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NOTES ON RIEMANNIAN MAPS

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In this paper, we first find necessary and sufficient conditions for the total space of a Riemannian map to be an Einstein manifold and then we obtain various inequalities in terms of the scalar curvatures of the base space, fibers and images. In the equality cases of those inequalities, we obtain harmonicity and totally geodesicity of such maps.

Keywords: Einstein manifold, Riemannian map, Harmonic map, Totally geodesic map, Generalized divergence theorem.
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1. Introduction

Smooth maps between Riemannian manifolds are useful for comparing geometric structures between two manifolds. Isometric immersions (Riemannian submanifolds) are basic such maps between Riemannian manifolds. In 1992, Fischer introduced Riemannian maps between Riemannian manifolds in [3] as a generalization of the notions of isometric immersions and Riemannian submersions. Let $F:(M,g_M)\to(N,g_N)$ be a smooth map between Riemannian manifolds such that $0<rankF<min\{m,n\}$, where $dimM=m$ and $dimN=n$. Then we denote the kernel space of F_* by $kerF_*$ and consider the orthogonal complementary space $\mathcal{H}=(kerF_*)^\perp$ to $kerF_*$. Then the tangent bundle of M has the following decomposition

$$TM=kerF_*\oplus\mathcal{H}.$$

We denote the range of F_* by $rangeF_*$ and consider the orthogonal complementary space $(rangeF_*)^\perp$ to $rangeF_*$ in the tangent bundle TN of N. Since $rankF<min\{m,n\}$, we always have $(rangeF_*)^\perp\neq\{0\}$. Thus the tangent bundle TN of N has the following decomposition

$$TN=(rangeF_*)\oplus(rangeF_*)^\perp.$$

Now, a smooth map $F:(M^m,g_M)\to(N^n,g_N)$ is called Riemannian map at $p_1\in M$ if the horizontal restriction $F^h_{*p_1}:(kerF_{*p_1})^\perp\to(rangeF_{*p_1})$ is a linear

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isometry between the inner product spaces $((kerF_{*p_1})^{\perp}, g_M(p_1)|_{(kerF_{*p_1})^{\perp}})$ and $(rangeF_{*p_1}, g_N(p_2)|_{(rangeF_{*p_1})})$, $p_2 = F(p_1)$. Therefore Fischer stated in [3] that a Riemannian map is a map which is as isometric as it can be. It follows that isometric immersions and Riemannian submersions are particular Riemannian maps with $kerF_* = \{0\}$ and $(rangeF_*)^{\perp} = \{0\}$. It is known that a Riemannian map is a subimmersion which implies that the rank of the linear map $F_{*p} : T_pM \to T_{F(p)}N$ is constant for p in each connected component of M [1] and [3]. A remarkable property of Riemannian maps is that a Riemannian map satisfies the generalized eikonal equation $\|F_*\|^2 = rankF$ which is a bridge between geometric optics and physical optics [3].

Riemannian maps between semi-Riemannian manifolds have been defined in [7] by putting some regularity conditions. On the other hand, affine Riemannian maps have been also investigated and decomposition theorems related to Riemannian maps and curvatures are obtained in [6] (For Riemannian maps and their applications in spacetime geometry, see: [7]). Riemannian maps and related topics are now very active research area in differential geometry, see: [4], [5], [10], [11].

In this paper, we first obtain necessary and sufficient conditions for the total space of a Riemannian maps to be an Einstein manifold. Then we calculate the scalar curvatures of the base, space, fibers and image $(rangeF_*)$ and obtain several inequalities. By using these inequalities, we obtain new conditions for a Riemannian map to be harmonic or totally geodesic.

2. Preliminaries

Let (M, g_M) and (N, g_N) be Riemannian manifolds and suppose that $F : M \to N$ is a smooth map between them. Let $p_2 = F(p_1)$ for each $p_1 \in M$. Suppose that $\overset{N}{\nabla}$ is the Levi-Civita connection on (N, g_N). Then the second fundamental form of F is given by

$$(\nabla F_*)(X,Y) = \overset{N}{\nabla}{}^F_X F_*(Y) - F_*(\overset{M}{\nabla}_X Y) \qquad (1)$$

for $X, Y \in \Gamma(TM)$, where $\overset{N}{\nabla}{}^F$ is the pulback connection of $\overset{N}{\nabla}$ It is known that the second fundamental form is symmetric. First note that in [12] we showed that the second fundamental form $(\nabla F_*)(X,Y)$, $\forall X, Y \in \Gamma((kerF_*)^{\perp})$, of a Riemannian map has no components in $rangeF_*$. More precisely we have the following.

