A NEW SCHEME FOR SOLVING OPTIMAL CONTROL OF THE VOLTERRA INTEGRAL EQUATIONS VIA BERNSTEIN'S APPROXIMATION

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In this study, a novel method is proposed to approximate the solution of optimal control problems governed by Volterra integral equations. The method is based on the Bernstein and parametrization approaches for discretizing of the problem. Several numerical examples are presented to show the efficiency and reliability of the proposed method.

Keywords: Optimal control problem; Volterra integral equation; Numerical method ; Bernstein's approximation.

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1. Introduction

Optimal control problems(OCPs) appear in a wide class of applications. It is well known that there are two main class of OCPs. The first class contains OCPs which governed by differential equations and the second class contains those governed by integral equations. The classical theory of optimal control was originally developed to deal with the problems of first class. But there are many problems in economics, biology, epidemiology and memory effects which belong to the second class.

During the past three decades, different types of techniques have been proposed for solving OCPs governed by the Volterra integral equations(VIEs).

Belbas has suggested a method based on parametrization of Hamilton Jacobi function and discretization of the original Volterra controlled system for optimal control of VIEs (See [3]) and in [4], proposed a method to solve OCPs of VIEs based on approximating the controlled VIEs by a sequence of systems of controlled ordinary differential equations.

The necessary and sufficient conditions for existence of solution to optimal control of VIEs have been considered in [16]. Existence and uniqueness of solution of the optimal control of systems governed by VIEs can be found in [1]. Recently homotopy perturbation method (HPM) [9], Legendre polynomials [15] and a hybrid method based on steepest descent and Newton methods [12] have been used for solving OCPs governed by VIEs. In this paper we propose a method based on combining Bernstein's approximation [10, 2] and control parameterization [11] where control

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and trajectory functions are considered as polynomials with unknown coefficients. Then a nonlinear programming is solved to determine the unknown coefficients.

The paper is organized as follows. In Section 2, Bernstein's approximation for solving OCP governed by VIE is presented. The convergence of the approach is investigated. in Section 3. In Section 4, numerical examples are presented.

2. Bernstein's approximation for OCP governed by VIE

In this section we present a numerical scheme to obtain an approximate solution for a class of OCP governed by VIE by using Bernstein's polynomials and parametrized control.
An OCP governed by VIE is formulated as the following minimization problem (See [9]).

$$\text{Minimize } J(u) = \int_0^T f(t, x(t), u(t))dt, \tag{1}$$

subject to:

$$x(t) = y(t) + \lambda \int_0^t k(s, t, u(s))x(s)ds, \ t \in [0, T] \tag{2}$$

where $k \in C([0,T] \times [0,T] \times W)$, $y \in C([0,T])$, $\lambda \in \mathbb{R}$ and $f \in C([0,T] \times \mathbb{R} \times W)$, in which $W \subseteq \mathbb{R}^n$ is a compact set. It is known that for a given continuous control $u(s)$, the VIE has a unique and continuous solution.
To determine an approximate solution of Eq. (2), we first define the following concepts.

Definition 2.1. A sequence $\{\varphi_l\}_0^\infty$ in $C([0,T])$ is called a basis for every f in $C([0,T])$, if there exists a unique sequence $\{\beta_l\}_0^\infty$ of real numbers such that:

$$f = \sum_{l=0}^{\infty} \beta_l \varphi_l. \tag{3}$$

A basis $\{\varphi_l\}_0^\infty$ of polynomials is called polynomial basis.

The Bernstein polynomial is defined as

$$B(t) = \sum_{i=0}^{n} a_i P_{n,i}(t), \tag{4}$$

where $P_{n,i}(t)(i = 0, 1, ..., n)$ are Bernstein basis polynomials of degree n defined on $[0, T]$, as

$$P_{n,i}(t) = \binom{n}{i} \frac{t^i (T - t)^{n-i}}{T^n}, \ i = 0, \cdots, n, \tag{5}$$

and $a_i(i = 0, 1, ..., n)$ are called the Bernstein coefficients. Without loss of generality we suppose $T = 1$.

Definition 2.2. The Bernstein's approximation of order n, $B_n(x)$ to a function $x : [0, 1] \to \mathbb{R}$ is the polynomial

$$B_n(x(t)) = \sum_{i=0}^{n} a_i P_{n,i}(t), \tag{6}$$

where $P_{n,i}$ is Bernstein basis polynomials of degree n, and $a_i = x(\frac{i}{n})$.

