longrightarrow A^*$ is weakly compact. Then by [5, Theorem 6.5.5], we have $D''(A^{**}) \subseteq A^*$ and hence, by Arens regularity of A, A^* is an A^{**}-submodule of $(A^{**})^*$ and $D''(A^{**}).A^{**} \subseteq A^*.A^{**} \subseteq A^*$. Then by [7, Theorem 7.1], $D'': A^{**} \longrightarrow A^{***}$ is a w^*-continuous derivation. Thus, there exists $a''' \in A^{***}$ such that $D''(F) = F.a''' - a'''.F$, for each $F \in A^{**}$. Now, let $E: A \longrightarrow A^{**}$ be the canonical map and set $f = E^*(a''')$, then $D(a) = a.f - f.a$, for all $a \in A$. This means that D is an inner w^*-continuous derivation. Thus the proof follows.

For any $n \in \mathbb{N}$, we denote the n-th dual of the Banach algebra A by $A^{(\mathrm{n})}$. In the following, we extend the [3, Corollary 2.8](i) to the general case as follows:

Lemma 2.1. *If $A^{(2n)}$ is a two-sided ideal in $A^{(2n+2)}$ and $A^{(2n+1)}$ factors, then $A^{(2n)}$ is Arens regular, where $n \in \mathbb{N} \cup \{0\}$.*

Theorem 2.6. *Let A be a Banach algebra such that A^{**} is an ideal in A^{****} and A^{***} factors. If A is weakly amenable, then $H^1_{w^*}(A^{**}, A^{***}) = \{0\}$.*

Proof. Lemma 2 implies that A^{**} is Arens regular. Now, let $D: A^{**} \longrightarrow A^{***}$ be a weak *- weak *-continuous derivation. First, we prove that A^{***} is a normal Banach A^{**}-bimodule. Let $(a''_\alpha)_\alpha$ be a net in A^{**} and $a''' \in A^{***}$. Then, by Arens regularity of A^{**}, for every $b'' \in A^{**}$ we have

$$\begin{aligned}
&\langle (w^* - \lim_\alpha a''_\alpha).a''', b''\rangle = \langle a''', b''.(w^* - \lim_\alpha a''_\alpha)\rangle \\
&= \lim_\alpha \langle a''', b''.a''_\alpha\rangle \\
&= \lim_\alpha \langle a''_\alpha.a''', b''\rangle \\
&= \langle w^* - \lim_\alpha (a''_\alpha.a'''), b''\rangle.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\langle a'''.(w^* - \lim_\alpha a''_\alpha), b''\rangle = \langle a''', w^* - \lim_\alpha a''_\alpha.b''\rangle \\
&= \lim_\alpha \langle a''', a''_\alpha, b''\rangle \\
&= \lim_\alpha \langle a'''.a''_\alpha, b''\rangle \\
&= \langle w^* - \lim_\alpha (a'''.a''_\alpha), b''\rangle.
\end{aligned}$$

Hence, the mappings $a'' \mapsto a''.a'''$ and $a'' \mapsto a'''.a''$ are weak *-weak *-continuous from A^{**} into A^{***}. Thus, A^{***} is a normal Banach A^{**}-bimodule. For each $a \in A$, we define $\overline{D}: A \longrightarrow A^*$ by

$$\overline{D}(a) = D(\hat{a})|_A,$$

where $\hat{a} \in A^{**}$ with $\hat{a}(a') = a'(a)$, for all $a \in A$. As the following equalities $\overline{D}$ is a continuous derivation from A into A^*.

$$\overline{D}(ab) = D(\widehat{ab}) = D(\hat{a}.\hat{b}) = a.D(\hat{b}) + D(\hat{a}).b = a.\overline{D}(b) + \overline{D}(a).b,$$

where $a, b \in A$. By weak amenability of A, we have $\overline{D}$ is inner. Then there exist $a''' \in A^{***}$ such that

$$D(\hat{a}) = \overline{D}(a) = a.a'''|_A - a'''|_A.a = \hat{a}.a'''|_A - a'''|_A.\hat{a}.$$

We consider the canonical mapping $E: A^* \longrightarrow A^{***}$. Then there exists $b''' \in A^{***}$ such that $E(a'''|_A) = b'''$. So

$$D(\hat{a}) = \hat{a}.b''' - b'''.\hat{a}.$$

Then D is inner. It follows that $H^1_{w^*}(A^{**}, A^{***}) = \{0\}$.

