INFINITELY MANY SOLUTIONS FOR A DIRICHLET BOUNDARY VALUE PROBLEM WITH IMPULSIVE CONDITION

Ghasem A. Afrouzi[1], Armin Hadjian[2], Saeid Shokooh[3]

In this paper, by employing a critical point theorem, we establish the existence of infinitely many solutions for second-order impulsive differential equations with Dirichlet boundary conditions, depending on two real parameters. We also provide some particular cases and a concrete example in order to illustrate the main abstract results of this paper.

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1. Introduction

In this paper, we investigate the existence of infinitely many solutions for the following nonlinear Dirichlet boundary-value problem

$$\begin{cases} -(|u'(t)|^{p-2}u'(t))' + |u(t)|^{p-2}u(t) = \lambda f(t,u(t)) + \mu g(t,u(t)), & \text{in } \Omega, \\ u(0) = u(T) = 0, \\ \Delta|u'(t_j)|^{p-2}u'(t_j) = I_j(u(t_j)), \quad j = 1,2,\dots,m, \end{cases} \tag{1}$$

where $p \in [2,+\infty)$, $T > 0$, $\lambda \in]0,+\infty[$, $\mu \in [0,+\infty[$, $\Omega := [0,T] \setminus \{t_1, t_2, \dots, t_m\}$, $f, g : [0,T] \times \mathbb{R} \to \mathbb{R}$ are L^1-Carathéodory functions, $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$, $\Delta|u'(t_j)|^{p-2}u'(t_j) = |u'(t_j^+)|^{p-2}u'(t_j^+) - |u'(t_j^-)|^{p-2}u'(t_j^-)$, where $u'(t_j^+)$ and $u'(t_j^-)$ denote the right and left limits, respectively, of $u'(t)$ at t_j, and $I_j : \mathbb{R} \to \mathbb{R}$ are continuous for every $j = 1,2,\dots,m$.

Many dynamical systems describing models in applied sciences have an *impulsive* dynamical behaviour due to abrupt changes at certain instants during the evolution process. The rigorous mathematical description of these phenomena leads to *impulsive differential equations*; they characterize various processes of the real

[1]Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran, e-mail addresses: `afrouzi@umz.ac.ir, saeid.shokooh@stu.umz.ac.ir`

[2]Department of Mathematics, Faculty of Basic Sciences, University of Bojnord, P.O. Box 1339, Bojnord 94531, Iran, e-mail address: `hadjian83@gmail.com`

3Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran, e-mail addresses: `afrouzi@umz.ac.ir, saeid.shokooh@stu.umz.ac.ir`

world described by models that are subject to *sudden changes* in their states. Essentially, impulsive differential equations correspond to a *smooth evolution* that may *change instantaneously* or even *abruptly*, as happens in various applications that describe mechanical or natural phenomena. These changes correspond to impulses in the smooth system, such as for example in the model of a mechanical clock. Impulsive differential equations also study models in physics, population dynamics, ecology, industrial robotics, biotechnology, economics, optimal control, chaos theory. Associated with this development, a theory of impulsive differential equations has been given extensive attention.

For a general second order differential equation $\mathcal{F}(t, u, u') = 0$, one can consider impulses in the *position* u and the *velocity* u'. However, as argued in [11], it is natural to consider in the motion of spacecraft only instantaneous impulses depending on the position that result in jump discontinuities, but with no change in position. The impulses only on the velocity occurs also in impulsive mechanics, see [12].

Recently, many researchers pay their attention to impulsive differential equations by variational method and critical point theory, and we refer the reader to [1, 2, 8, 11, 16, 17, 18, 19] and references cited therein. Meanwhile, some people begin to study p-Laplacian differential equations with impulsive effects, for example, see [3, 4, 7, 19]. In [3, 4], Bai and Dai utilize Ricceri's three critical point theorem and the mountain pass theorem to investigate the existence of solutions for an impulsive boundary value problem involving the p-Laplacian operator. Chen and Tang [7] adopt the least action principle and the saddle point theorem to obtain some existence theorems for second-order p-Laplacian systems with or without impulsive effects under weak sublinear growth conditions. In [9], they also consider that a class of second-order impulsive differential equations with Dirichlet problems has one or infinitely many solutions under more relaxed assumptions on their nonlinearity f, which satisfies a kind of new superquadratic and subquadratic condition. Authors in [19] discuss the existence of weak solutions for a p-Laplacian problem with impulsive conditions by topological degree theory and critical point theory.

The purpose of this paper is to show the variational structure underlying of a class of nonlinear impulsive differential equations. We take as a model a Dirichlet problem with impulses. For an excellent overview of the most significant mathematical methods employed in this paper we refer to Ciarlet [10].