Lemma 2.1. [12] *Let F be a Riemannian map from a Riemannian manifold (M, g_M) to a Riemannian manifold (N, g_N). Then*

$$g_N((\nabla F_*)(X,Y), F_*(Z)) = 0, \forall X,Y,Z \in \Gamma((kerF_*)^\perp).$$

Let F be a Riemannian map from a Riemannian manifold (M, g_M) to a Riemannian manifold (N, g_N). Then we define T and A as

$$\mathsf{A}_E F = \mathsf{H}\overset{M}{\nabla}_{\mathsf{H}E}\mathsf{V}F + \mathsf{V}\overset{M}{\nabla}_{\mathsf{H}E}\mathsf{H}F, \mathsf{T}_E F = \mathsf{V}\overset{M}{\nabla}_{\mathsf{V}E}\mathsf{V}F + \mathsf{V}\overset{M}{\nabla}_{\mathsf{V}E}\mathsf{H}F,$$

for vector fields E, F on M, where $\overset{M}{\nabla}$ is the Levi-Civita connection of g_M. In fact, one can see that these tensor fields are O'Neill's tensor fields which were defined for Riemannian submersions. For any $E \in \Gamma(TM)$, T_E and A_E are skew-symmetric operators on $(\Gamma(TM), g)$ reversing the horizontal and the vertical distributions. It is also easy to see that T is vertical, $\mathsf{T}_E = \mathsf{T}_{\mathsf{V}E}$ and A is horizontal, $\mathsf{A}_E = \mathsf{A}_{\mathsf{H}E}$.

We now state the following curvature relations between the base manifold (N, g_N) and the total manifold (M, g), [2], [9] and [13].

$$g(R(U,V)W,F) = g(\hat{R}(U,V)W,F) - g(\mathsf{T}_U F, \mathsf{T}_V W) + g(\mathsf{T}_V F, \mathsf{T}_U W) \quad (2)$$

$$g(R(X,V)Y,W) = -g((\nabla_X \mathsf{T})_V W, Y) - g((\nabla_V \mathsf{A})_X Y, W) + g(\mathsf{T}_V X, \mathsf{T}_W Y) - g(\mathsf{A}_X V, \mathsf{A}_Y W) \quad (3)$$

$$g_N(\overset{N}{R}(F_*X, F_*Y)F_*Z, F_*T) = g_M(\overset{M}{R}(X,Y)Z,T) + g_N((\nabla F_*)(X,Z), (\nabla F_*)(Y,T)) - g_N((\nabla F_*)(Y,Z), (\nabla F_*)(X,T)). \quad (4)$$

$$g(R(U,V)W,X) = g((\nabla_U \mathsf{T})_V W, X) - g((\nabla_V \mathsf{T})_U W, X) \quad (5)$$

$$g(R(X,Y)Z,V) = -g((\nabla_Z \mathsf{A})_X Y, V) - g(\mathsf{T}_V Z, \mathsf{A}_X Y) - g(\mathsf{A}_X Z, \mathsf{T}_V Y) + g(\mathsf{A}_Y Z, \mathsf{T}_V X) \quad (6)$$

for $X,Y,Z,H,T \in \in (M)$ and $U,V,W \in \chi^v(M)$.

In this section, we finally recall the following relation for a Riemannian map.

Lemma 2.2. [7] *Let $F:(M,g) \to (B, g_B)$ be a Riemannian map between Riemannian manifolds (M,g) and (B, g_B). Then we have*

$$\|(\nabla F_*)\|^2(p) = -div(\tau(F)) + \sum_{i,j=r+1}^{m} \{g_N(\overset{N}{R}(F_*(x_i), F_*(x_j))F_*(x_i), F_*(x_j))\}$$

$$-\sum_{i=r+1}^{m} \overset{M}{Ric}(x_i, \overset{M}{R}(x_i, x_j)x_j). \qquad (7)$$

at $p \in M$, *where* $\{e_1,...,e_r,e_{r+1},...,e_m\}$ *be an orthonormal basis of* $\Gamma(TM)$ *such that* $\{e_1,...,e_r\}$ *is an orthonormal basis of* $(kerF_*)$ *and* $\{e_{r+1},...,e_m\}$ *is an orthonormal basis of* $(kerF_*)^{\perp}$, $\overset{M}{Ric}$ *is Ricci tensor of* M *and* $\|(\nabla F_*)\|^2$ *is the square of the length of the second fundamental form.*

3. Einstein Metrics on the Total space of a Riemannian Map

In this section, we are going to find necessary and sufficient conditions for the total space of a Riemannian map to be Einstein manifold. By using (2), (3), (4), (5) and (6), we have the following result for the Ricci tensor.