Theorem 2.1. For all functions x in $C[0,1]$, the sequence $\{B_n(x); n = 1, \cdots\}$ converges uniformly to x.
Proof: See [14].

Let $u(t) \in C([0,1])$ be a given function, By using theorem 1, we approximate $x(t)$ as

$$\mathrm{x(t)} = \sum_{\mathrm{i=0}}^{\mathrm{n}} \mathrm{a_i P_{n,i}(t)}. \tag{7}$$

Obviously by substituting (7) in Eq. (2) we have

$$\sum_{\mathrm{i=0}}^{\mathrm{n}} \mathrm{a_i} \ (\binom{\mathrm{n}}{\mathrm{i}} \ (\mathrm{t^i(1-t)^{n-i}} - \lambda \int_0^{\mathrm{t}} \mathrm{k(s,t,u(s))\ s^i(1-s)^{n-i}ds})\ ) = \mathrm{y(t)}. \tag{8}$$

Assume that t_0 and t_n are taken, respectively, near to 0 and 1, such that $0 < t_0 < t_n < 1$. We let

$$t_j = t_0 + j\frac{\Delta}{n},\ j = 0, \cdots, n,\ \Delta = t_n - t_0.$$

Substituting $t = t_j,\ j = 0, ..., n$ in the Volterra integration equation (8), we obtain:

$$\mathrm{E^n\ a = z}, \tag{9}$$

where the $(n+1) \times (n+1)$ matrix $E^n(u) = [{}^n e_{i,j}]$ is defined as follows:

$$\mathrm{{}^n e_{i,j}} = \binom{\mathrm{n}}{\mathrm{i}} \ (\mathrm{t_j{}^i(1-t_j)^{n-i}} - \lambda \int_0^{\mathrm{t_j}} \mathrm{k(s,t_j,u(s))\ s^i(1-s)^{n-i}ds}), \tag{10}$$

$$a = [a_0, \cdots a_n]^T,\ z = [y(t_0), \cdots, y(t_n)]^T.$$

By solving system (9) and computing $a_i, (i = 0, \cdots, n)$, an approximate solution of Eq. (2) is defined as follows:

$$\mathrm{x_n(t;u)} := \sum_{\mathrm{i=0}}^{\mathrm{n}} \mathrm{a_i P_{n,i}(t)} \tag{11}$$

In notation $x_n(t; u)$, u is appeared to emphasize that the solution is obtained for the certain u.
Let $\{s^l\}_0^\infty$ be a basis for $C[0,1]$. To obtain an appropriate continuous control function, we use the following approximation:

$$\mathrm{u_k(s)} = \sum_{\mathrm{l=0}}^{\mathrm{k}} \mathrm{b_l s^l}. \tag{12}$$

By substituting Eq. (12) in system (9), we obtain the following nonlinear system

$$\mathrm{E_k^n(b)\ a = z}, \tag{13}$$

where $E_k^n(b) = [{}_k^n\bar{e}_{i,j}]$ is defined as follows:

$${}_k^n\bar{e}_{i,j} = \binom{n}{i} \ ({t_j}^i(1-t_j)^{n-i} - \lambda \int_0^{t_j} k(s, t_j, \sum_{l=0}^{k} b_l s^l)\ s^i(1-s)^{n-i}ds),$$

$$a = [a_0, \cdots a_n]^T,\ z = [y(t_0), \cdots, y(t_n)]^T,\ b = [b_0, \cdots, b_n]^T.$$

The approximate control and state functions can be obtained by solving the following problem

$$\mathrm{Minimize}_{a,b}\ \bar{J}_k(a, b) = \int_0^1 f(t, \sum_{i=0}^{n} a_i P_{n,i}(t), \sum_{l=0}^{k} b_l t^l)dt, \quad (14)$$

subject to:

$$E_k^n(b)\ a = z.$$

The optimization problem is solved with the optimization toolbox in MATLAB to obtain the unknown coefficients a and b. By substituting these coefficients in Eqs. (11) and (12), respectively, the approximate trajectory and control functions are computed. Assuming $\bar{J}_k$ as optimal value of (14) in the kth iteration, a stopping criterion is considered in the following relation:

$$|\bar{J}_k - \bar{J}_{k-1}| < \epsilon, \quad (15)$$

where small positive number ϵ can be chosen according to the desired accuracy. In the sequel based on above results the algorithm is described in two stages, initialization step and main step.

