ABOUT MOLECULES IN DISTRIBUTIVE LATTICES (I)

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Scopul acestei lucrări este de a prezenta unele proprietăți ale moleculelor în laticile distributive finite.

The aim of this paper is to present some properties of molecules in finite distributive latices

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1. Definitions. Preliminaries

Let $A$ be a bounded distributive lattice.

Definition 1.1 In $A$ an element $x$ is an atom if $x \neq 0$ and $y \leq x \Rightarrow y = 0$ or $y = x$. An element $x$ is join-irreducible if $x = y \lor z \Rightarrow x = y$ or $x = z$. Note by $J(A)$ the set of nonzero join – irreducible elements and by $At(A)$ the set of atoms of $A$.

We have $At(A) \subseteq J(A)$

If $A$ is a Boolean algebra $At(A) = J(A)$, and this equality characterizes finite Boolean algebras.

Another generalization of atoms was introduced by Abian [1].

Definition 1.2 [1] An element $m$ in $A$ is a molecule if $m \neq 0$ and $x, y \leq m$, $x, y \neq 0 \Rightarrow x \land y \neq 0$. Note by $M(A)$ the set of molecules. We have $At(A) \subseteq M(A)$ and in a Boolean algebra $At(A) = M(A)$.

There are no relations between molecules and join-irreducible elements in arbitrary lattices.

The notion of molecule was studied by Yaqub [6] in Postalgebras. He proved.

Proposition 1.3 [6] In a Postalgebra $A$ the following conditions are equivalent

(i) $m$ is a molecule

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and the principal ideal \((m)\) is prime \(\Leftrightarrow \overline{m}\) is a molecule.

This means \(\overline{(m)} = [\overline{-m}]\) is a prime filter \(\Leftrightarrow \overline{m}\) is a molecule so \(m\) is a molecule \(\Leftrightarrow m \in J(A)\).

Hence in Postalgebras molecule and join-irreducible element is a same notion. We have the following generalization.

**Proposition 1.4**[3] In a \(n\)-valued Lukasiewicz-Moisil algebra \(A\) (an algebra without negation) the following are equivalent:

(i) \(m\) is a molecule;
(ii) \(\varphi_{m^{-1}} m \in \text{AtC}(A)\)
(iii) \(m \in J(A)\)

But \(\varphi_{m^{-1}} : A \to A\) is a Boolean multiplicative closure, that is a multiplicative closure operator and \(\text{Im} \varphi_{m^{-1}} = \text{C}(A)\). We have the following

**Proposition 1.5**[5] If \(A\) is a bounded distributive lattice, the following are equivalent:

(i) \(A\) has a Boolean multiplicative closure
(ii) \(A\) is a Stone Algebra (That is \(A\) is pseudocomplemented lattice such that \(x^* \vee x^{**} = 1\), for any \(x \in A\). \(x^*\) is the pseudocomplement of \(x\), and \(x \mapsto x^{**}\) is the Boolean multiplicative closure.

Then the Proposition 1.4 allows us the following generalization:

**Proposition 1.6:** In a Stone algebra \(A\) the following are equivalent:

(i) \(m\) is a molecule
(ii) \(m^{**} \in \text{AtC}(A)\)
\(m \in J(A) \Rightarrow m\) is a molecule.

**Proof.** (i) \(\Rightarrow\) (ii) If \(m^{**}\) is a not an atom in \(C(A)\) there is \(a \in C(A)\), such that \(0 < a < m^{**}\), so \(m^{**} \notin a\), that is \(m^{**} \wedge a^* \neq 0\). Hence \((m \wedge a^*)^{**} \neq 0\). Therefore \(m \wedge a^* \neq 0\). But \((m \wedge a)^{**} = m^{**} \wedge a = a \neq 0\) so \(m \wedge a \neq 0\). Since \(m\) is molecule, it follows that \(m \wedge a \wedge a^* \neq 0\), a contradiction.

(ii) \(\Rightarrow\) (i): Consider \(x, y \neq 0, x, y \leq m\). We have \(x^{**} \neq 0\) and \(y^{**} \neq 0\), so \(x^{**} = y^{**} = m^{**}\). This implies \((x \wedge y)^{**} = x^{**} \wedge y^{**} \neq 0\). \(x \wedge y \neq 0\).

If \(m \in J(A), [m]\) is a prime filter. Consider \([m^{**}]_{\text{C}(A)}\) and \(x, y \in C(A)\), such
that \( x \lor y \geq m^* \). Therefore \( x \lor y \geq m \). Since \( \left[ m \right] \) is prime it follows \( x \geq m \), or \( y \geq m \) so \( x \geq m^* \) or \( y \geq m^* \). Hence \( m^* \) is prime. So \( m^* \in J(C(A)) = AtC(A) \).

Therefore in a Stone algebra \( J(A) \subseteq M(A) \). In the next section we prove the converse of Propositions 1.4 and 1.6 in the finite case.

2. Molecules in finite distributive lattices

In a infinite distributive lattice there are no relations between molecules and atoms. But in a finite lattice the situation is very different and we have a satisfactory characterization of molecules.

Consider now \( A \) a finite distributive lattice.

**Proposition 2.1** \( m \in A \) is a molecule iff \( (\exists a)(a \in At(A) \text{and } a \leq m) \)

**Proof.** If \( m \in M(A) \) then there exists an atom \( a \leq m \), since \( A \) is finite. If \( b \leq m, b \in At(A) \) and \( b \neq a \) we have \( a \land b = 0 \), a contradiction.

If \( m \) is an element such that there exists an unique atom \( a, a \leq m \), consider \( 0 \neq x \leq m \). We have an atom \( b \leq x \) so \( b \leq m \). Hence \( a = b \). If \( x, y \leq m, x, y \neq 0 \), then \( a \leq x \land y \), so \( x \land y \neq 0 \).

