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THE GENERALIZED RICCATI EQUATION TOGETHER WITH THE (G'/G)-EXPANSION METHOD FOR THE (3+1)-DIMENSIONAL MODIFIED KDV-ZAKHAROV-KUZNETSOV EQUATION

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We construct new exact travelling wave solutions including solitons and periodic solutions of the (3+1)-dimensional modified KdV Zakharov-Kuznetsov equation involving parameter by applying the generalized Riccati equation together with the (G'/G) -expansion method. In addition, in this method, $G' = n + lG + mG^2$ is used, as the auxiliary equation, called the generalized Riccati equation, where l, m and n are arbitrary constants. Further, the solutions are expressed in terms of the hyperbolic function, the trigonometric function and the rational functional form. Moreover, some of obtained traveling wave solutions are presented in the figures with the aid of commercial software Maple.

Keywords: The generalized Riccati equation, the (3+1)-dimensional modified KdV Zakharov-Kuznetsov equation, the (G'/G)-expansion method, exact traveling wave solutions, nonlinear evolution equations.

1. Introduction

The study of exact traveling wave solutions for the nonlinear partial differential equations (PDEs) is one of the attractive and remarkable research fields in all areas of science and engineering, such as, plasma physics, chemical physics, optical fibres, solid state physics, fluid mechanics, chemistry and many others [1-41]. In the recent years, many researchers implemented various methods to study different nonlinear differential equations for searching traveling wave solutions. For example, the inverse scattering method [1, 2], the homogeneous balance method [3], the Backlund transformation method [4, 5], the Jacobi elliptic function expansion method [6], the tanh-coth method [7, 8], the Hirota's bilinear transformation method [9], the direct algebraic method [10], the Exp-function method [11-14] and others [15-18].

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Recently, Wang *et al.* [19] introduced the (G'/G)-expansion method to construct traveling wave solutions for the nonlinear evolution equations (NLEEs). In the method, they employed $G'' + \lambda G' + \mu G = 0,$ as an auxiliary equation, where λ and μ are arbitrary constants. After that many researchers studied various nonlinear PDEs by using (G'/G)-expansion method for obtaining exact traveling wave solutions, for example [20-30].

Zhu [31] presented the generalized Riccati equation mapping method to construct non- traveling wave solutions for the (2+1)-dimensional Boiti-Leon-Pempinelle equation. In generalized Riccati equation mapping method, the auxiliary equation $G' = n + l\,G + m\,G^2$ is considered. Salas [32] applied the projective Riccati equation to obtain exact solutions for a type of generalized Sawada-Kotera equation while Gomez *et al*. [33] studied the higher-order nonlinear KdV equation to construct traveling wave solutions by using this method. In Ref. [34], Li and Dai implemented the generalized Riccati equation mapping with the (G'/G)-expansion method to construct traveling wave solutions for the higher dimensional Jimbo-Miwa equation whilst Guo *et al.* [35] investigated diffusion-reaction and modified KdV equation with variable coefficient by using the Riccati equation mapping method. Bekir and Cevikel [36] constructed analytical solutions via the tanh-coth method combined with the Riccati equation of nonlinear coupled equation in mathematical physics. Naher and Abdullah [37] executed the generalized Riccati equation together with the basic (G'/G)-expansion method to generate traveling wave solutions for the modified Benjamin-Bona-Mahony equation whereas they [38] constructed exact solutions of the (2+1)-dimensional modified Zakharov-Kuznetsov equation by applying the same method and so on.

Many researchers used different methods for obtaining exact traveling wave solutions for the (3+1)-dimensional modified KdV-Zakharov-Kuznetsov equation. Such as, Xu [39] utilized an elliptic equation method and its applications to obtain exact solutions of this equation. Naher *et al.* [40] applied the Exp-function method to construct traveling wave solutions for the same equation. Zayed [41] investigated this equation by using the (G'/G)-expansion method, whereas, he applied the second order linear ordinary differential equation (LODE) as an auxiliary equation. But, to the best of our knowledge, the generalized Riccati equation together with the (G'/G)-expansion method has not been applied to generate traveling wave solutions for the (3+1)-dimensional modified KdV-Zakharov-Kuznetsov equation.