Corollary 2.4. *Let $A^{(2n+2)}$ be a two sided ideal in $A^{(2n+4)}$ and $A^{(2n+3)}$ factors. If $A^{(2n)}$ is weakly amenable, then $H^1_{w^*}(A^{(2n+2)}, A^{(2n+3)}) = \{0\}$.*

Proof. Apply Lemma 2 and Theorem 2.

Weak *-continuous derivations from dual Banach algebras into their ideals are studied in [8].

Remark 2.1. *If M is subspace of A and N is subspace of A^*, then $M^\perp = \{x^* \in X^*: \langle x^*, x\rangle = 0, \quad \forall x \in M\}$ and $^\perp N = \{x \in A: \langle x^*, x\rangle = 0, \quad \forall x^* \in N\}$. If A is a dual Banach algebra and I is w^* $-$closed ideal of A, then I is dual with predual $I_* = \frac{A_*}{^\perp I}$ that $(I_*)^* = (\frac{A_*}{^\perp I})^* = (^\perp I)^\perp = I$ and $I^* = \frac{A^*}{I^\perp}$, see [5].*

Proposition 2.4. *Let A be a dual Banach algebra and I be an arbitrary w^*-closed ideal of A such that $H^1(A, I^{**}) = \{0\}$. Then $H^1_{w^*}(A, I) = \{0\}$.*

Proof. Let $D \in Z^1_{w^*}(A, I)$ and $E: I \to I^{**}$ be the natural embedding. Then $E \circ D: A \to I^{**}$ is a bounded derivation. Since $H^1(A, I^{**}) = \{0\}$, there exists $a^{**} \in I^{**}$ such that $E \circ D = \delta_{a^{**}}$. Consider the decomposition $I^{**} = I \bigoplus I_*^\perp$ as an A-bimodule. If $P: I^{**} \to I$ is a projection, we have $D = \delta_{p(a^{**})}$. Then $H^1_{w^*}(A, I) = \{0\}$.

A Banach algebra A is without of order if for any $a, b \in A$, $ab = 0$ implies that $a = 0$ or $b = 0$. Semisimple and unital Banach algebras are without of order Banach algebras. Now by Proposition 2 and [8, Theorem 3.1], we have the following result.

Corollary 2.5. *Let A be a dual Banach algebra and I be a closed two-sided ideal in A such that I is without order. If $H^1(A, I^{**}) = \{0\}$, then $H^1_{w^*}(I, I) = \{0\}$.*

Example 2.5. (i) *Let G be a locally compact group. A linear subspace $S^1(G)$ of $L^1(G)$ is said to be a Segal algebra, if it satisfies the following conditions:*

(S1) *$S^1(G)$ is dense in $L^1(G)$;*

(S2) *If $f \in S^1(G)$, then $L_x f \in S^1(G)$, i.e. $S^1(G)$ is left translation invariant;*

(S3) *$S^1(G)$ is a Banach space under some norm $\|\cdot\|_S$ and $\| L_x f \|_S = \| f \|_S$, for all $f \in S^1(G)$ and $x \in G$;*

(S4) *$x \mapsto L_x f$ from G into $S^1(G)$ is continuous.*

For more details about Segal algebras, see [22, 23]. Now, let G be an *abelian locally compact group. Then* $H^1(L^1(G), S^1(G)^{**}) = \{0\}$. *Then by Proposition 2 and Corollary 2, we have* $H^1_{w^*}(L^1(G), S^1(G)) = \{0\}$ *and* $H^1_{w^*}(S^1(G), S^1(G)) = \{0\}$.

(ii) *Let* Λ *be a non-empty, totally ordered set, and regard it as a semigroup by defining the product of two elements to be their maximum. The resulting semigroup, which we denote by* $\Lambda_\vee$, *is a semilattice. We may then form the* ℓ^1-*convolution algebra* $\ell^1(\Lambda_\vee)$. *For every* $t \in \Lambda_\vee$ *we denote the point mass concentrated at* t *by* e_t. *The definition of multiplication in* $\ell^1(\Lambda_\vee)$ *ensures that* $e_s e_t = e_{max(s,t)}$ *for all* s *and* t.