Initialization step: Choose $\epsilon > 0$ for the accuracy, i_n and i_k for maximum iteration of n and k respectively. Set $n = 3$, $k = 1$ and go to the main step.

Main step:

Step 1. Compute $\bar{J}_k$ by (14) . If $k = 1$ go to step 3; Otherwise go to step 2.

Step 2. If the stopping criterion (15) holds, then stop; Otherwise go to step 3.

Step 3. Set $k = k + 1$. if $k \leq i_k$, go step 1; Otherwise go to step 4.

Step 4. Set $n = n + 1$. If $n > i_n$, then stop; Otherwise set $k = 1$ and go to step 1.

3. Convergence of the approach

In this section we study the convergence of the mentioned approach.

The set of admissible control functions is defined as follows:

$$U = \{u : [0, 1] \longrightarrow W|\ u(.) \in C[0, 1]\}.$$

Definition 3.1. A trajectory-control pair $(x(.), u(.))$ is called admissible if the following conditions hold:

i): The trajectory function $x(.)$ is continuous in $[0, 1]$.

ii): The control function $u(.)$ is continuous in $[0, 1]$ and $u(.) \in U$.

iii): The pair $(x(.), u(.))$ satisfies in Eq. (2).

Let $\xi \subseteq C([0, 1]) \times C([0, 1])$ denote the set of all admissible pairs $(x(.), u(.))$. We define ξ^n and ξ_k^n as follows:

$$\xi^n := \{(x_n(.; u), u(.))|\ u \in U\},\ \xi_k^n := \{(x_n(.; u_k), u_k(.))|\ u_k \in P_k \cap U\}, \quad (16)$$

where P_k is the set of all polynomials of degree at most k.

Let

$$I(x, u) := \int_0^1 f(t, x(t), u(t))dt,$$

thus the problem can be converted to the following problem:

$$inf_{(x,u)\in\xi} I(x, u).$$

We define

$$\alpha_k^n := \inf_{(x_n,u_k)\in\xi_k^n} I(x_n,u_k),\ \alpha^n := \inf_{(x_n,u)\in\xi^n} I(x_n,u)$$

Assumption 3.1. We assume $\alpha_k^n,\ \alpha^n$ exists for all $n,k\in N$.
Assumption 3.2.
i) $\ddot{x}(.;u)$ is a uniformly bounded function. It means

$$\exists\ c_1>0,\ \forall u\in U:\|\ddot{x}(.;u)\|\leqslant c_1.$$

ii) The sequence $\{\ddot{x_n}(.;u)\}$ is a uniformly bounded sequence. It means

$$\exists M_1,\forall u\in U,n\geqslant M_1:\|\ddot{x_n}(.;u)\|\leqslant M_1.$$

iii) $\|(E^n)^{-1}(u)\|$ is a uniformly bounded sequence, which means

$$\exists M_2,\forall u\in U,n\geqslant M_2:\|(E^n)^{-1}(u)\|\leqslant M_2.$$

Lemma 3.1. The following relation holds

$$\alpha_1^n\geq\alpha_2^n\geq\cdots\geq\alpha_k^n\geq\cdots\geq\alpha^n.$$

Proof: By relations (16)

$$\xi_1^n\subseteq\xi_2^n\subseteq,\cdots,\subseteq\xi_k^n\cdots\subseteq\xi^n,$$

hence by infimum property the relation is established. □

Lemma 3.2. The following equality holds

$$\lim_{k\to\infty}\ \alpha_k^n=\alpha^n.$$

Proof: Since $\{\alpha_k^n\}$ is a non-increasing and bounded sequence, then it is convergent. Let $\tilde{\alpha}$ be the limit point of the non-increasing sequence $\{\alpha_k^n\}$, as $k\to\infty$. From Lemma 3.1, we have $\alpha^n\leq\tilde{\alpha}\leq\alpha_k^n$, for any $k\in N$. Assume $\tilde{\alpha}\neq\alpha^n$, let $\epsilon=\frac{\tilde{\alpha}-\alpha^n}{2}>0$, because of the continuity of f and density of $\bigcup P_k$ in $C([0,1])$, there exists $(x_n(.;u_k),u_k(.))\in\xi_k^n$, for sufficiently large k and n, such that $I(x_n(.;u_k),u_k(.))<\alpha^n+\epsilon$, thus $I(x_n(.;u_k),u_k(.))<\tilde{\alpha}$ which is a contradiction to the definition of $\tilde{\alpha}$. □

Proposition 3.1. Let $u(t)$ be a given control function and the solution of the equation (2) belong to $(C^\alpha\cap L^2)[0,1]$ for some $\alpha>2$. We have

$$\sup_{t_i\in[0,T]}|x(t_i)-B_n(x_n(t_i))|\leq\frac{1}{8n}((1+c_0)\|(E^n)^{-1}(u)\|\|\ddot{x}\|+\|\ddot{x}_n\|)\qquad(17)$$

where $c_0=\sup_{s,t\in[0,T],w\in W}|k(s,t,w)|$.
Proof. The proof is similar to Theorem 2 in [10].