If \( a \in At(A) \) note by \( M_a = \{ m \in M \mid \text{supp} m = a \} \), where \( \text{supp} m \) is the unique atom of Proposition 2.1

**Proposition 2.2** The following properties hold:

(i) \( x \in M_a, 0 \neq y \leq x \Rightarrow y \in M_a \)

(ii) \( x, y \in M_a \Rightarrow x \land y \in M_a \)

(iii) \( x, y \in M_a \Rightarrow x \lor y \in M_a \)

(iv) \( M_a \) is a sub lattice, where \( a \) is the least element

(v) \( x \in M_a, x \in M_b \ a \neq b \Rightarrow x \land y = 0 \)

\( M = + M_a \) (cardinal sum).

**Proof**:  

(i) If \( x \in M_a \) then \( \text{supp} x = a \), and \( b \leq y \),  
\( b \in At(A) \Rightarrow b \leq x \Rightarrow b = a \Rightarrow y \in M_a \)

(ii) By (i)

(iii) Consider an atom \( b \leq x \lor y \). As \( b \) is join-irreducible, we have \( b \leq x \) or
\[ b \leq y, \text{ so } b = a \]

(iv) By (ii) and (iii)

(v) If \( x \land y \neq 0 \), then \( x \land y \in M_a, M_b \), a contradiction.

**Lemma 2.3** We have the following cases:

(a) \( \text{card } At(A) = 1 \Rightarrow A \) is a Stone algebra.

(b) \( \text{card } At(A) > 1 \). The following are equivalent:

(i) \( J(A) \subseteq M(A) \)

(ii) \( [M]_0 = A \)

(iii) For any \( x \neq 0 \), there exists a set \( \{x_1, \ldots, x_n\} \subseteq M \) such that

\[ i \neq j \Rightarrow x_i \land x_j = 0 \quad \text{and} \quad x = \bigvee_{i=1}^n x_i \text{ or } x \in M \]

\[ 1 = \bigwedge_{a \in At(A)} 1_a, \text{ where } 1_a \text{ is the greatest element in } M_a \]

\[ A \cong \prod_{a \in At} [I_a] \]

(v) \( A \) is a Stone algebra

**Proof:**

If \( A \) has an unique atom, \( A \) is a dense lattice, that is a Stone algebra.

Suppose now, \( \text{card } At(A) > 1 \).

(i) \( \Rightarrow \) (ii): Any \( x \in A, x \neq 0 \) is a join of nonzero join-irreducible elements, because \( A \) is finite, and \( 0 = a_1 \land a_2, a_1, a_2 \in At(A) \) so \( [J(A)]_0 = A \)

(ii) \( \Rightarrow \) (iii): By Proposition 2.2 (ii), (iii), (v)

(iii) \( \Rightarrow \) (iv): By Proposition 2.2 (iv) \( M_a \) is a finite lattice, that is it has a greatest element

(iv) \( \Rightarrow \) (v): The set \( \{1_a\}_{a \in At(A)} \), contains disjunct elements with join 1, so we have the decomposition

\[ A = \prod_{a \in At} [I_a] \]  

(\text{By [2], p.68})

(v) \( \Rightarrow \) (vi): \( A \big| [I_a] \cong 1_a = 0 \oplus M_a \) as \( M_a \) is a lattice (Proposition 2.2 (iv)) it follows that the factors are dense lattices, hence Stone Algebras

(vi) \( \Rightarrow \) (i): By Proposition 1.6

**Corollary 2.4** For a finite distributive lattice, the following are equivalent:

(i) \( A \) is a Stone algebra

(ii) \( J(A) \subseteq M(A) \)
And the decomposition of Lemma 2.3 (iii) is unique

**Proposition 2.5** In a finite distributive lattice, \( J(A) \supseteq M(A) \) iff \( M_a \) is a chain for any \( a \in \text{At}(A) \)

**Proof:**

If any molecule is join-irreducible, consider \( x, y \in M_a \). By Proposition 2.2 (iii) \( x \vee y \in M_a \), so \( x \vee y \in J(A) \). Hence \( x \vee y = x \) or \( x \vee y = y \) and \( x, y \) are comparable. If \( M_a \) is a chain, consider \( x \in M_a \). \( x \) is a join of nonzero join – irreducible elements. By Proposition 2.2 (i), these elements belong to \( M_a \), and their join is a join – irreducible element.

**Theorem 2.6** If \( A \) is finite distributive lattice, the following are equivalent:

(i) \( J(A) = M(A) \)

(ii) \( A \) has a structure of Lukasiewicz-Moisil algebra

(iii) \( J(A) = +L_i \) where \( L_i \) are chains

**Proof:**

(i) \( \Rightarrow \) (ii): By Lemma 2.3 \( A \) is direct product of factors \( 0 \oplus M_a \) and by Proposition 2.5 \( M_a \) are chains, so \( A \) is direct product of chains and has a structure of Lukasiewicz-Moisil algebra

(ii) \( \Rightarrow \) (iii): By [4], p.277

(iii) \( \Rightarrow \) (i): If \( x \in J(A) \), then \( x \) belongs to a unique maximal chain \( L_i \); so the least element in \( L_i \) is an atom \( a \) and \( x \geq a \) (\( a \) is unique by hypothesis). Hence \( x \in M_a \).

If \( x \in M \), then there exist an unique atom \( a \), such that \( x \in M_a \), and \( x \) is a join of nonzero join – irreducible elements in \( M_a \). These elements are in \( L_i \) for some \( i \), so their join is in \( L_i \).

**Remark** Proposition 2.1, 2.2 and Lemma 2.3 hold if \( A \) is a distributive lattice that satisfies the descending chain condition.

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