The semilattice $\Lambda_\vee$, *is a commutative semigroup in which every element is idempotent. If we denote the set of idempotent elements of* $\Lambda_\vee$ *by* $E(\Lambda_\vee)$, *then* $E(\Lambda_\vee) = \Lambda_\vee$. *The* ℓ^1-*convolution algebras of semilattices provide interesting examples of commutative Banach algebras. By [14, Proposition 3.3],* $H^1(\ell^1(\Lambda_\vee), I^{**}) = \{0\}$, *for any closed two-sided* I *of* $\ell^1(\Lambda_\vee)$. *Then by Corollary 2,* $H^1_{w^*}(I, I) = \{0\}$, *for any closed two-sided* I *of* $\ell^1(\Lambda_\vee)$.

(iii) *Let* K *be an infinite compact metric space,* $\alpha \in (0,1)$ *and* $lip_\alpha K$ *be the small Lipschitz algebra (see Example 2). By [14, Proposition 3.4],* $H^1(lip_\alpha K, I^{**}) = \{0\}$, *for any closed two-sided* I *of* $lip_\alpha K$. *Then by Corollary 2,* $H^1_{w^*}(I, I) = \{0\}$, *for any closed two-sided* I *of* $lip_\alpha K$.

3. Representations of derivations and Arens regularity

Let A be a Banach algebra and B be a Banach A-bimodule with the module action "$\bullet$". Then for every $b \in B$, we define

$$L_b(a) = b \bullet a \quad \text{and} \quad R_b(a) = a \bullet b,$$

for every $a \in A$. These are the operation of left and right multiplication by b on A. In the following by using the super-amenability of Banach algebra A, we give a representation for $Z^1(A, C)$, where C is a Banach A-bimodule.

For a Banach A-bimodule B and for a derivation $D: A \to B^*$, we show that the left module action $\pi_\ell: A \times B \to B$ is Arens regular whenever $D'': A^{**} \to B^{***}$ is a derivation and $B^* \subseteq D''(A^{**})$. On the other hand, if A is a left strongly Arens irregular and A^{**} is amenable Banach algebra with respect to the first Arens product, then A is unital. Moreover, if A is a dual Banach algebra, it follows that A is reflexive.

Theorem 3.1. *Assume that* A *is an amenable Banach algebra. Then there are Banach* A-*bimodules* C, D *and elements* $\mathfrak{a}, \mathfrak{b} \in A^{**}$ *such that*

$$Z^1(A, C^*) = \{R_{D''(\mathfrak{a})}: D \in Z^1(A, C^*)\} = \{L_{D''(\mathfrak{b})}: D \in Z^1(A, D^*)\}.$$

Proof. Suppose that B is a Banach A-bimodule with a module action $\bullet$. Every amenable Banach algebra has a BAI [25, Proposition 2.2.1], so A has a BAI such as $(e_\alpha)_\alpha$. Then by Cohen factorization Theorem we have $B \bullet A = B = A \bullet B$, i.e., for every $b \in B$, there are $y, z \in B$ and $a, t \in A$ such that $y \bullet a = b = t \bullet z$. Then we have

$$\lim_\alpha b \bullet e_\alpha = \lim_\alpha (y \bullet a) \bullet e_\alpha = \lim_\alpha y \bullet (a e_\alpha) = y \bullet a = b \tag{1}$$

and

$$\lim_\alpha e_\alpha \bullet b = \lim_\alpha e_\alpha \bullet (t \bullet z) = \lim_\alpha (t e_\alpha) \bullet z = t \bullet z = b. \tag{2}$$

It follows that B has a BAI as $(e_\alpha)_\alpha \subseteq A$. Let e'' and f'' be the right and left unit for A^{**}, respectively such that $e_\alpha \overset{w^*}{\rightarrow} e''$ and $e_\alpha \overset{w^*}{\rightarrow} f''$ in A^{**}.