2. **Preliminaries**

We shall prove our results applying the following smooth version of Theorem 2.1 of [6], which is a more precise version of Ricceri's variational principle [15, Theorem 2.5]. We point out that Ricceri's variational principle generalizes the celebrated three critical point theorem of Pucci and Serrin [13, 14] and is an useful result that gives alternatives for the multiplicity of critical points of certain functions depending on a parameter.

Lemma 2.1. *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous and coercive, and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let*

$$\varphi(r) := \inf_{u \in \Phi^{-1}(-\infty, r)} \frac{\left(\sup_{v \in \Phi^{-1}(-\infty, r)} \Psi(v)\right) - \Psi(u)}{r - \Phi(u)},$$

$$\gamma := \liminf_{r \to +\infty} \varphi(r), \qquad \text{and} \qquad \delta := \liminf_{r \to (\inf_X \Phi)^+} \varphi(r).$$

Then the following properties hold:

(a) *For every $r > \inf_X \Phi$ and every $\lambda \in (0, 1/\varphi(r))$, the restriction of the functional*

$$I_\lambda := \Phi - \lambda \Psi$$

to $\Phi^{-1}(-\infty, r)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.

(b) *If $\gamma < +\infty$, then for each $\lambda \in (0, 1/\gamma)$, the following alternative holds: either*

(b$_1$) *I_λ possesses a global minimum, or*

(b$_2$) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that*

$$\lim_{n \to +\infty} \Phi(u_n) = +\infty.$$

(c) *If $\delta < +\infty$, then for each $\lambda \in (0, 1/\delta)$, the following alternative holds: either*

(c$_1$) *there is a global minimum of Φ which is a local minimum of I_λ, or*

(c$_2$) *there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of I_λ that converges weakly to a global minimum of Φ.*

In the following, we use the Sobolev space $X := W_0^{1,p}(0, T)$, equipped with the norm

$$\|u\| := \left(\int_0^T |u'(t)|^p \, dt + \int_0^T |u(t)|^p \, dt \right)^{1/p}.$$

Lemma 2.2. *For any $u \in X$, there exists a constant $c := 2^{1/q} \max\{T^{-1/p}, T^{1/q}\}$, $1/p + 1/q = 1$, such that*

$$\|u\|_\infty \leq c \, \|u\|, \tag{2}$$

where $\|u\|_\infty := \max_{t \in [0,T]} |u(t)|$.

Proof. It can be proved in an elementary way by the Mean Value Theorem and the Hölder inequality. □

Let $f, g : [0,T] \times \mathbb{R} \to \mathbb{R}$ be two L^1-Carathéodory functions. We recall that $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function if

(a) the mapping $t \mapsto f(t,x)$ is measurable for every $x \in \mathbb{R}$;
(b) the mapping $x \mapsto f(t,x)$ is continuous for almost every $t \in [0,T]$;
(c) for every $\rho > 0$ there exists a function $l_\rho \in L^1([0,T])$ such that

$$\sup_{|x| \le \rho} |f(t,x)| \le l_\rho(t)$$

for almost every $t \in [0,T]$.

We say that $u \in \{w \in C([0,T]) : w|_{[t_j,t_{j+1}] \in W^{2,p}([t_j,t_{j+1}])}\}$ is a *classical solution* of problem (1), if it satisfies the equation in (1) a.e. on Ω, the limits $u'(t_j^+), u'(t_j^-)$, $j = 1, \ldots, m$, exist, satisfy the impulsive condition $\Delta |u'(t_j)|^{p-2} u'(t_j) = I_j(u(t_j))$ and the boundary condition $u(0) = u(T) = 0$. We say that a function $u \in X$ is a *weak solution* of problem (1), if u satisfies

$$\begin{aligned} &\int_0^T |u'|^{p-2} u' v' \, dt + \int_0^T |u|^{p-2} u v \, dt \\ &- \lambda \int_0^T f(t,u) v \, dt - \mu \int_0^T g(t,u) v \, dt + \sum_{j=1}^m I_j(u(t_j)) v(t_j) = 0, \end{aligned} \tag{3}$$

for any $v \in X$.

By the same argument as in the proof of [5, Lemma 2.3], we can prove the following lemma.

Lemma 2.3. *The function $u \in X$ is a weak solution of problem* (1) *if and only if u is a classical solution of* (1).

We will use the following lemma in the proof of our main result.