In this article, we have obtained twenty seven new traveling wave solutions herein include soliton solutions, periodic solutions, and rational solutions for the (3+1)-dimensional modified KdV-Zakharov-Kuznetsov equation involving parameters by using the generalized Riccati equation together with the (G'/G) -expansion method combined.

3. Description of the generalized Riccati equation together with the (G'/G) -expansion method Equations

Suppose the general nonlinear partial differential equation:

$$P\left(u,u_t,u_x,u_y,u_z,u_{tt},u_{xt},u_{xx},u_{xy},u_{yy},u_{yt},u_{zz},u_{zt},u_{zx},u_{zy},...\right)=0, \tag{1}$$

where $u=u(x,y,z,t)$ is an unknown function (the subscripts denote the partial derivatives), P is a polynomial of $u=u(x,y,z,t)$ and various partial derivatives of $u=u(x,y,z,t)$ including higher order derivatives and the nonlinear terms.

The main steps of the generalized Riccati equation together with the (G'/G) -expansion method [19, 31] are:

Step 1. Consider the traveling wave variable:

$$u(x,y,z,t)=w(\eta), \quad \eta=x+y+z-S\,t, \tag{2}$$

where S is the wave speed. Now using Eq. (2), Eq. (1) is converted into an ordinary differential equation for $w(\eta)$:

$$H\left(w,w',w'',w''',...\right)=0, \tag{3}$$

where the superscripts stand for the ordinary derivatives with respect to η.

Step 2. According to possibility, Eq. (3) is integrated term by term one or more times, thus it yields constant(s) of integration. For simplicity, the integral constant(s) may be zero.

Step 3. Suppose that the traveling wave solution of Eq. (3) can be expressed in the form [19, 31]:

$$w(\eta)=\sum_{k=0}^{r} a_k\left(\frac{G'}{G}\right)^k \tag{4}$$

where $a_k\ (k=0,1,2,...,r)$ and $a_r\neq 0$, with $G=G(\eta)$, the solution of the Riccati equation:

$$G'=n+l\,G+m\,G^2, \tag{5}$$

where l,m,n are arbitrary constants and $m\neq 0$.

Step 4. To determine the positive integer r, taking the homogeneous balance between the highest order nonlinear terms and the highest order derivatives appearing in Eq. (3).

Step 5. Substitute Eq. (4) along with Eq. (5) into the Eq. (3), then collect all the coefficients with the same order; the left hand side of Eq. (3) converts into polynomials in $G^{i}(\eta)$ and $G^{-i}(\eta),(i=0,1,2,...)$. Then equating each coefficient of the polynomials to zero, yields a set of algebraic equations for $a_k\ (k=0,1,2,...,r), l, m, n$ and S.

Step 6. Solve the system of algebraic equations obtained in step 5 with the aid of algebraic software Maple and we obtain values for $a_k\ (k=0,1,2,...,r)$ and S. Then, substituting the obtained values in Eq. (4) along with Eq. (5) with the value of r, obtained at step 4, we get exact traveling wave solutions of Eq. (1).

We have the following twenty seven solutions including four different types' solutions of Eq. (5).

Family 2.1: When $l^2-4mn>0$ and $lm\neq 0$ or $mn\neq 0$, the solutions of Eq. (5) are:

$$Q_1=\frac{-1}{2m}\left(l+\sqrt{l^2-4mn}\,\tanh\left(\frac{\sqrt{l^2-4mn}}{2}\eta\right)\right),$$

$$Q_2=\frac{-1}{2m}\left(l+\sqrt{l^2-4mn}\,\coth\left(\frac{\sqrt{l^2-4mn}}{2}\eta\right)\right),$$

$$Q_3=\frac{-1}{2m}\left(l+\sqrt{l^2-4mn}\left(\tanh\left(\sqrt{l^2-4mn}\,\eta\right)\pm i\,\mathrm{sec}h\left(\sqrt{l^2-4mn}\,\eta\right)\right)\right),$$