Take $C = B$ and define a module action "$\cdot$" as $a \cdot x = 0$ and $x \cdot a = x \bullet a$, for all $a \in A$ and $x \in C$. Clearly, $(C, \cdot)$ is a Banach A-bimodule. Suppose that $D \in Z^1(A, C^*)$. Then there is an element $c \in C^*$ such that $D = \delta_c$. Then for every $a \in A$, we have

$$D(a) = \delta_c(a) = a \cdot c - c \cdot a = a \bullet c.$$

From (1) and module actions of C, for any $x \in C$ and $x' \in C^*$, we have

$$\lim_\alpha \langle x, x' \cdot e_\alpha \rangle = \lim_\alpha \langle e_\alpha \cdot x, x' \rangle = 0 \tag{3}$$

and

$$\lim_\alpha \langle x, e_\alpha \cdot x' \rangle = \lim_\alpha \langle x \cdot e_\alpha, x' \rangle = \lim_\alpha \langle x \bullet e_\alpha, x' \rangle = \langle x, x' \rangle. \tag{4}$$

It follows that $e_\alpha \cdot x' \overset{w^*}{\rightarrow} x'$ in C^*. Since D'' is a weak *-to-weak * continuous linear operator, we have

$$D''(e'') = D''(w^* - \lim_\alpha e_\alpha) = w^* - \lim_\alpha D''(e_\alpha) = w^* - \lim_\alpha D(e_\alpha)$$
$$= w^* - \lim_\alpha (e_\alpha x') = x'.$$

Thus we conclude that $D(a) = a \cdot D''(e'') = a \bullet D''(e'')$ for all $a \in A$. It follows that $D = R_{D''(e'')}$. On the other hand, since for every derivation $D \in Z^1(A, C^*)$, $R_{D''(e'')} \in Z^1(A, C^*)$, the result holds.

Now, again consider B as a Banach A-bimodule with the module action "$\bullet$" and set $D = B$ with the module action $\lhd$ such that $a \lhd y = a \bullet y$ and $y \lhd a = 0$, for all $a \in A$ and $y \in D$. By a similar argument that we have discussed above, and setting $\mathfrak{b} = f''$, the proof completes.

Example 3.1. (i) *Let G be an amenable locally compact group. Then by Johnson Theorem $H^1(L^1(G), X^*) = \{0\}$, for every Banach A-bimodule X. Then by defining the similar module actions of $L^\infty(G)$ as a Banach $L^1(G)$-bimodule in the proof of Theorem 3 and by this Theorem, we have*

$$Z^1(L^1(G), L^\infty(G)) = \{R_{D(e'')} : D \in Z^1(L^1(G), L^\infty(G))\}$$

$$= \{L_{D(f'')}: D \in Z^1(L^1(G), L^\infty(G))\},$$

where e'' and f'' are the left and right units of $L^1(G)^{**}$, indeed they w^*-accumulations of the BAI of $L^1(G)$.

(ii) *Let G be locally compact group. Then by [20, Corollary 1.2] $H^1(L^1(G), M(G)) = \{0\}$. Then by applying the module actions defined in the proof of Theorem 3, we can see $M(G)$ as a Banach $L^1(G)$-bimodule. Then by Theorem 3, we have*

$$Z^1(L^1(G), L^\infty(G)) = \{R_{D(e'')}: D \in Z^1(L^1(G), L^\infty(G))\}$$
$$= \{L_{D(f'')}: D \in Z^1(L^1(G), L^\infty(G))\},$$

where e'' and f'' are the left and right units of $L^1(G)^{**}$.

Theorem 3.2. *Let A be a Banach algebra, B be a Banach A-bimodule and $D: A \longrightarrow B^*$ be a continuous derivation. If $D'': A^{**} \longrightarrow B^{***}$ is a derivation and $B^* \subseteq D''(A^{**})$, then $Z^\ell_{A^{**}}(B^{**}) = B^{**}$.*

Proof. Since $D'': A^{**} \to B^{***}$ is a derivation, by [26, Theorem 4.2], $D''(A^{**})B^{**} \subseteq B^*$. Due to $B^* \subseteq D''(A^{**})$, we have $B^*B^{**} \subseteq B^*$. Let $(a''_\alpha)_\alpha \subseteq A^{**}$ such that $a''_\alpha \overset{w^*}{\to} a''$ in A^{**}. Assume that $b'' \in B^{**}$. Then for every $b' \in B^*$, since $b'b'' \in B^*$, we have

$$\langle b''.a''_\alpha, b'\rangle = \langle a''_\alpha b', b''\rangle \to \langle a'', b'b''\rangle = \langle b''.a'', b'\rangle.$$

Thus $b''.a''_\alpha \overset{w^*}{\to} b''.a''$ is in B^{**}, and so $b'' \in Z^\ell_{A^{**}}(B^{**})$.