Proposition 3.2. We have

$$\forall\epsilon>0,\ \exists N,\ \forall u\in U,\ n\geqslant N\ |I(x_n(.;u),u(.))-I(x(.),u(.))|<\epsilon.$$

Proof. By the Assumption 3.2
$\exists K, \forall u \in U,\ ((1+c_0)\|(E^n)^{-1}\|\|\ddot{x}\|+\|\ddot{x}_n\|) < K$, then $\forall u \in U,\ \lim_{n\longrightarrow\infty}\sup|x(t_i) - B_n(x_n(t_i))| = 0$, and by Theorem 2.1, we have

$$\forall u \in U \ \lim_{n\longrightarrow\infty}\|x_n(t_i) - x(t_i)\| = 0.$$

Theorem 3.1. If $\lim_{n\to\infty}\alpha^n$ exists, then $\lim_{n\to\infty}\ \lim_{k\to\infty}\alpha_k^n = \alpha$, where $\alpha = inf_{(x,u)\in\xi}I(x,u) = I(x^*,u^*)$.
Proof: By Lemma 3.2, $\lim_{k\to\infty}\ \alpha_k^n = \alpha^n$, then it sufficient to prove

$$\lim_{n\to\infty}\alpha^n = \alpha.$$

Let $\lim_{n\to\infty}\alpha^n = \hat{\alpha}$, we will show $\alpha = \hat{\alpha}$. We have

$$\forall n\ \alpha^n = \inf\ I(x_n(.;u),u(.)) \le I(x_n(.;u),u^*(.)).$$

Then, if we take a limit from above relation as $n\to\infty$, we will have

$$\lim_{n\to\infty}\alpha^n = \hat{\alpha} \le \lim_{n\to\infty} I(x_n(.;u^*),u^*(.)) = I(x^*(.),u^*(.)) = \alpha,$$

where x^* and u^* are exact trajectory and control vectors which are also admissible. By contradiction, if $\hat{\alpha} < \alpha$, then $\epsilon = \frac{\alpha-\hat{\alpha}}{2} > 0$, since $\lim_{n\to\infty}\alpha^n = \hat{\alpha}$ we have:

$$\exists \mathrm{N}':\ \mathrm{n} > \mathrm{N}',\ |\alpha^{\mathrm{n}} - \hat{\alpha}| < \epsilon/2, \Longrightarrow\ \hat{\alpha} - \frac{\epsilon}{2} < \alpha^{\mathrm{n}} < \hat{\alpha} + \frac{\epsilon}{2}. \tag{18}$$

By Proposition 3.2, we have

$$\exists \mathrm{N}:\ \mathrm{n} > \mathrm{N},\ |\mathrm{I}(\mathrm{x_n}(.,\mathrm{u}),\mathrm{u}(.)) - \mathrm{I}(\mathrm{x}(.),\mathrm{u}(.))| < \epsilon, \tag{19}$$

$\exists m\ > \max\{N, N'\}$

$$\alpha^{\mathrm{m}} = \inf\ \mathrm{I}(\mathrm{x_m}(.,\mathrm{u}),\mathrm{u}(.)) \Longrightarrow\ \exists\tilde{\mathrm{u}},\ \mathrm{I}(\mathrm{x_m}(.,\tilde{\mathrm{u}}),\tilde{\mathrm{u}}(.)) < \alpha^{\mathrm{m}} + \frac{\epsilon}{2}, \tag{20}$$

from (18) and (19)

$$I(x(.),\tilde{u}(.)) < I(x_m(.;\tilde{u}),\tilde{u}(.)) + \epsilon < \alpha^m + \frac{\epsilon}{2} + \epsilon < \hat{\alpha} + 2\epsilon < \alpha,$$

where a contradiction is concluded.