$$Q_4=\frac{-1}{2m}\left(l+\sqrt{l^2-4mn}\left(\coth\left(\sqrt{l^2-4mn}\,\eta\right)\pm \mathrm{csc}h\left(\sqrt{l^2-4mn}\,\eta\right)\right)\right),$$

$$Q_5=\frac{-1}{4m}\left(2l+\sqrt{l^2-4mn}\left(\tanh\left(\frac{\sqrt{l^2-4mn}}{4}\,\eta\right)+\cot h\left(\frac{\sqrt{l^2-4mn}}{4}\eta\right)\right)\right),$$

$$Q_6=\frac{1}{2m}\left(-l+\frac{\sqrt{\left(M^2+N^2\right)\left(l^2-4mn\right)}-M\sqrt{l^2-4mn}\,\cosh\left(\sqrt{l^2-4mn}\,\eta\right)}{M\sinh\left(\sqrt{l^2-4mn}\,\eta\right)+N}\right),$$

$$Q_7=\frac{1}{2m}\left(-l-\frac{\sqrt{\left(N^2-M^2\right)\left(l^2-4mn\right)}+M\sqrt{l^2-4mn}\,\sinh\left(\sqrt{l^2-4mn}\,\eta\right)}{M\cosh\left(\sqrt{l^2-4mn}\,\eta\right)+N}\right),$$

where M and N are two non-zero real constants and satisfy $N^2-M^2>0$.

$$Q_8=\frac{2n\cosh\left(\frac{\sqrt{l^2-4mn}}{2}\eta\right)}{\sqrt{l^2-4mn}\sinh\left(\frac{\sqrt{l^2-4mn}}{2}\eta\right)-l\cosh\left(\frac{\sqrt{l^2-4mn}}{2}\eta\right)},$$

$$Q_9=\frac{-2n\sinh\left(\frac{\sqrt{l^2-4mn}}{2}\eta\right)}{l\sinh\left(\frac{\sqrt{l^2-4mn}}{2}\eta\right)-\sqrt{l^2-4mn}\cosh\left(\frac{\sqrt{l^2-4mn}}{2}\eta\right)},$$

$$Q_{10}=\frac{2n\cosh\left(\sqrt{l^2-4mn}\;\eta\right)}{\sqrt{l^2-4mn}\sinh\left(\sqrt{l^2-4mn\eta}\right)-l\cosh\left(\sqrt{l^2-4mn}\,\eta\right)\pm i\sqrt{l^2-4mn}},$$

$$Q_{11}=\frac{2n\sinh\left(\sqrt{l^2-4mn}\;\eta\right)}{-l\sinh\left(\sqrt{l^2-4mn\eta}\right)+\sqrt{l^2-4mn}\cosh\left(\sqrt{l^2-4mn}\,\eta\right)\pm\sqrt{l^2-4mn}},$$

$$Q_{12}=\frac{4n\sinh\left(\frac{\sqrt{l^2-4mn}}{4}\eta\right)\cosh\left(\frac{\sqrt{l^2-4mn}}{4}\eta\right)}{-2l\sinh\frac{\sqrt{l^2-4mn}}{4}\xi\cosh\left(\frac{\sqrt{l^2-4mn}}{4}\eta\right)+}$$

$$\times\frac{1}{2\sqrt{l^2-4mn}\cosh^2\left(\frac{\sqrt{l^2-4mn}}{4}\eta\right)-\sqrt{l^2-4mn}},$$