Corollary 3.1. *Let A be a Banach algebra and $D: A \longrightarrow A^*$ be a continuous derivation such that $A^* \subseteq D''(A^{**})$. If $D'': A^{**} \longrightarrow A^{***}$ is a derivation, then A is Arens regular.*

Example 3.2. *Let G be an infinite locally compact group. Thus, $L^1(G)$ is not Arens regular. Then Corollary 3 implies that there is no $D \in Z^1(L^1(G), L^1(G)^*)$ such that $L^1(G)^* \subseteq D''(L^1(G)^{**})$ and its second transpose D'' is a derivation.*

Lemma 3.1. *Let B be a Banach left A-module and B^{**} has a LBAI with respect to A^{**}. Then B^{**} has a left unit with respect to A^{**}.*

Proof. Assume that $(e''_\alpha)_\alpha \subseteq A^{**}$ is a LBAI for B^{**}. By passing to a suitable subnet, we may suppose that there is an $e'' \in A^{**}$ such that $e''_\alpha \overset{w^*}{\to} e''$ in A^{**}. Then for every $b'' \in B^{**}$ and $b' \in B^*$, we have

$$\langle \pi_\ell^{***}(e'', b''), b'\rangle = \langle e'', \pi_\ell^{**}(b'', b')\rangle = \lim_\alpha \langle e''_\alpha, \pi_\ell^{**}(b'', b')\rangle$$
$$= \lim_\alpha \langle \pi_\ell^{***}(e''_\alpha, b''), b'\rangle = \langle b'', b'\rangle.$$

It follows that $\pi_\ell^{***}(e'', b'') = b''$.

Theorem 3.3. *Let A be a left strongly Arens irregular and suppose that A^{**} is an amenable Banach algebra. Then we have the following assertions.*

(i) A has an identity.

(ii) *If A is a dual Banach algebra, then A is reflexive.*

Proof. (i) Amenability of A^{**} implies that it has a BAI. By using Lemma 3, A^{**} has an identity say that e''. So, the mapping $x'' \to e''.x'' = x''$ is weak *-to-weak * continuous from A^{**} into A^{**}. It follows that $e'' \in Z_1(A^{**}) = A$. This means that A has an identity.

(ii) Assume that E is a predual of A. Then we have $A^{**} = A \oplus E^{\perp}$. Since A^{**} is amenable, by [12, Theorem 1.8] or [13, Theorem 2.3], A is amenable, and so $E^{\perp}$ is amenable. Thus $E^{\perp}$ has a BAI such as $(e''_\alpha)_\alpha \subseteq E^{\perp}$. Since $E^{\perp}$ is a closed and weak *-closed subspace of A^{**}, without loss generality, there is $e'' \in E^{\perp}$ such that

$$e''_\alpha \xrightarrow{w^*} e'' \qquad \text{and} \qquad e''_\alpha \xrightarrow{\|\cdot\|} e''.$$

Then e'' is a left identity for $E^{\perp}$. On the other hand, for every $x'' \in E^{\perp}$, since $E^{\perp}$ is an ideal in A^{**}, we have $x''.e'' \in E^{\perp}$. Thus, for every $a' \in A^*$,

$$\langle x''.e'', a'\rangle = \lim_\alpha \langle (x''.e'').e''_\alpha, a'\rangle = \lim_\alpha \langle x''.(e''.e''_\alpha), a'\rangle$$
$$= \lim_\alpha \langle x''.e''_\alpha, a'\rangle = \langle x'', a'\rangle.$$

It follows that $x''.e'' = x''$, and so e'' is a right identity for $E^{\perp}$. Consequently, e'' is a two-sided identity for $E^{\perp}$. Now, let $a'' \in A^{**}$. Then

$$e''.a'' = (e''.a'').e'' = e''.(a''.e'') = a''.e''.$$

Hence $e'' \in Z_1(A^{**}) = A$. It follows that $e'' = 0$, and so $E^{\perp} = 0$. This implies that $A^{**} = A$.