Lemma 2.4 ([5, Lemma 3.1])**.** *Assume that*

(A1) *there exist constants $\alpha, \beta > 0$ and $\sigma \in [0,1[$ such that*

$$|I_j(x)| \le \alpha + \beta |x|^\sigma \quad \text{for all } x \in \mathbb{R},\ j = 1, 2, \ldots, m\,.$$

Then, for any $u \in X$, we have

$$\left| \sum_{j=1}^m \int_0^{u(t_j)} I_j(x) \, dx \right| \le m \Big(\alpha \|u\|_\infty + \frac{\beta}{\sigma+1} \|u\|_\infty^{\sigma+1} \Big). \tag{4}$$

Finally, put

$$k := \frac{2T^{p-1}(p+1)}{c^p \big[4^p (p+1) + T^p (p+2) \big]}, \qquad \Gamma_a := \frac{\alpha}{a} + \Big(\frac{\beta}{\sigma+1} \Big) a^{\sigma-1},$$

where α, β, σ are given by (A1), c is defined in Lemma 2.2 and a is a positive constant.

3. **Main results**

In this section we establish the main abstract result of this paper. Let

$$A := \liminf_{\xi\to+\infty} \frac{\displaystyle\int_0^T \max_{|x|\le\xi} F(t,x)\,dt}{\xi^p},$$

$$B := \limsup_{\xi\to+\infty} \frac{\displaystyle\int_{T/4}^{3T/4} F(t,\xi)\,dt}{\xi^p},$$

where $F(t,x) := \displaystyle\int_0^x f(x,\xi)\,d\xi$ for all $(t,x)\in[0,T]\times\mathbb{R}$, and

$$\lambda_1 := \frac{1}{pc^pkB}, \qquad \lambda_2 := \frac{1}{pc^pA}.$$

With the above notations we establish the following multiplicity property.

Theorem 3.1. *Assume that* (A1) *holds and* $f : [0,T]\times\mathbb{R}\to\mathbb{R}$ *be an* L^1*-Carathéodory function, whose potential* $F(t,x) := \int_0^x f(x,\xi)\,d\xi$ *for all* $(t,x)\in[0,T]\times\mathbb{R}$*, satisfies the following conditions*

(A2) $F(t,\xi)\ge 0$ *for all* $(t,\xi)\in([0,\frac{T}{4}]\cup[\frac{3T}{4},T])\times\mathbb{R}$;

(A3) $A < kB$.

Then, for every $\lambda\in(\lambda_1,\lambda_2)$ *and for every arbitrary* L^1*-Carathéodory function* $g : [0,T]\times\mathbb{R}\to\mathbb{R}$*, whose potential* $G(t,x) := \int_0^x g(x,\xi)\,d\xi$ *for all* $(t,x)\in[0,T]\times\mathbb{R}$*, is a nonnegative function satisfying the condition*

$$G_\infty := \limsup_{\xi\to+\infty} \frac{\displaystyle\int_0^T \max_{|x|\le\xi} G(t,x)\,dt}{\xi^p} < +\infty, \tag{5}$$

if we put

$$\mu_{G,\lambda} := \frac{1}{pc^pG_\infty}\left(1-\lambda pc^pA\right),$$

where $\mu_{G,\lambda} = +\infty$ *when* $G_\infty = 0$*, problem* (1) *has an unbounded sequence of classical solutions for every* $\mu\in[0,\mu_{G,\lambda})$ *in* X*.*

Proof. Our aim is to apply Lemma 2.1(b) to problem (1). To this end, fix $\overline{\lambda}\in(\lambda_1,\lambda_2)$ and g satisfying our assumptions. Since $\overline{\lambda}<\lambda_2$, we have

$$\mu_{G,\overline{\lambda}} := \frac{1}{pc^pG_\infty}\left(1-\overline{\lambda}pc^pA\right) > 0.$$

Now fix $\overline{\mu}\in(0,\mu_{G,\overline{\lambda}})$ and set

$$J(t,x) := F(t,x) + \frac{\overline{\mu}}{\overline{\lambda}}G(t,x)$$

for all $(t,x) \in [0,T] \times \mathbb{R}$. For each $u \in X$, let the functionals $\Phi, \Psi : X \to \mathbb{R}$ be defined by

$$\Phi(u) := \frac{1}{p}\|u\|^p,$$
$$\Psi(u) := \int_0^T J(t,u(t))\,dt - \frac{1}{\overline{\lambda}}\sum_{j=1}^m \int_0^{u(t_j)} I_j(x)\,dx,$$

and put

$$E_{\overline{\lambda},\overline{\mu}}(u) := \Phi(u) - \overline{\lambda}\Psi(u), \qquad u \in X.$$

Using the property of f, g and the continuity of I_j, $j = 1,2,\dots,m$, we obtain that $\Phi, \Psi \in C^1(X,\mathbb{R})$ and for any $v \in X$, we have

$$\Phi'(u)(v) = \int_0^T |u'(t)|^{p-2}u'(t)v'(t)\,dt + \int_0^T |u(t)|^{p-2}u(t)v(t)\,dt$$

and

$$\Psi'(u)(v) = \int_0^T f(t,u(t))v(t)\,dt + \frac{\overline{\mu}}{\overline{\lambda}}\int_0^T g(t,u(t))v(t)\,dt - \frac{1}{\overline{\lambda}}\sum_{j=1}^m I_j(u(t_j))v(t_j).$$

So, with standard arguments, we deduce that the critical points of the functional $E_{\overline{\lambda},\overline{\mu}}$ are the weak solutions of problem (1) and so they are classical. We first observe that the functionals Φ and Ψ satisfy the regularity assumptions of Lemma 2.1.