Family 2.2: When $l^2-4mn<0$ and $lm\neq 0$ or $mn\neq 0$, the solutions of Eq. (5) are:

$$Q_{13}=\frac{1}{2m}\left(-l+\sqrt{4mn-l^2}\;\tan\left(\frac{\sqrt{4mn-l^2}}{2}\eta\right)\right),$$

$$Q_{14}=\frac{-1}{2m}\left(l+\sqrt{4mn-l^2}\;\cot\left(\frac{\sqrt{4mn-l^2}}{2}\eta\right)\right),$$

$$Q_{15}=\frac{1}{2m}\left(-l+\sqrt{4mn-l^2}\left(\tan\left(\sqrt{4mn-l^2}\;\eta\right)\pm\sec\left(\sqrt{4mn-l^2}\;\eta\right)\right)\right),$$

$$Q_{16}=\frac{-1}{2m}\left(l+\sqrt{4mn-l^2}\left(\cot\left(\sqrt{4mn-l^2}\;\eta\right)\pm\csc\left(\sqrt{4mn-l^2}\;\eta\right)\right)\right),$$

$$Q_{17}=\frac{1}{4m}\left(-2l+\sqrt{4mn-l^2}\left(\tan\left(\frac{\sqrt{4mn-l^2}}{4}\,\eta\right)-\cot\left(\frac{\sqrt{4mn-l^2}}{4}\,\eta\right)\right)\right),$$

$$Q_{18}=\frac{1}{2m}\left(-l+\frac{\sqrt{\left(M^2-N^2\right)\left(4mn-l^2\right)}\;-M\sqrt{4mn-l^2}\,\cos\left(\sqrt{4mn-l^2}\;\eta\right)}{M\sin\left(\sqrt{4mn-l^2}\,\eta\right)+N}\right),$$

$$Q_{19}=\frac{1}{2m}\left(-l+\frac{\sqrt{\left(M^2-N^2\right)\left(4mn-l^2\right)}\;+M\sqrt{4mn-l^2}\cos\left(\sqrt{4mn-l^2}\,\eta\right)}{M\sin\left(\sqrt{4mn-l^2}\,\eta\right)+N}\right),$$

where M and N are two non-zero real constants and satisfy $M^2-N^2>0.$

$$Q_{20}=\frac{-2n\cos\left(\frac{\sqrt{4mn-l^2}}{2}\eta\right)}{\sqrt{4mn-l^2}\sin\left(\frac{\sqrt{4mn-l^2}}{2}\eta\right)+l\cos\left(\frac{\sqrt{4mn-l^2}}{2}\eta\right)},$$

$$Q_{21}=\frac{2n\sin\left(\frac{\sqrt{4mn-l^2}}{2}\eta\right)}{-l\sin\left(\frac{\sqrt{4mn-l^2}}{2}\eta\right)+\sqrt{4mn-l^2}\cos\left(\frac{\sqrt{4mn-l^2}}{2}\eta\right)},$$

$$Q_{22}=\frac{-2n\cos\left(\sqrt{4mn-l^2}\,\eta\right)}{\sqrt{4mn-l^2}\sin\left(\sqrt{4mn-l^2}\,\eta\right)+l\cos\left(\sqrt{4mn-l^2}\eta\right)\pm\sqrt{4mn-l^2}},$$

$$Q_{23}=\frac{2n\sin\left(\sqrt{4mn-l^2}\,\eta\right)}{-l\sin\left(\sqrt{4mn-l^2}\,\eta\right)+\sqrt{4mn-l^2}\cos\left(\sqrt{4mn-l^2}\,\eta\right)\pm\sqrt{4mn-l^2}},$$

$$Q_{24}=\frac{4n\sin\left(\frac{\sqrt{4mn-l^2}}{4}\eta\right)\cos\left(\frac{\sqrt{4mn-l^2}}{4}\eta\right)}{-2l\sin\left(\frac{\sqrt{4mn-l^2}}{4}\,\eta\right)\cos\left(\frac{\sqrt{4mn-l^2}}{4}\eta\right)+}$$

$$\times\frac{1}{2\sqrt{4mn-l^2}\cos^2\left(\frac{\sqrt{4mn-l^2}}{4}\eta\right)-\sqrt{4mn-l^2}}.$$

Family 2.3: when $n=0$ and $lm\neq 0,$ the solutions of Eq. (5) become:

$$Q_{25}=\frac{-lc}{m\left(c+\cosh\left(l\eta\right)-\sinh\left(l\eta\right)\right)},$$

$$Q_{26}=\frac{-l\left(\cosh\left(l\eta\right)+\sinh\left(l\eta\right)\right)}{m\left(c+\cosh\left(l\eta\right)+\sinh\left(l\eta\right)\right)},$$

where c is an arbitrary constant.

