SOME COINCIDENCE POINT THEOREMS FOR SEMI-NONSELF HYBRID PAIR WITH ERROR ESTIMATES OF f-PICARD SEQUENCES

Muhammad Usman Ali[1], Quanita Kiran[2], Tayyab Kamran[3]

In this paper, we prove some coincidence and common fixed point theorems for semi-nonself and for self hybrid pairs. Our results not only provide the iterative scheme to locate the coincidence point but also provide the error estimates of f-Picard sequence. In the support of our results we give some nontrivial examples. The results of this paper, extend and generalize many existing results in literature. The observation of Haghi [R. H. Haghi, Sh. Rezapour, N. Shahzad, Some fixed point generalization are not real generalizations, Nonlinear Anal., 74 (2011) 1799-1803.] in general is not applicable for our results.

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Keywords: f-orbit, Lower semi-continuous, Bianchini-Grandolfi gauge function.
MSC2010: 47H10, 54H25.

1. **Introduction**

Markin [1] and Nadler [2] initiated the study of fixed point theorems for multivalued mappings. Assad and Kirk [3] gave sufficient condition for nonself multivalued mapping to have a fixed point. Ahmed and Khan [4] proved some common fixed point theorems for nonself hybrid pair of mappings. Subsequently, Singh and Mishra [5] and Ciric *et al*. [6] also proved some coincidence and common fixed point theorems for nonself hybrid pair in metrically convex metric space. In this paper, we prove some coincidence point and common fixed point theorems for semi-nonself and self hybrid pairs with error estimates of f-Picard sequence in a complete metric space.

Let (X, d) be a metric space and $D \subseteq X$. For each $x \in X$ and $A \subseteq X$, $d(x, A) = \inf\{d(x, y) : y \in A\}$. We denote by $CL(X)$ the class of all nonempty closed subsets of X. A point $x \in X$ is said to be a fixed point of $T : X \to CL(X)$ if $x \in Tx$. A point $x \in D$ is said to be a coincidence point of $f : D \to X$ and $T : X \to CL(X)$ if $fx \in Tx$. A point $x \in D$ is said to be a common fixed point of $f : D \to X$ and $T : X \to CL(X)$ if $x = fx \in Tx$. If for $x_0 \in X$, there exists a sequence $\{x_n\}$ in X such that $x_n \in Tx_{n-1}$, then $O(T, x_0) = \{x_0, x_1, x_2, \cdots\}$ is said to be the orbit of $T : X \to CL(X)$. If for $x_0 \in X$, $f : X \to X$ and $T : X \to CL(X)$, then a sequence $\{fx_n\}$ in fX of the form $fx_n \in Tx_{n-1}$ is

[1] Department of Mathematics, COMSATS Institute of Information Technology, Attock Pakistan, e-mail: muh_usman_ali@yahoo.com (M. U. Ali)
[2] School of Electrical Engineering and Computer Sciences, National University of Sciences and Technology H-12, Islamabad Pakistan, e-mail: quanita.kiran@seecs.nust.edu.pk
[3] Department of Mathematics, Quaid-i-Azam University, Islamabad Pakistan, e-mail: tayyabkamran@gmail.com (T. Kamran)

said to be an f-Picard sequence. If for $x_0 \in D$ and $f : D \to X$ there exists a sequence $\{fx_n\}$ in fD such that $fx_n \in Tx_{n-1}$, then $O_f(x_0) = \{fx_1, fx_2, fx_3, \cdots\}$ is said to be an f-orbit of $T : X \to CL(X)$. A mapping $g : X \to \mathbb{R}$ is said to be lower semi-continuous at ξ if for any sequence $\{x_n\}$ in X with $x_n \to \xi$, implies $g(\xi) \leq \liminf_{n\to\infty} g(x_n)$. Throughout this paper J denotes an interval on $\mathbb{R}_+$ containing 0, that is an interval of the form $[0, A], [0, A)$ or $[0, \infty)$ and $S_n(t)$ denotes the polynomial $S_n(t) = 1 + t + ... + t^{n-1}$. We use the abbreviation φ^n for the nth iterate of a function $\varphi : J \to J$.

Definition 1.1. *[7] Let $r \geq 1$. A function $\varphi : J \to J$ is said to be a gauge function of order r on J if it satisfies the following conditions:*

- *(i) $\varphi(\lambda t) \leq \lambda^r \varphi(t)$ for all $\lambda \in (0, 1)$ and $t \in J$;*
- *(ii) $\varphi(t) < t$ for all $t \in J - \{0\}$.*

It is easy to see that the first condition of Definition 1.1 is equivalent to the following: $\varphi(0) = 0$ and $\varphi(t)/t^r$ is nondecreasing on $J - \{0\}$.