4. **Numerical experiments**

To investigate the efficiency of the proposed method, we present results of numerical experiments through three examples. In these examples the approximate solutions is compared with the exact solution. In Example 1, the obtained results are compared with the results of the presented method in [9], which is based on HPM.
Example 1: In the first example, we consider the following OCP governed by VIE which minimizes the functional [9]

$$J(x,u) = \int_0^1 (x(t) - t - 1)^2 + (u(t) - t^2 - t)^2 dt,$$

subject to:

$$x(t) = y(t) + \int_0^t st^2 u(s)x(s)ds,$$

where,

$$y(t) = -\frac{1}{5}t^7 - \frac{1}{2}t^6 - \frac{1}{3}t^5 + t + 1.$$

The exact optimal trajectory and control functions are: $x^*(t) = t+1,\ u^*(t) = t^2+t$. Choosing $t_0 = 10^{-10}$, $\epsilon = 10^{-8}$.

$$t_j = t_0 + j\Delta/n,\ j = 0, \cdots, n,\ t_n = 1 - t_0.$$

The results of applying the algorithm are presented in Table 1. Figure 1 and Figure 2 show the comparison of the exact and approximate control and trajectory in some iterations.

k	n	$u(t)_{Approx}$	$\bar{J}_k$	$\bar{J}_k$ (HPM)
1	3	$-0.1672 + 1.9996t$	0.0055	0.0056
2	3	$-0.0006 + 1.0004t + 0.9996t^2$	$1.0002e-009$	$1.8336e-007$
3	3	$0.0000 + 1.0000t + 1.0000t^2 + 0.0000t^3$	$9.5502e-016$	$1.3452e-009$

Table 1. Results of the numerical experiments and comparison with (HPM) in example 1.

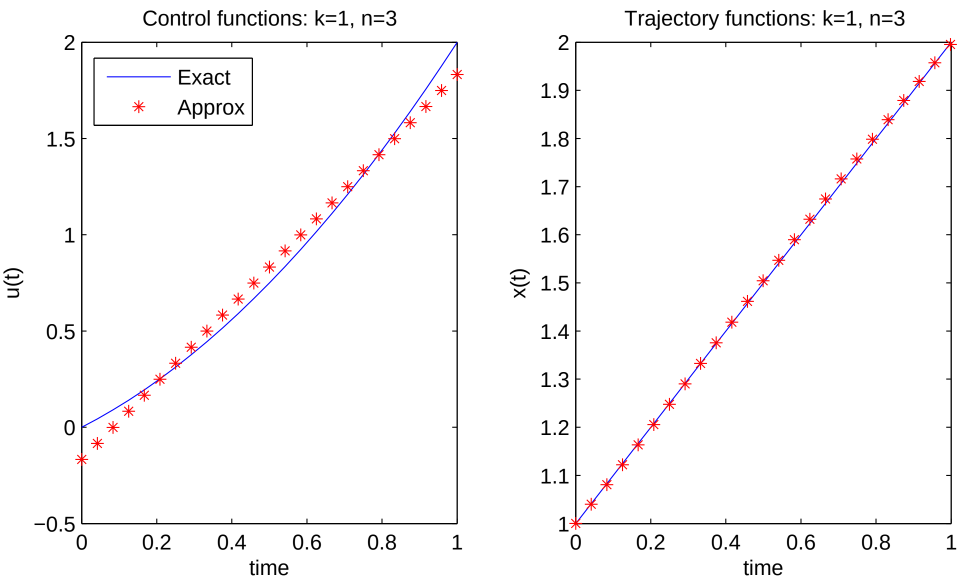

FIGURE 1. The exact and approximate control and trajectory functions in example 1.

Example 2: In this example an OCP as

$$\text{Minimize } J(x,u) = \frac{1}{8}\int_0^1 (x(t) - u(t))^2 dt,$$

governed by VIE

$$x(t) = e^t - t^2 + \int_0^t te^{-2s}u(s)x(s)ds,$$

is considered.
Choosing $t_0 = 10^{-10}$, $\epsilon = 10^{-5}$.