First of all, we show that $\overline{\lambda} < 1/\gamma$. Hence, let $\{\xi_n\}$ be a sequence of positive numbers such that $\lim_{n\to+\infty}\xi_n = +\infty$ and

$$\lim_{n\to+\infty}\frac{\int_0^T \max_{|x|\le\xi_n} F(t,x)\,dx}{\xi_n^p} = A.$$

Put $r_n := \frac{1}{pc^p}\xi_n^p$ for all $n \in \mathbb{N}$. Then, for all $v \in X$ with $\Phi(v) < r_n$, taking (2) into account, one has $\|v\|_\infty < \xi_n$. Note that $\Phi(0) = \Psi(0) = 0$. Then, for all $n \in \mathbb{N}$,

$$\begin{aligned}
\varphi(r_n) &= \inf_{u\in\Phi^{-1}(-\infty,r_n)}\frac{\left(\sup_{v\in\Phi^{-1}(-\infty,r_n)}\Psi(v)\right) - \Psi(u)}{r_n - \Phi(u)} \\
&\le \frac{\sup_{v\in\Phi^{-1}(-\infty,r_n)}\Psi(v)}{r_n} \\
&\le \frac{\int_0^T \max_{|x|\le\xi_n} J(t,x)\,dt + \frac{m}{\overline{\lambda}}\left(\alpha\xi_n + \frac{\beta}{\sigma+1}\xi_n^{\sigma+1}\right)}{\frac{1}{pc^p}\xi_n^p} \\
&\le pc^p\left[\frac{\int_0^T \max_{|x|\le\xi_n} F(t,x)\,dt}{\xi_n^p} + \frac{\overline{\mu}}{\overline{\lambda}}\frac{\int_0^T \max_{|x|\le\xi_n} G(t,x)\,dt}{\xi_n^p} + \frac{m}{\overline{\lambda}}\frac{\Gamma_{\xi_n}}{\xi_n^{p-2}}\right].
\end{aligned}$$

Since $\lim_{n\to+\infty}\frac{\Gamma_{\xi_n}}{\xi_n^{p-2}}=0$, from the assumption (A3) and the condition (5), we have

$$\gamma \le \liminf_{n\to+\infty}\varphi(r_n)\le pc^p\Big(A+\frac{\overline{\mu}}{\overline{\lambda}}G_\infty\Big)<+\infty. \tag{6}$$

The assumption $\overline{\mu}\in(0,\mu_{G,\overline{\lambda}})$ immediately yields

$$\gamma \le pc^p\Big(A+\frac{\overline{\mu}}{\overline{\lambda}}G_\infty\Big)<pc^pA+\frac{1-pc^p\overline{\lambda}A}{\overline{\lambda}}.$$

Hence,

$$\overline{\lambda}=\frac{1}{pc^pA+\big(1-pc^p\overline{\lambda}A\big)/\overline{\lambda}}<\frac{1}{\gamma}.$$

Let $\overline{\lambda}$ be fixed. We claim that the functional $E_{\overline{\lambda},\overline{\mu}}$ is unbounded from below. Since

$$\frac{1}{\overline{\lambda}}<pc^pkB,$$

there exist a sequence $\{\eta_n\}$ of positive numbers and $\tau>0$ such that $\lim_{n\to+\infty}\eta_n=+\infty$ and

$$\frac{1}{\overline{\lambda}}<\tau<pc^pk\frac{\displaystyle\int_{T/4}^{3T/4}F(t,\eta_n)\,dt}{\eta_n^p} \tag{7}$$

for each $n\in\mathbb{N}$ large enough. For all $n\in\mathbb{N}$ define $w_n\in X$ by

$$w_n(t):=\begin{cases}\dfrac{4\eta_n}{T}t, & t\in[0,T/4],\\ \eta_n, & t\in]T/4,3T/4],\\ \dfrac{4\eta_n}{T}(T-t), & t\in]3T/4,T].\end{cases} \tag{8}$$

For any fixed $n\in\mathbb{N}$, one has

$$\Phi(w_n)=\left[\frac{4^p}{2pT^{p-1}}+\frac{(p+2)T}{2p(p+1)}\right]\eta_n^p=\frac{1}{pc^pk}\eta_n^p. \tag{9}$$