Example 3.3. *Let G be a locally compact group. If $M(G)^{**}$ is amenable, then by Theorem 3(ii), because $C_0(G)^* = M(G)$, we conclude that $M(G)$ is reflexive. This means that G is a finite group, moreover see [12, Corollary 1.4].*

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R E F E R E N C E S

[1] *R. E. Arens*, The adjoint of a bilinear operation, Proc. Amer. Math. Soc., **2** (1951), No. 6, 839-848.

[2] *W. G. Bade, P. C. Curtis and H. G. Dales*, Amenability and weak amenability for Beurling and Lipschitz algebras, Proc. London Math. Soc., **55** (1987), No. 3, 359-377.

[3] *A. Bagheri Vakilabad, K. Haghnejad Azar and A. Jabbari*, Arens regularity of module actions and weak amenability of Banach algebras, Period. Math. Hung., **71** (2015), No. 2, 224-235.

[4] *A. Connes*, Classification of injective factors, Ann. Math., **104** (1976),73-114.

[5] *J. B. Conway*, A course in functional analysis, Springer-Verlag, New York, 1985.
[6] *H. G. Dales*, Banach algebra and automatic continuity, Oxford 2000.
[7] *H. G. Dales, A. Rodr13̆053′fgues-Palacios and M. V. Velasco*, The second transpose of a derivation, J. London. Math. Soc., **64** (2001), No. 2, 707-721.
[8] *A. Ebadian and A. Jabbari*, Weak *-continuous derivations on module extension of dual Banach algebras, South. Asian Bull. Math., **39** (2015), No. 3, 347â€“363.
[9] *M. Eshaghi Gordji and M. Filali*, Arens regularity of module actions, Studia Math., **181** (2007), No. 3, 237-254.
[10] *M. Eshaghi Gordji and M. Filali*, Weak amenability of the second dual of a Banach algebra, Studia Math., **182** (2007), No. 3, 205-213.
[11] *F. Ghahramani and J. Laali*, Amenability and topological centers of the second duals of Banach algebras, Bull. Austral. Math. Soc., **65** (2002), 191-197.
[12] *F. Ghahramani, R. J. Loy and G. A. Willis*, Amenability and weak amenability of second conjugate Banach algebras, Proc. Amer. Math. Soc., **129** (1996), 1489-1497.
[13] *F. Gourdeau*, Amenability and the second dual of a Banach algebra, Studia Math., **125** (1997), No. 1, 75-81.
[14] *A. Jabbari*, On ideal amenability of Banach algebras, J. Math. Physics, Analysis, Geometry, **8** (2012), No. 2, 135-143.
[15] *B. E. Johnson*, Cohomology in Banach algebra, Mem. Amer. Math. Soc., **127**, 1972.
[16] *B. E. Johnson*, Weak amenability of group algebra, Bull. London. Math. Soc., **23** (1991), No. 3, 281-284.
[17] *B. E. Johnson, R. V. Kadison, and J. Ringrose*, Cohomology of operator algebras, III, Bull. Soc. Math. France., **100** (1972),73-79.
[18] *A. Ya. Helemskii*, Homological essence of amenability in the sense of A. Connes: the injectivity of the predual bimodule (traslated from the Russian), Math.USSR-Sb, **68** (1991), 555-566.
[19] *A. T. Lau and A. Ülger*, Topological center of certain dual algebras, Trans. Amer. Math. Soc., **348** (1996), 1191-1212.
[20] *V. Losert*, The derivation problem for group algebra, Ann. Math., **168** (2008), 221-246.
[21] *S. Mohamadzadih and H. R. E. Vishki*, Arens regularity of module actions and the second adjoint of a derivation, Bull. Aust. Math. Soc., **77** (2008), 465-476.
[22] *H. Reiter*, L^l-algebras and Segal algebras, Lecture notes in Mathematics, Vol. **231**, Springer, Berlin, 1971.
[23] *H. Reiter and J. D. Stegeman*, Classical harmonic analysis and locally compact groups, London Math. Soc. Mono. Ser. Vol. **22**, Oxford, 2000.
[24] *V. Runde*, Amenability for dual Banach algebras, Studia Math., **148** (2001), 47-66.
[25] *V. Runde*, Lectures on amenability, Lecture Note in Mathematics, Vol. **1774**, Springer Verlag, 2001.
[26] *A. Ülger*, Arens regularity of weakly sequentialy compact Banach algebras, Proc. Amer. Math. Soc., **127** (1999), No. 11, 3221-3227.