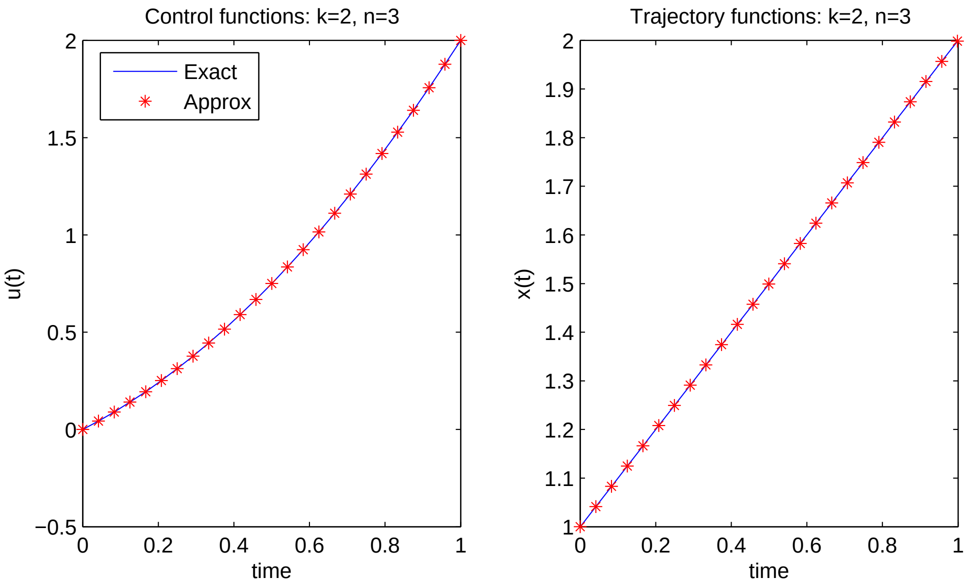

FIGURE 2. The exact and approximate control and trajectory functions in example 1.

$x^*(t) = u^*(t) = e^t$ are the exact optimal trajectory and control functions.
The numerical results can be found in Table 2. In Figure 3 and Figure 4 the exact and approximate control and trajectory functions are shown in some iterations.

k	n	$u(t)_{Approx}$	J_k
1	3	$0.8664 + 1.6890t$	$4.9415e - 004$
2	3	$1.0125 + 0.8547t + 0.8360t^2$	$3.5680e - 006$
3	3	$0.9899 + 1.1289t + 0.1489t^2 + 0.4593t^3$	$1.4816e - 006$

Table 2. Results of numerical experiments in example2.

Example 3:
Consider the following OCP governed by VIE

$$\text{Minimize } J(x,u) = \frac{1}{3}\int_0^1 (\frac{x(t)-\sin(t)}{2})^2 + (\frac{u(t)-t^3}{2})^2 dt,$$

subject to:

$$x(t) = \sin(t) + \int_0^t t^2(u(s)-s^3)x(s)ds.$$

Choosing $t_0 = 10^{-10}$, $\epsilon = 10^{-3}$.
$x^*(t) = \sin(t)$ and $u^*(t) = t^3$ consider as the exact optimal trajectory and control functions respectively. The results of numerical experiments are shown in Table 3. The comparison of obtained approximate control and trajectory with exact ones are shown in Figure 5, Figure 6 and Figure 7.

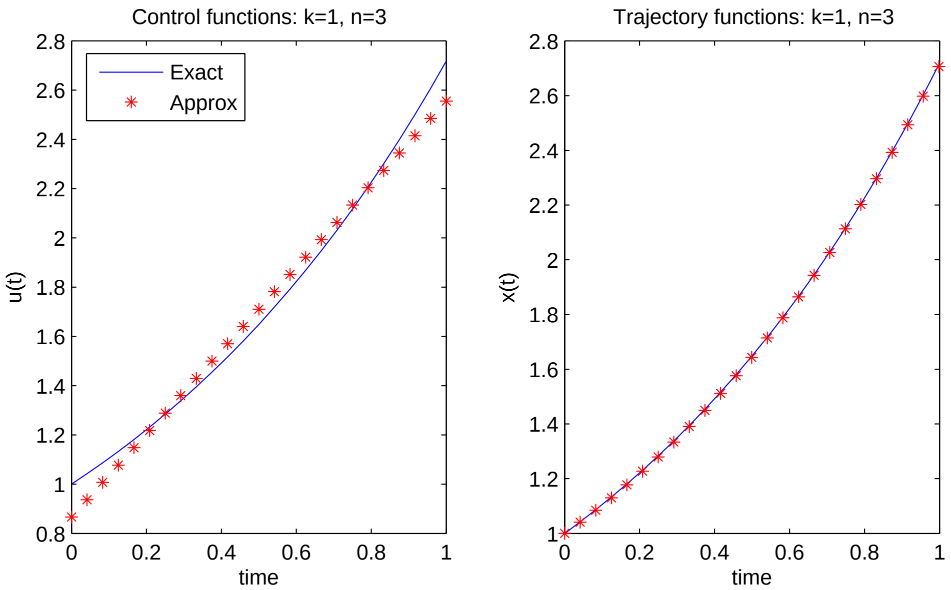

FIGURE 3. The exact and approximate control and trajectory functions in example 2.