On the other hand, by (A2) and since G is nonnegative, from the definition of Ψ, we infer

$$\Psi(w_n)\ge\int_{T/4}^{3T/4}F(t,\eta_n)\,dt-\frac{1}{\overline{\lambda}}\frac{m}{\sqrt[p]{k^2}}\eta_n^2\,\Gamma_{(\eta_n/\sqrt[p]{k})}. \tag{10}$$

By (7), (9) and (10), we see that

$$\begin{aligned}E_{\overline{\lambda},\overline{\mu}}(w_n)&\le\frac{1}{pc^pk}\eta_n^p-\overline{\lambda}\int_{T/4}^{3T/4}F(t,\eta_n)\,dt+\frac{m}{\sqrt[p]{k^2}}\eta_n^2\,\Gamma_{(\eta_n/\sqrt[p]{k})}\\&<\frac{1}{pc^pk}\eta_n^p(1-\overline{\lambda}\tau)+\frac{m}{\sqrt[p]{k^2}}\eta_n^2\,\Gamma_{(\eta_n/\sqrt[p]{k})}\end{aligned}$$

for every $n\in\mathbb{N}$ large enough. Since $\sigma<1$, $\overline{\lambda}\tau>1$ and $\lim_{n\to+\infty}\eta_n=+\infty$, we have

$$\lim_{n\to+\infty}E_{\overline{\lambda},\overline{\mu}}(w_n)=-\infty.$$

Then, the functional $E_{\overline{\lambda},\overline{\mu}}$ is unbounded from below, and it follows that $E_{\overline{\lambda},\overline{\mu}}$ has no global minimum. Therefore, by Lemma 2.1(b), there exists a sequence $\{u_n\}$ of critical points of $E_{\overline{\lambda},\overline{\mu}}$ such that

$$\lim_{n\to+\infty} \|u_n\| = +\infty,$$

and the conclusion is achieved. □

Remark 3.1. Under the conditions $A = 0$ and $B = +\infty$, from Theorem 3.1 we see that for every $\lambda > 0$ and for each $\mu \in \left[0, \frac{1}{pc^p G_\infty}\right)$, problem (1) admits a sequence of classical solutions which is unbounded in X. Moreover, if $G_\infty = 0$, the result holds for every $\lambda > 0$ and $\mu \geq 0$.

Now, we present a concrete example of application of Theorem 3.1.

Example 3.1. Let $I(u(t_1)) = 1 - \sqrt[3]{u(t_1)}$ for some $t_1 \in (0,1)$. Then $I : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the sublinear growth condition (A1) with $\alpha = \beta = 1$ and $\sigma = \frac{1}{3}$. Now, put

$$a_n := \frac{2n!(n+2)! - 1}{4(n+1)!}, \qquad b_n := \frac{2n!(n+2)! + 1}{4(n+1)!},$$

for every $n \in \mathbb{N}$, and define the nonnegative continuous function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(\xi) = \begin{cases} \frac{32(n+1)!^2[(n+1)!^6 - n!^6]}{\pi} \sqrt{\frac{1}{16(n+1)!^2} - \left(\xi - \frac{n!(n+2)}{2}\right)^2}, & \text{if } \xi \in \bigcup_{n\in\mathbb{N}} [a_n, b_n], \\ 0, & \text{otherwise.} \end{cases}$$

One has

$$\int_{n!}^{(n+1)!} f(t)\,dt = \int_{a_n}^{b_n} f(t)\,dt = (n+1)!^6 - n!^6$$

for every $n \in \mathbb{N}$. Then, one has

$$\lim_{n\to+\infty} \frac{F(a_n)}{a_n^6} = 0, \quad \text{and} \quad \lim_{n\to+\infty} \frac{F(b_n)}{b_n^6} = 64.$$

Therefore, by a simple computation, we obtain

$$\liminf_{\xi\to+\infty} \frac{F(\xi)}{\xi^6} = 0, \quad \text{and} \quad \limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^6} = 64.$$

Also, let

$$g(t) = \begin{cases} 0, & \text{if } t < 0, \\ \frac{t^5}{32}, & \text{if } t \geq 0. \end{cases}$$

Putting $G(t) := \int_0^t g(\xi)\,d\xi$ for all $t \in \mathbb{R}$, we have $\limsup_{\xi\to+\infty} \frac{G(\xi)}{\xi^6} = \frac{1}{192}$. So, for every $\lambda > \frac{1195}{112}$ and $\mu \in [0,1)$, the nonlinear problem

$$\begin{cases} -(|u'(t)|^4 u'(t))' + |u(t)|^4 u(t) = \lambda f(u(t)) + \mu g(u(t)), & \text{a.e. in } [0,1], \\ u(0) = u(1) = 0, \\ \Delta |u'(t_1)|^4 u'(t_1) = 1 - \sqrt[3]{u(t_1)}, \end{cases}$$

has a sequence of classical solutions which is unbounded in $W_0^{1,6}(0,1)$.