Family 2.4: when $m\neq 0$ and $n=l=0,$ solution of Eq. (5) becomes:

$$Q_{27}=\frac{-1}{m\eta+p_1},$$

where p_1 is an arbitrary constant.

3. Applications of the method

In this section, we construct new exact traveling wave solutions by using this powerful and straightforward method. We consider the (3+1)-dimensional modified KdV-Zakharov-Kuznetsov equation with parameter considered by Zayed [41]:

$$u_t+\alpha u^2u_x+u_{xxx}+u_{xyy}+u_{xzz}=0, \tag{6}$$

where α is nonzero constant.

Now, we use the transformation Eq. (2) into the Eq. (6). This yield:

$$-S\,w'+\alpha\,w^2\,w'+3w'''=0, \tag{7}$$

now integrating once the above equation with respect to η, it yields:

$$K-S\,w+\frac{1}{3}\,\alpha w^3+3w''=0, \tag{8}$$

where K is constant of integration that could be determined later. We considered $r=1$, taking the homogeneous balance between the highest order nonlinear terms and the highest order derivatives appearing in Eq. (8). Therefore, the solution of Eq. (8) is of the form:

$$w\left(\eta\right)=a_1\left(G'/G\right)+a_0. \tag{9}$$

Using Eq. (5) with Eq. (9), it gives:

$$w\left(\eta\right)=a_1\left(l+nG^{-1}+mG\right)+a_0, \tag{10}$$

where l,m and n are free parameters. By substituting Eq. (10) into Eq. (8), collecting all coefficients of G^i and G^{-i} $\left(i=0,1,2,...\right)$ setting them equal to zero, we obtain a set of algebraic equations for a_0,a_1,l,m,n,K and S (for simplicity, algebraic equations are not displayed). Solving the system of algebraic equations with the aid of algebraic software Maple, we obtain

$$a_0 = \mp \frac{3}{2} l\, i \sqrt{\frac{2}{\alpha}}, \quad a_1 = \pm 3i \sqrt{\frac{2}{\alpha}}, \quad S = \frac{-3}{2} l^2 - 12 m n, \quad K = \pm i 18 l\, m\, n \sqrt{\frac{2}{\alpha}},$$

where l, m, n are free parameters and $\alpha \neq 0$.

Family 3.1: The soliton and soliton-like solutions of Eq. (6) (when $l^2 - 4mn > 0$ and $lm \neq 0$ or $mn \neq 0$) are:

$$w_1 = \pm 3i \sqrt{\frac{2}{\alpha}} \; \frac{2\Phi^2 \operatorname{sec}h^2(\Phi\eta)}{l + 2\Phi \tanh(\Phi\eta)} \mp \frac{3}{2} l\, i \sqrt{\frac{2}{\alpha}},$$

where $\Phi = \frac{1}{2}\sqrt{l^2 - 4mn}$, $\eta = x + y + z + \left(\frac{3}{2} l^2 + 12 m n\right) t$ and l, m, n are arbitrary constants.

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$$w_{12} = \pm 3i \sqrt{\frac{2}{\alpha}} \frac{2\Phi^2 \operatorname{csc}h(\Phi\eta)}{2\Phi \cosh(\Phi\eta) - l \sinh(\Phi\eta)} \mp \frac{3}{2} l i \sqrt{\frac{2}{\alpha}}.$$

$$w_7 = \mp 3i \sqrt{\frac{2}{\alpha}} \frac{4M\Phi^2 \left(M - N \sinh(2\Phi\eta) + \sqrt{(M^2 + N^2)} \cosh(2\Phi\eta)\right)}{\left(lM \sinh(2\Phi\eta) + lN + 2\Phi \sqrt{(M^2 + N^2)} + 2M\Phi \cosh(2\Phi\eta)\right)}$$

$$\times \frac{1}{(M \sinh(2\Phi\eta) + N)} \mp \frac{3}{2} l i \sqrt{\frac{2}{\alpha}},$$

where M and N are two non-zero real constants and satisfy $N^2 - M^2 > 0$.