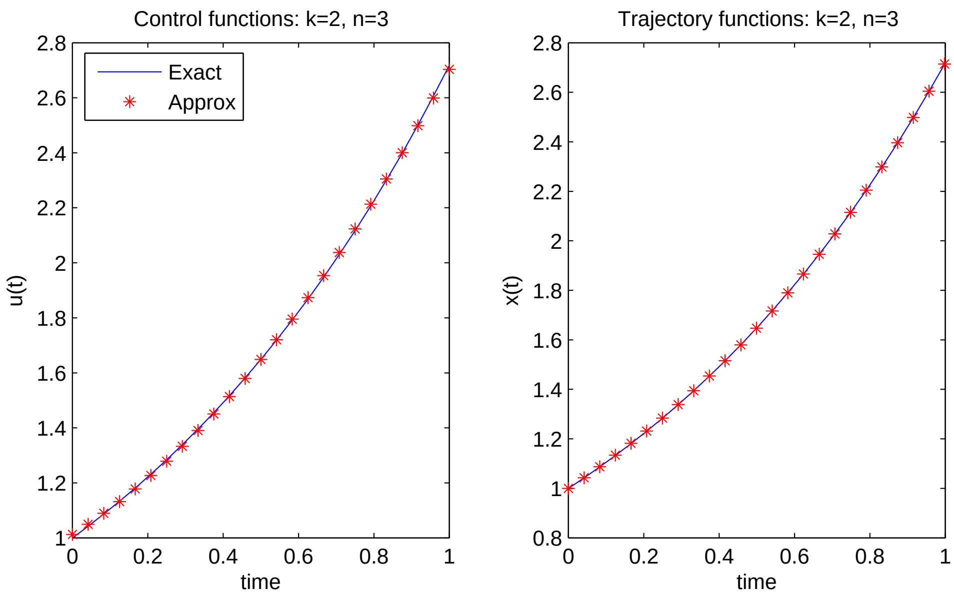

FIGURE 4. The exact and approximate control and trajectory functions in example 2.

k	n	$u(t)_{Approx}$	$\bar{J}_k$
1	3	$-0.2005 + 0.8998t$	0.0011
2	3	$-0.0410 - 0.0863t + 1.0014t^2$	$1.4670e - 004$
3	3	$0.0000 + 0.0000t + 0.0000t^2 + 1.0000t^3$	$2.2034e - 009$

Table 3. Results of the numerical experiments in example 3.

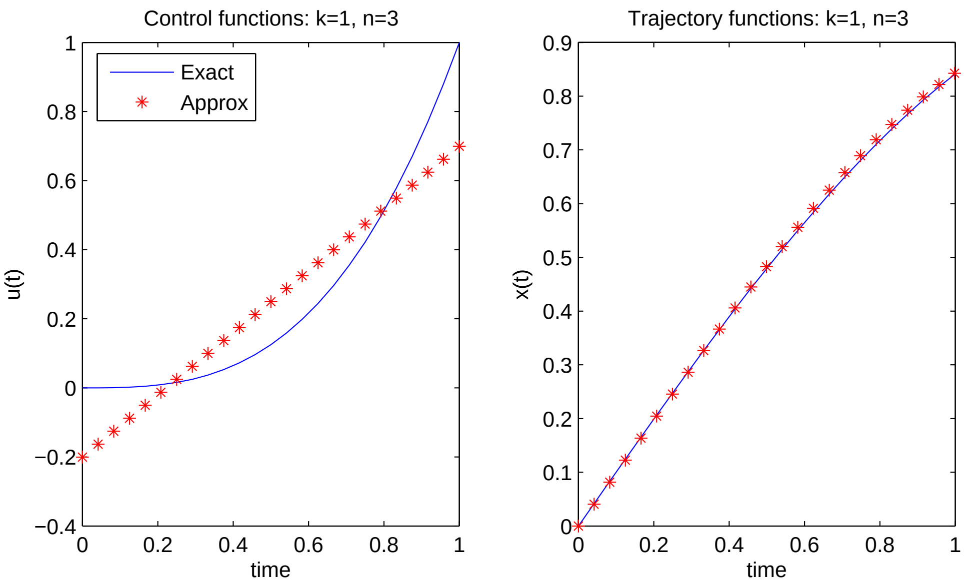

FIGURE 5. The exact and approximate control and trajectory functions in example 3.