Now we state several useful consequences and particular cases of Theorem 3.1.

Corollary 3.1. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function. Suppose that hypotheses* (A1), (A2) *are fulfilled and*

$$A < \frac{1}{pc^p}, \qquad B > \frac{1}{pc^p k}.$$

Then, for every arbitrary L^1-Carathéodory function $g : [0,T] \times \mathbb{R} \to \mathbb{R}$, whose potential $G(t,x) := \int_0^x g(t,\xi)\,d\xi$ for all $(t,x) \in [0,T] \times \mathbb{R}$, is a nonnegative function satisfying the condition (5), *if we put*

$$\mu_G := \frac{1}{pc^p G_\infty}\left(1 - pc^p A\right),$$

where $\mu_G = +\infty$ when $G_\infty = 0$, the problem

$$\begin{cases} -(|u'(t)|^{p-2} u'(t))' + |u(t)|^{p-2} u(t) = f(t,u(t)) + \mu g(t,u(t)), & \text{in } \Omega, \\ u(0) = u(T) = 0, \\ \Delta |u'(t_j)|^{p-2} u'(t_j) = I_j(u(t_j)), \quad j = 1,2,\ldots,m, \end{cases}$$

has an unbounded sequence of classical solutions for every $\mu \in [0, \mu_G)$ in X.

Corollary 3.2. *Let* (A1) *holds and $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function. Put $F(\xi) := \int_0^\xi f(t)\,dt$ for all $\xi \in \mathbb{R}$ and assume that*

$$\liminf_{\xi\to+\infty} \frac{F(\xi)}{\xi^3} < \frac{4T^2}{c^3(5T^3+256)} \limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^3}.$$

Then, for each

$$\lambda \in \left] \frac{5T^3+256}{(12T^2)\limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^3}}, \frac{1}{(3c^3)\liminf_{\xi\to+\infty} \frac{F(\xi)}{\xi^3}} \right[,$$

for every arbitrary nonnegative continuous function $g : \mathbb{R} \to \mathbb{R}$, whose potential $G(\xi) := \int_0^\xi g(t)\,dt$ for all $\xi \in \mathbb{R}$, satisfies the condition

$$\limsup_{\xi\to+\infty} \frac{G(\xi)}{\xi^3} < +\infty,$$

and for every

$$\mu \in \left[0, \frac{1}{(3Tc^3)\limsup_{\xi\to+\infty}\frac{G(\xi)}{\xi^3}}\left(1-(3Tc^3)\lambda\liminf_{\xi\to+\infty}\frac{F(\xi)}{\xi^3}\right)\right[,$$

the nonlinear problem

$$\begin{cases} -(|u'(t)|u'(t))' + |u(t)|u(t) = \lambda f(u(t)) + \mu g(u(t)), & \text{in } \Omega, \\ u(0) = u(T) = 0, \\ \Delta|u'(t_j)|u'(t_j) = I_j(u(t_j)), & j = 1, 2, \dots, m, \end{cases}$$

has a sequence of classical solutions which is unbounded in $W_0^{1,3}(0,T)$.

Corollary 3.3. *Let* (A1) *holds and let* $f : \mathbb{R} \to \mathbb{R}$ *be a nonnegative continuous function. Put* $F(\xi) := \int_0^\xi f(t)\,dt$ *for all* $\xi \in \mathbb{R}$ *and assume that*

$$\liminf_{\xi\to+\infty}\frac{F(\xi)}{\xi^p} = 0, \qquad \limsup_{\xi\to+\infty}\frac{F(\xi)}{\xi^p} = +\infty.$$

Then, for every arbitrary nonnegative continuous function $g : \mathbb{R} \to \mathbb{R}$ *satisfying the condition*

$$g_\infty := \lim_{\xi\to+\infty}\frac{\int_0^\xi g(x)\,dx}{\xi^p} < +\infty,$$

and for every $\mu \in [0, \frac{1}{Tpc^p g_\infty}[$, *the problem*

$$\begin{cases} -(|u'(t)|^{p-2}u'(t))' + |u(t)|^{p-2}u(t) = f(u(t)) + \mu g(u(t)), & \text{in } \Omega, \\ u(0) = u(T) = 0, \\ \Delta|u'(t_j)|^{p-2}u'(t_j) = I_j(u(t_j)), & j = 1, 2, \dots, m, \end{cases}$$

admits infinitely many distinct pairwise classical solutions.

Now, we point out a special situation of our main result when $\mu = 0$ and the nonlinear term has separated variables. To be precise, let $l \in L^1([0,T])$ such that $l(t) \geq 0$ a.e. $t \in [0,T]$, $l \not\equiv 0$, and let $h : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function.