Lemma 3.1. *Let* $F:(M,g)\to(B,g_B)$ *be a Riemannian map. Then, we have*

$$\overset{M}{Ric}(W_1,W_2) = \overset{\vee}{Ric}(W_1,W_2) - rg(H, \mathcal{T}_{W_1}W_2)$$
$$+ \sum_{j=r+1}^{m} g((\nabla_{e_j}\mathcal{T})_{W_1}W_2, e_j) + g(\mathcal{A}_{e_j}W_1, \mathcal{A}_{e_j}W_2), \qquad (8)$$

where $\overset{\vee}{Ric}(W_1,W_2)$ *and* H *are Ricci tensor and the mean curvature vector field of any fibre,*

$$\overset{M}{Ric}(X,Y) = \sum_{i=1}^{r} g((\nabla_X\mathcal{T})_{u_i}u_i, Y) + g((\nabla_{u_i}\mathcal{A})_X Y, u_i) - g(\mathcal{T}_{u_i}X, \mathcal{T}_{u_i}Y)$$
$$+ g(\mathcal{A}_X u_i, \mathcal{A}_Y u_i) - \sum_{r+1=1}^{m} g_N((\nabla F_*)(e_j,Y),(\nabla F_*)(X,e_j))$$
$$+ g_N((\nabla F_*)(X,Y), \tau^{((kerF_*)^{\perp})}) + \overset{(rangeF_*)}{Ric}(F_*(X),F_*(Y)), \qquad (9)$$

where $\overset{(rangeF_*)}{Ric}(F_*(X),F_*(Y))$ *and* $\tau^{((kerF_*)^{\perp})}$ *are Ricci tensor of* $rangeF_*$ *and* $((kerF_*)^{\perp})-$ *component of the tension field* τ ,

$$\overset{M}{Ric}(X,U) = \sum_{i=1}^{r} g((\nabla_U\mathcal{T})_{u_i}u_i, X) - g((\nabla_{u_i}\mathcal{T})_U u_i, X) + \sum_{j=r+1}^{m} g((\nabla_{e_j}\mathcal{A})_{e_j}X, U)$$
$$+ 2g(\mathcal{T}_U e_j, \mathcal{A}_{e_j}X) \qquad (10)$$

for $W_1,W_2,U \in \Gamma(kerF_*)$ *and* $X,Y \in \Gamma((kerF_*)^{\perp})$, *where* $\{u_1,...,u_r\}$ *and* $\{e_{r+1},...,e_m\}$ *are orthonormal frames of* $(kerF_*)$ *and* $(kerF_*)^{\perp}$.

Proposition 3.1. *Let* $F:(M,g)\to(B,g_B)$ *be a Riemannian map with totally geodesic fibers. Then,* (M,g_M) *is Einstein if and only if the following conditions are satisfied:*

$$\overset{\vee}{Ric}(W_1,W_2)+\sum_{j=r+1}^{m}g(\mathrm{A}_{e_j}W_1,\mathrm{A}_{e_j}W_2)=\frac{s}{m}g(U,V) \tag{11}$$

$$\sum_{i=1}^{r}g(\mathrm{A}_X u_i,\mathrm{A}_Y u_i)+\overset{(rangeF_*)}{Ric}(F_*(X),F_*(Y))+g_N((\nabla F_*)(X,Y),\tau^{((kerF_*)^{\perp})})$$
$$-\sum_{j=r+1}^{m}g_N((\nabla F_*)(e_j,Y),(\nabla F_*)(X,e_j))=\frac{s}{m}g(X,Y) \tag{12}$$

and
$$\sum_{j=r+1}^{m}g((\nabla_{e_j}\mathrm{A})_{e_j}X,U)=0. \tag{13}$$

We note that the above conditions (11) and (13) are the same as the conditions given for Riemannian submersions in [2, Page:144]. The only difference is the condition (12). Using (8) and (9) we have the scalar curvature of the total space as follows.