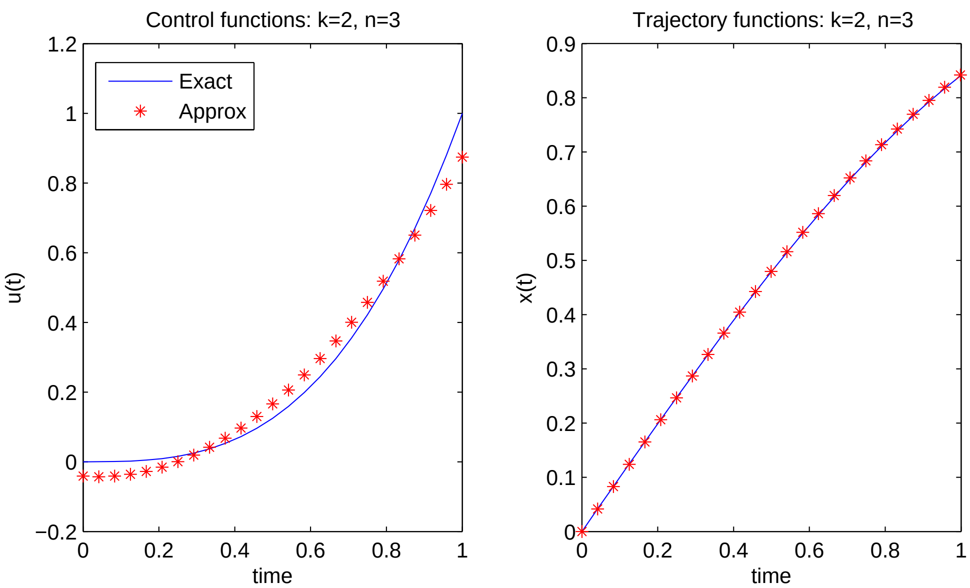

FIGURE 6. The exact and approximate control and trajectory functions in example 3.

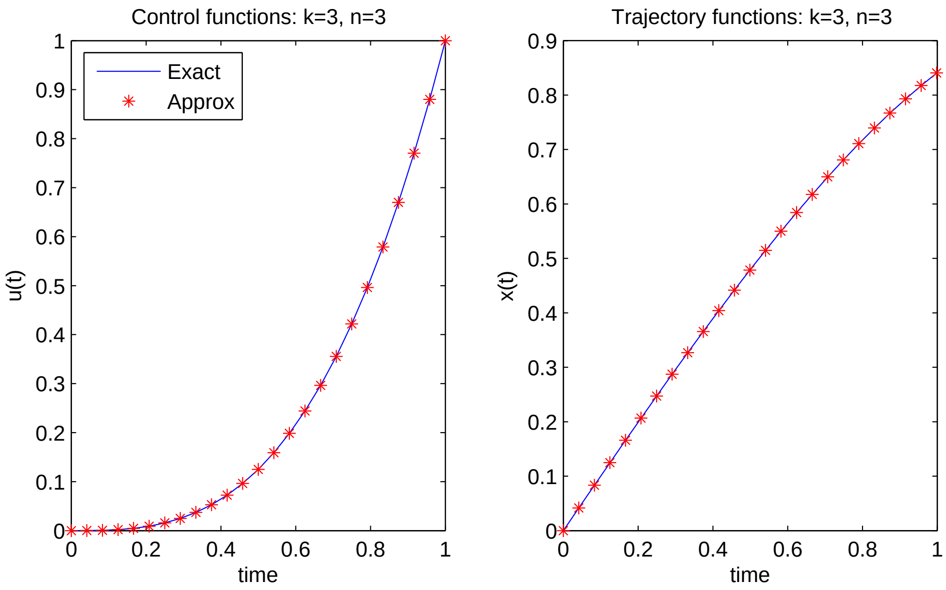

FIGURE 7. The exact and approximate control and trajectory functions in example 3.

5. Conclusion

In this paper we used Bernstein's approximation and parameterized control to approximate the solution of OCPs governed by VIEs. The achieved results in this paper show that this approach is very effective and efficient. Convergence and uniqueness of the approach have been proved and efficiency of it, discussed in three examples.

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