Consider the following nonlinear Dirichlet boundary-value problem

$$\begin{cases} -u''(t) + u(t) = \lambda l(t)h(u(t)), & \text{in } \Omega, \\ u(0) = u(T) = 0, \\ \Delta u'(t_j) = I_j(u(t_j)), & j = 1, 2, \dots, m. \end{cases} \tag{11}$$

Put $H(\xi) := \int_0^\xi h(x)\,dx$ for all $\xi \in \mathbb{R}$, and set

$$\|l\|_1 := \int_0^T l(t)\,dt, \qquad l_0 := \int_{T/4}^{3T/4} l(t)\,dt.$$

Corollary 3.4. *Let (A1) holds. Moreover, suppose that*

$$\liminf_{\xi\to+\infty}\frac{H(\xi)}{\xi^2}<\frac{3Tl_0}{c^2(24+2T^2)\|l\|_1}\limsup_{\xi\to+\infty}\frac{H(\xi)}{\xi^2}.$$

Then, for each

$$\lambda\in\left]\frac{12+T^2}{(3Tl_0)\limsup\limits_{\xi\to+\infty}\frac{H(\xi)}{\xi^2}},\frac{1}{(2c^2\|l\|_1)\liminf\limits_{\xi\to+\infty}\frac{H(\xi)}{\xi^2}}\right[,$$

problem (11) *has an unbounded sequence of classical solutions.*

Put

$$\begin{aligned}\Im_a &:= \sum_{j=1}^m \min_{|\xi|\le a}\int_0^\xi I_j(x)\,dx, \qquad \text{for all } a>0,\\ A' &:= \liminf_{\xi\to 0^+}\frac{\int_0^T \max\limits_{|x|\le\xi}F(t,x)\,dt}{\xi^p},\\ B' &:= \limsup_{\xi\to 0^+}\frac{\int_{T/4}^{3T/4} F(t,\xi)\,dt}{\xi^p},\end{aligned}$$

and

$$\lambda_1':=\frac{1}{pc^pkB'},\qquad \lambda_2':=\frac{1}{pc^pA'}.$$

Using Lemma 2.1(c) and arguing as in the proof of Theorem 3.1, we can obtain the following multiplicity result.

Theorem 3.2. *Let $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ be an L^1-Carathéodory function and $I_j(x)\le 0$ for all $x\in\mathbb{R}$, $j=1,\dots,m$. Moreover, assume that* (A2) *and*

(A4) $A'<kB'$.

are satisfied. Then, for every $\lambda\in(\lambda_1',\lambda_2')$ and for every arbitrary L^1-Carathéodory function $g:[0,T]\times\mathbb{R}\to\mathbb{R}$, whose potential $G(t,x):=\int_0^x g(x,\xi)\,d\xi$ for all $(t,x)\in[0,T]\times\mathbb{R}$, is a nonnegative function satisfying the condition

$$G_0:=\limsup_{\xi\to 0^+}\frac{\int_0^T \max\limits_{|x|\le\xi}G(t,x)\,dt}{\xi^p}<+\infty, \tag{12}$$

if we put

$$\mu_{G,\lambda}':=\frac{1}{pc^pG_0}\left(1-\lambda pc^pA'\right),$$

where $\mu_{G,\lambda}'=+\infty$ when $G_0=0$, for every $\mu\in[0,\mu_{G,\lambda}')$ problem (1) *has a sequence of classical solutions, which strongly converges to zero in X.*

Proof. Fix $\overline{\lambda} \in (\lambda'_1, \lambda'_2)$ and let g be a function that satisfies the condition (12). Since $\overline{\lambda} < \lambda'_2$, we obtain

$$\mu'_{G,\overline{\lambda}} := \frac{1}{pc^p G_0}\left(1 - \overline{\lambda} pc^p A'\right) > 0.$$

Now fix $\overline{\mu} \in (0, \mu'_{G,\overline{\lambda}})$ and set

$$J(t,x) := F(t,x) + \frac{\overline{\mu}}{\overline{\lambda}} G(t,x),$$

for all $(t,x) \in [0,T] \times \mathbb{R}$. We take Φ, Ψ and $E_{\overline{\lambda},\overline{\mu}}$ as in the proof of Theorem 3.1. Now, as it has been pointed out before, the functionals Φ and Ψ satisfy the regularity assumptions required in Lemma 2.1. As first step, we will prove that $\overline{\lambda} < 1/\delta$. Then, let $\{\xi_n\}$ be a sequence of positive numbers such that $\lim_{n\to+\infty} \xi_n = 0$ and

$$\lim_{n\to+\infty} \frac{\displaystyle\int_0^T \max_{|x|\le\xi_n} F(t,x)\, dx}{\xi_n^p} = A'.$$