Theorem 3.1. *Let* $F:(M,g)\to(B,g_B)$ *be a Riemannian map. Then, we have*

$$s=\hat{s}+s^{(rangeF_*)}+\left\|\tau^{((kerF_*)^{\perp})}\right\|^2-r^2\|\mathrm{H}\|^2-\sum_{j,l=r+1}^{m}\left\|\nabla F_*)(e_j,e_l)\right\|^2$$
$$+2\sum_{j=r+1}^{m}\sum_{k=1}^{r}g((\nabla_{e_j}\mathrm{T})_{u_k}u_k,e_j)+\left\|\mathrm{A}_{e_j}u_k\right\|^2+\sum_{l=r+1}^{m}\sum_{i=1}^{r}g((\nabla_{u_i}\mathrm{A})_{e_l}e_l,u_i)$$
$$-\left\|\mathrm{T}_{u_i}e_l\right\|^2 \tag{14}$$

where s, $\hat{s}$ *and* $s^{(rangeF_*)}$ *denote the scalar curvature of* M, *the scalar curvature of the fiber and the scalar curvature of* $(rangeF_*)$.

Using (7), we have also the following result.

Corollary 3.1. *Let* $F:(M^m,g_M)\to(B,g_B)$ *be a Riemannian map. Then, we have*

$$s=\hat{s}+s^{\mathrm{H}}+div\,\tau(F)+\|(\nabla F_*)\|^2+\left\|\tau^{((kerF_*)^{\perp})}\right\|^2-r^2\|\mathrm{H}\|^2$$
$$-\sum_{j,l=r+1}^{m}\left\|\nabla F_*)(e_j,e_l)\right\|^2+2\sum_{j=r+1}^{m}\sum_{k=1}^{r}g((\nabla_{e_j}\mathrm{T})_{u_k}u_k,e_j)+\left\|\mathrm{A}_{e_j}u_k\right\|^2$$
$$+\sum_{l=r+1}^{m}\sum_{i=1}^{r}g((\nabla_{u_i}\mathrm{A})_{e_l}e_l,u_i)-\left\|\mathrm{T}_{u_i}e_l\right\|^2 \tag{15}$$

at $p \in M$ *, where* $s^{\mathcal{H}}$ *denotes the scalar curvature of* $(\ker F_*)^{\perp}$ *and* $\{\mu_1,...,\mu_m\}$ *is an orthonormal frame of* M *.*

From Theorem 3.1, we have the following results.

Corollary 3.2. *Let* $F:(M,g)\to(B,g_B)$ *be a Riemannian map with totally geodesic fibers. Then*

$$s \le \hat{s} + s^{(rangeF_*)} + 2\sum_{j=r+1}^{m}\sum_{k=1}^{r}\left\|\mathcal{A}_{e_j}u_k\right\|^2 + \left\|\tau^{((kerF_*)^{\perp})}\right\|^2,$$

and the equality is satisfied if and only if F *is totally geodesic.*

Corollary 3.3. *Let* $F:(M,g)\to(B,g_B)$ *be a Riemannian map with totally geodesic fibers. Then*

$$s \ge \hat{s} + s^{(rangeF_*)} - \sum_{j,l=r+1}^{m}\left\|\nabla F_*)(e_j,e_l)\right\|^2,$$

and the equality is satisfied if and only if F *is harmonic and the horizontal distribution is integrable.*

From Corollary 3.1, we have the following result.

Corollary 3.4. *Let* $F:(M,g)\to(B,g_B)$ *be a Riemannian map with totally geodesic fibers. Then, we have*

$$s = \hat{s} + s^{\mathcal{H}} + div\,\tau(F) + \left\|(\nabla F_*)\right\|^2 + \left\|\tau^{((kerF_*)^{\perp})}\right\|^2 - \sum_{j,l=r+1}^{m}\left\|\nabla F_*)(e_j,e_l)\right\|^2$$
$$+\sum_{j=r+1}^{m}\sum_{k=1}^{r}\left\|\mathcal{A}_{e_j}u_k\right\|^2 + \sum_{l=r+1}^{m}\sum_{i=1}^{r} g((\nabla_{u_i}\mathcal{A})_{e_l}e_l,u_i)$$

at $p \in M$ *.*

By using the generalized divergence theorem of a map (see:[7, Theorem 3.3.1, page:70], we have the following sufficient condition for a Riemannian map F to be harmonic.