… … … … … … … … … … … … … … … …

$$w_{12} = \pm 3i \sqrt{\frac{2}{\alpha}} \frac{2\Phi^2 \operatorname{csc}h(\Phi\eta)}{2\Phi \cosh(\Phi\eta) - l \sinh(\Phi\eta)} \mp \frac{3}{2} l i \sqrt{\frac{2}{\alpha}}.$$

Family 3.2: The periodic form solutions of Eq. (6) (when $l^2 - 4mn < 0$ and $lm \neq 0$ or $mn \neq 0$):

$$w_{13} = \pm 3i \sqrt{\frac{2}{\alpha}} \; \frac{2\Delta^2 \sec^2(\Delta\eta)}{-l + 2\Delta \tan(\Delta\eta)} \mp \frac{3}{2} l i \sqrt{\frac{2}{\alpha}},$$

where $\Delta = \frac{1}{2}\sqrt{4 m n - l^2}$, $\eta = x + y + z + \left(\frac{3}{2} l^2 + 12 m n\right) t$ and l, m, n are arbitrary constants.

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$$w_{12} = \pm 3i \sqrt{\frac{2}{\alpha}} \frac{2\Phi^2 \operatorname{csc}h(\Phi\eta)}{2\Phi \cosh(\Phi\eta) - l \sinh(\Phi\eta)} \mp \frac{3}{2} l i \sqrt{\frac{2}{\alpha}}.$$

$$w_{19}=\mp 3i\sqrt{\frac{2}{\alpha}}\frac{4M\Delta^2\left(\sqrt{M^2-N^2}\cos(2\Delta\eta)+N\sin(2\Delta\eta)+M\right)\left(M\sin(2\Delta\eta)+N\right)}{\left(lM\sin(2\Delta\eta)-2M\Delta\cos(2\Delta\eta)+lN-2\Delta\sqrt{M^2-N^2}\right)}$$
$$\times\frac{1}{\left(M^2\cos^2(2\Delta\eta)-M^2-N^2-2MN\sin(2\Delta\eta)\right)}\mp\frac{3}{2}li\sqrt{\frac{2}{\alpha}},$$

where M and N are two non-zero real constants and satisfy $M^2-N^2>0.$

$$w_{20}=\mp 3i\sqrt{\frac{2}{\alpha}}\frac{2\Delta^2\sec(\Delta\eta)\left(2\Delta\sin(\Delta\eta)+l\cos(\Delta\eta)\right)}{2\left(l^2-2mn\right)\cos^2(\Delta\eta)+4l\Delta\sin(\Delta\eta)\cos(\Delta\eta)+4\Delta^2}\mp\frac{3}{2}li\sqrt{\frac{2}{\alpha}},$$

… … … … … … … … … … … … … … … …

$$w_{24}=\pm 3i\sqrt{\frac{2}{\alpha}}\frac{2\Delta^2\csc(\Delta\eta)\left(l\sin(\Delta\eta)-2\Delta\cos(\Delta\eta)\right)}{2\left(l^2-2mn\right)\cos^2(\Delta\eta)+4l\Delta\sin(\Delta\eta)\cos(\Delta\eta)-l^2}\mp\frac{3}{2}li\sqrt{\frac{2}{\alpha}},$$