By the fact that $\inf_X \Phi = 0$ and the definition of δ, we have $\delta = \liminf_{r\to0^+} \varphi(r)$. Put $r_n := \frac{1}{pc^p}\xi_n^p$ for all $n \in \mathbb{N}$. Then, for all $v \in X$ with $\Phi(v) < r_n$, taking (2) into account, one has $\|v\|_\infty < \xi_n$. Thus, for all $n \in \mathbb{N}$,

$$\begin{aligned}
\varphi(r_n) &\le \frac{\displaystyle\sup_{v\in\Phi^{-1}(-\infty,r_n)} \Psi(v)}{r_n} \\
&\le \frac{\displaystyle\int_0^T \max_{|x|\le\xi_n} J(t,x)\, dt - \frac{1}{\overline{\lambda}}\Im_{\xi_n}}{\frac{1}{pc^p}\xi_n^p} \\
&\le pc^p \left[\frac{\displaystyle\int_0^T \max_{|x|\le\xi_n} F(t,x)\, dt}{\xi_n^p} + \frac{\overline{\mu}}{\overline{\lambda}} \frac{\displaystyle\int_0^T \max_{|x|\le\xi_n} G(t,x)\, dt}{\xi_n^p} - \frac{1}{\overline{\lambda}} \frac{\Im_{\xi_n}}{\xi_n^p} \right].
\end{aligned}$$

Since $\lim_{n\to+\infty} \frac{\Im_{\xi_n}}{\xi_n^p} = 0$, from the assumption (A4) and the condition (12), we have

$$\delta \le \liminf_{n\to+\infty} \varphi(r_n) \le pc^p \left(A' + \frac{\overline{\mu}}{\overline{\lambda}} G_0 \right) < +\infty.$$

From $\overline{\mu} \in (0, \mu'_{G,\overline{\lambda}})$, the following inequalities hold

$$\delta \le pc^p \left(A' + \frac{\overline{\mu}}{\overline{\lambda}} G_0 \right) < pc^p A' + \frac{1 - \overline{\lambda} pc^p A'}{\overline{\lambda}}.$$

Therefore

$$\overline{\lambda} = \frac{1}{pc^p A' + \left(1 - pc^p \overline{\lambda} A'\right)/\overline{\lambda}} < \frac{1}{\delta}.$$

Let $\overline{\lambda}$ be fixed. We claim that the functional $E_{\overline{\lambda},\overline{\mu}}$ does not have a local minimum at zero. Since

$$\frac{1}{\overline{\lambda}} < pc^p kB',$$

there exists a sequence $\{\eta_n\}$ of positive numbers and $\tau > 0$ such that $\lim_{n\to+\infty} \eta_n = 0$ and

$$\frac{1}{\overline{\lambda}} < \tau < pc^p k \frac{\displaystyle\int_{T/4}^{3T/4} F(t,\eta_n)\, dt}{\eta_n^p}$$

for each $n \in \mathbb{N}$ large enough. For all $n \in \mathbb{N}$, let $w_n \in X$ defined by (8) with the above η_n. Note that $\overline{\lambda}\tau > 1$. Then, since $I_j(x) \leq 0$ for all $x \in \mathbb{R}$, $j = 1, \ldots, m$, we obtain

$$\begin{aligned} E_{\overline{\lambda},\overline{\mu}}(w_n) &\leq \frac{1}{pc^p k}\eta_n^p - \overline{\lambda}\int_{T/4}^{3T/4} F(t,\eta_n)\, dt + \sum_{j=1}^{m} \int_0^{w_n(t_j)} I_j(x)\, dx \\ &< \frac{1}{pc^p k}\eta_n^p(1 - \overline{\lambda}\tau) < 0, \end{aligned}$$

for every $n \in \mathbb{N}$ large enough. Then, since

$$\lim_{n\to+\infty} E_{\overline{\lambda},\overline{\mu}}(w_n) = E_{\overline{\lambda},\overline{\mu}}(0) = 0,$$

we see that zero is not a local minimum of $E_{\overline{\lambda},\overline{\mu}}$. This, together with the fact that zero is the only global minimum of Φ, we deduce that the energy functional $E_{\overline{\lambda},\overline{\mu}}$ does not have a local minimum at the unique global minimum of Φ. Therefore, by Lemma 2.1(c), there exists a sequence $\{u_n\}$ of critical points of $E_{\overline{\lambda},\overline{\mu}}$ which converges weakly to zero. In view of the fact that the embedding $X \hookrightarrow C^0([0,T])$ is compact, we know that the critical points converge strongly to zero, and the proof is complete. □

Remark 3.2. Applying Theorem 3.2, results similar to Corollaries 3.1, 3.2, 3.3 and 3.4 can be obtained. We omit the discussions here.

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