Corollary 3.5. *Let* (M,g) *be an oriented compact Riemannian manifold with Riemannian volume form* μ_M *and let* (N,g_N) *be a Riemanian manifold. If* $F:M\to N$ *is a Riemannian map with totally geodesic fibers and the following inequality*

$$s \ge \hat{s} + s^{\mathcal{H}} + \left\|(\nabla F_*)\right\|^2 + \sum_{j=r+1}^{m}\sum_{k=1}^{r}(\left\|\mathcal{A}_{e_j}u_k\right\|^2 + g((\nabla_{u_k}\mathcal{A})_{e_j}e_j,u_k))$$

is satisfied, then F *is harmonic.*

Proof. Let F be a Riemannian map with totally geodesic fibers. Since $\partial M = \emptyset$, from generalized divergence theorem and corollary 3.1 we have

$$\int_M \{s-\hat{s}-s^{\mathcal{H}}-\|(\nabla F_*)\|^2+\sum_{j,l=r+1}^{m}\|\nabla F_*)(e_j,e_l)\|^2$$
$$-\sum_{j=r+1}^{m}\sum_{k=1}^{r}(\|\mathcal{A}_{e_j}u_k\|^2+g((\nabla_{u_k}\mathcal{A})_{e_j}e_j,u_k))\}\mu_M=0,$$

which gives the assertion.

Moreover, for the totally geodesicity of F, we have the following result.

Corollary 3.6. *Let (M,g) be an oriented compact Riemannian manifold with Riemannian volume form μ_M and let (N,g_N) be a Riemanian manifold. If $F:M\to N$ is a Riemannian map with totally geodesic fibers and the following inequality*

$$s\le\hat{s}+s^{\mathcal{H}}-\sum_{j,l=r+1}^{m}\|\nabla F_*)(e_j,e_l)\|^2+\sum_{l=r+1}^{m}\sum_{i=1}^{r}g((\nabla_{u_i}\mathcal{A})_{e_l}e_l,u_i)$$

is satisfied, then F is totally geodesic.

From Corollary 3.4, we have the following results.

Corollary 3.7. *Let $F:(M,g)\to(B,g_B)$ be a Riemannian map with totally geodesic fibers. Then, we have*

$$s\ge\hat{s}+s^{\mathcal{H}}+div\tau(F)-\sum_{j,l=r+1}^{m}\|\nabla F_*)(e_j,e_l)\|^2+\sum_{j=r+1}^{m}\sum_{k=1}^{r}\|\mathcal{A}_{e_j}u_k\|^2$$
$$+\sum_{l=r+1}^{m}\sum_{i=1}^{r}g((\nabla_{u_i}\mathcal{A})_{e_l}e_l,u_i)$$

at $p\in M$. The equality is satisfied if and only if F is totally geodesic. In the equality case, it takes the following form

$$s=\hat{s}+s^{\mathcal{H}}+\sum_{j=r+1}^{m}\sum_{k=1}^{r}\|\mathcal{A}_{e_j}u_k\|^2+\sum_{l=r+1}^{m}\sum_{i=1}^{r}g((\nabla_{u_i}\mathcal{A})_{e_l}e_l,u_i),\quad \text{at } p\in M.$$

Corollary 3.8. *Let $F:(M,g)\to(B,g_B)$ be a Riemannian map with totally geodesic fibers. Then, we have*

$$s\ge\hat{s}+s^{\mathcal{H}}+div\tau(F)-\sum_{j,l=r+1}^{m}\|(\nabla F_*)(e_j,e_l)\|^2+\sum_{l=r+1}^{m}\sum_{i=1}^{r}g((\nabla_{u_i}\mathcal{A})_{e_l}e_l,u_i)$$

at $p\in M$, the equality is satisfied if and only if F is totally geodesic and the horizontal distribution is integrable. In the equality case, it takes the following form

$$s=\hat{s}+s^{\mathcal{H}}. \tag{16}$$

Corollary 3.9. *Let $F:(M,g)\to(B,g_B)$ be a Riemannian map with totally geodesic fibers. Then, we have*

$$s \geq \hat{s} + s^{\mathcal{H}} + div\tau(F) + \sum_{i,j=1}^{r} g_B((\nabla F_*)(u_i,u_j),(\nabla F_*)(u_i,u_j))$$
$$+ \sum_{j=r+1}^{m}\sum_{k=1}^{r}\left\|\mathcal{A}_{e_j}u_k\right\|^2 + \sum_{l=r+1}^{m}\sum_{i=1}^{r} g((\nabla_{u_i}\mathcal{A})_{e_l}e_l,u_i)$$

at $p \in M$ *, the equality is satisfied if and only if* F *is harmonic.*

4. Conclusion

The theory of Riemannian maps is a new research area and it is a generalization of Riemannian submanifolds and Riemannian submersions. Einstein conditions and curvature relations for submanifolds and Riemannian submersions have been widely investigated. This paper is an attempt to investigate Riemannian maps by using curvature relations. In this direction, we first investigate Einstein conditions for the total manifold of a Riemannian map. Then we obtain the harmonicity and totally geodesicity of Riemannian maps by comparing the curvatures of the total manifold, the base manifold and the fiber, and by using the Bochner identity.

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