Family 3.3: The soliton and soliton-like solutions of Eq. (6) (when $n=0$ and $lm\neq 0$):

$$w_{25}=\pm 3i\sqrt{\frac{2}{\alpha}}\frac{l\left(\cosh(l\eta)-\sinh(l\eta)\right)}{p+\cosh(l\eta)-\sinh(l\eta)}\mp\frac{3}{2}li\sqrt{\frac{2}{\alpha}},$$
$$w_{26}=\pm 3i\sqrt{\frac{2}{\alpha}}\frac{lp}{p+\cosh(l\eta)+\sinh(l\eta)}\mp\frac{3}{2}li\sqrt{\frac{2}{\alpha}},$$

where p is an arbitrary constant and $\eta=x+y+z+\left(\frac{3}{2}l^2+12mn\right)t.$

Family 3.4: The rational function solution **(**when $m\neq 0$ and $n=l=0$):

$$w_{27}=\pm 3i\sqrt{\frac{2}{\alpha}}\frac{-m}{m\eta+d_1},$$

where d_1 is an arbitrary constant $\eta=x+y+z.$

4. Results and discussion:

It is imperative to declare that one of our solutions is in good agreement with already published results which are described in the table 4.1. Furthermore, some of newly obtained solutions are illustrated in the figure 1 to figure 8 in the following subsection.

Table 4.1

Comparison between Zayed [41] solutions and newly obtained solutions

Zayed [41] solutions	Present solutions
i. If $A=0, B=1, \alpha=-1$ and $V=-1,$ from section 3 in example 4, solution Eq. (3.39) becomes:	i. If $d_1=0, m=1, \alpha=-1, S=-1$ and $w_{27}(\eta)=u(\xi)$, solution w_{27} becomes:

$u(x,y,z,t)=\pm 3\sqrt{2}\left(\frac{1}{x+y+z+t}\right).$	$u(x,y,z,t)=\pm 3\sqrt{2}\left(\frac{1}{x+y+z+t}\right).$
ii. If $A=0, B=1, \alpha=-1$ and $V=1$, from section 3 in example 4, solution Eq. (3.39) becomes: $u(x,y,z,t)=\pm 3\sqrt{2}\left(\frac{1}{x+y+z-t}\right).$	ii. If $d_1=0, m=1, \alpha=-1, S=1$ and $w_{27}(\eta)=u(\xi)$, solution w_{27} becomes: $u(x,y,z,t)=\pm 3\sqrt{2}\left(\frac{1}{x+y+z-t}\right).$
iii. If $A=0, B=1, \alpha=1$ and $V=-\frac{1}{2}$, from section 3 in example 4, solution Eq. (3.39) becomes: $u(x,y,z,t)=\pm i3\sqrt{2}\left(\frac{2}{2(x+y+z)+t}\right).$	iii. If $d_1=0, m=1, \alpha=1, S=-\frac{1}{2}$ and $w_{27}(\eta)=u(\xi)$, solution w_{27} becomes: $u(x,y,z,t)=\pm i3\sqrt{2}\left(\frac{2}{2(x+y+z)+t}\right).$
iv. If $A=0, B=1, \alpha=1$ and $V=\frac{1}{2}$, from section 3 in example 4, Eq. (3.39) becomes: $u(x,y,z,t)=\pm i3\sqrt{2}\left(\frac{2}{2(x+y+z)-t}\right).$	iv. If $d_1=0, m=1, \alpha=1, S=\frac{1}{2}$ and $w_{27}(\eta)=u(\xi)$, solution w_{27} becomes: $u(x,y,z,t)=\pm i3\sqrt{2}\left(\frac{2}{2(x+y+z)-t}\right).$

Beyond the table, we have constructed new exact traveling wave solutions w_1 to w_{26} which have not been reported in the previous literature.

4.2 Graphical representations of the solutions

The graphical descriptions of the solutions are shown in the figures with the aid of commercial software Maple:

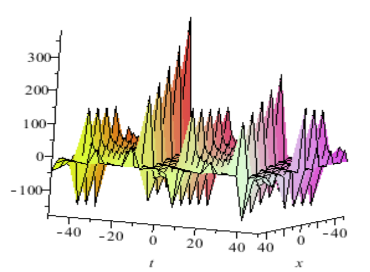

Figure 1: Solitons solutions for $l=5$, $m=4$, $n=2$, $\alpha=2$

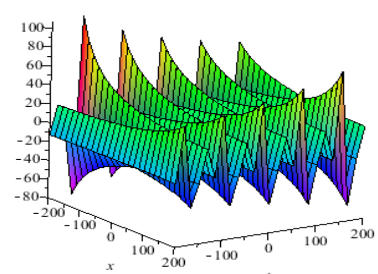

Figure 2: Periodic solutions for $l=2$, $m=3$, $n=3$, $\alpha=3$

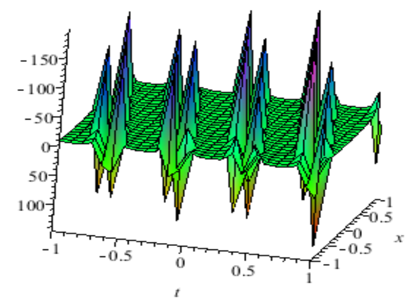

Figure 3: Solitons solutions for $l=3,\ m=2,\ n=2,\ \alpha=3$

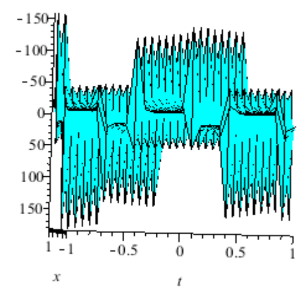

Figure 4: Solitons solutions for $l=1,\ m=2,\ n=1,\ \alpha=2$

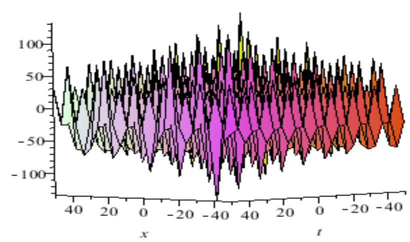

Figure 5: Solitons solutions for $l=3,\ m=3,\ n=3,\ \alpha=1$

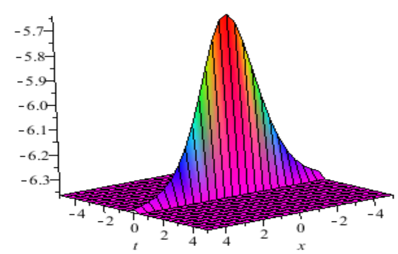

Figure 6: Solitons solutions for $l=3,\ m=1,\ n=2,\ \alpha=1$

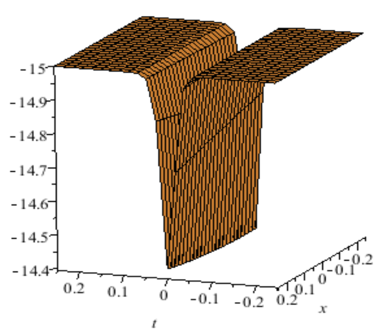

Figure 7: Periodic solutions for $l=5,\ m=6,\ n=1,\ \alpha=0.5$

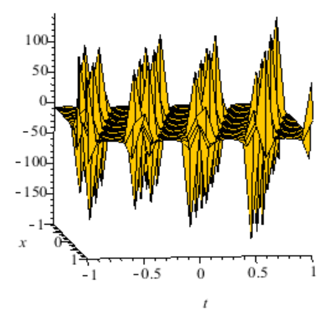

Figure 8: Solitons solutions for $l=3,\ m=4,\ n=1,\ \alpha=3$

5. Conclusions

In this article, we have applied the generalized Riccati equation together with the (G'/G)-expansion method to construct abundant exact traveling wave solutions of the (3+1)-dimensional modified KdV-Zakharov-Kuznetsov equation.

The solutions of the higher dimensional nonlinear evolution equations have many potential applications in Mathematical Physics, Engineering Sciences and other technical arena. The traveling wave solutions are described herein for four different families including solitons and periodic solutions. In addition, it is worth declaring that one of our solutions is in good agreement with already published results and some are new. Therefore, the extended generalized Riccati equation mapping method will be more effectively used to investigate many nonlinear evolution equations which frequently arise in scientific real time application fields.

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