Definition 1.2. *[7, 8] A nondecreasing function $\varphi : J \to J$ is said to be a Bianchini-Grandolfi gauge function on J if*

$$\sum_{n=0}^{\infty} \varphi^n(t) < \infty, \text{ for all } t \in J.$$

Remark 1.1. *Let the nondecreasing function $\varphi : J \to J$ is such that*

$$\sigma(t) = \sum_{n=0}^{\infty} \varphi^n(t) < \infty, \text{ for all } t \in J. \tag{1}$$

Then Ptak [9] called $\varphi : J \to J$ a rate of convergence on J and noticed that φ satisfies the following functional equation

$$\sigma(t) = \sigma(\varphi(t)) + t. \tag{2}$$

Remark 1.2. *[7] Every gauge function of order $r \geq 1$ on J is a Bianchini-Grandolfi gauge function on J.*

Lemma 1.1. *[2] Let (X, d) be a metric space. Let $B \in CL(X)$ and $x \in X$. Then for each $\epsilon > 0$, there exists $b \in B$ such that $d(x, b) \leq d(x, B) + \epsilon$.*

Lemma 1.2. *[7] Let φ be a gauge function of order $r \geq 1$ on J. If ϕ is a nonnegative and nondecreasing function on J satisfying*

$$\varphi(t) = t\phi(t) \text{ for all } t \in J, \tag{3}$$

then it has the following properties:

- *(i) $0 \leq \phi(t) < 1$ for all $t \in J$;*
- *(ii) $\phi(\lambda t) \leq \lambda^{r-1}\phi(t)$ for all $\lambda \in (0, 1)$ and $t \in J$.*

Moreover, for each $n \geq 0$ we have

- *(iii) $\varphi^n(t) \leq t\phi(t)^{S_n(r)}$ for all $t \in J$,*
- *(iv) $\phi(\varphi^n(t)) \leq \phi(t)^{r^n}$ for all $t \in J$.*

Definition 1.3. *[10] Let (X, d) be a metric space. A real number $\alpha > 0$ is called an order of convergence of $\{x_n\}$, provided $x_n \to \xi$ and there exists $\lambda > 0$ such that*

$$\lim_{n\to\infty} \frac{d(x_{n+1}, \xi)}{(d(x_n, \xi))^\alpha} = \lambda.$$

2. **Main Results**

We start this section with the following theorem:

Theorem 2.1. *Let (X,d) be a metric space, let D be a nonempty closed subset of X and let φ be a Bianchini-Grandolfi gauge function on an interval J. Let $T : X \to CL(X)$ and $f : D \to X$ be two mappings such that $TD \subset fD$, $Tx \cap D \neq \emptyset$ and*

$$d(fy, Ty \cap D) \leq \varphi(d(fx, fy)), \tag{4}$$

for all $x \in D$ and $fy \in Tx \cap D$ with $d(fx, fy) \in J$. Moreover, the strict inequality holds when $d(fx, fy) \neq 0$. Let fD be a complete metric subspace of X. Suppose that there exists $x_0 \in D$ such that $d(fx_0, fz) \in J$ for some $fz \in Tx_0 \cap D$. Then:

- **:** *(i) there exists an f-orbit $\{fx_n\}$ of T in D and $f\xi \in D$ such that $\lim_n fx_n = f\xi$;*
- **:** *(ii) ξ is a coincidence point of f and T if and only if the function $g(x) := d(fx, Tx \cap D)$ is lower semi-continuous at ξ;*
- **:** *(iii) if $ff\xi = f\xi$ then f and T have a common fixed point.*

Proof. By hypothesis, we have $x_0 \in D$ such that $fx_1 \in Tx_0 \cap D$ and $d(fx_0, fx_1) \in J$. We assume that $d(fx_0, fx_1) \neq 0$, for otherwise x_0 is a coincidence point of f and T. Define $\rho_0 = \sigma(d(fx_0, fx_1))$, where σ is defined by (1). From (2), $\sigma(t) \geq t$. We have

$$d(fx_0, fx_1) \leq \rho_0. \tag{5}$$

Notice that $fx_1 \in \overline{S}(fx_0, \rho_0) = \{fx \in fD : d(fx_0, fx) \leq \rho_0\}$. It follows from (4) that $d(fx_1, Tx_1 \cap D) < \varphi(d(fx_0, fx_1))$. We choose $\epsilon_1 > 0$ such that

$$d(fx_1, Tx_1 \cap D) + \epsilon_1 \leq \varphi(d(fx_0, fx_1)). \tag{6}$$

It follows from Lemma 1.1 that there exists $fx_2 \in Tx_1 \cap D$ such that

$$d(fx_1, fx_2) \leq d(fx_1, Tx_1 \cap D) + \epsilon_1. \tag{7}$$

We assume that $d(fx_1, fx_2) \neq 0$, otherwise x_1 is a coincidence point of f and T. From inequalities (6) and (7), we have

$$d(fx_1, fx_2) \leq \varphi(d(fx_0, fx_1)). \tag{8}$$

Note that $d(fx_1, fx_2) \in J$. Further, $fx_2 \in \overline{S}(fx_0, \rho_0)$, since

$$\begin{aligned} d(fx_0, fx_2) &\leq d(fx_0, fx_1) + d(fx_1, fx_2) \\ &\leq d(fx_0, fx_1) + \varphi(d(fx_0, fx_1)) \\ &\leq d(fx_0, fx_1) + \sigma(\varphi(d(fx_0, fx_1))) \\ &= \sigma(d(fx_0, fx_1)) \text{ (using (2))} \\ &= \rho_0. \end{aligned}$$

Again choose $\epsilon_2 > 0$ such that

$$d(fx_2, Tx_2 \cap D) + \epsilon_2 \leq \varphi(d(fx_1, fx_2)). \tag{9}$$

It again follows from Lemma 1.1 that there exists $fx_3 \in Tx_2 \cap D$ such that

$$d(fx_2, fx_3) \leq d(fx_2, Tx_2 \cap D) + \epsilon_2. \tag{10}$$

We assume that $d(fx_2, fx_3) \neq 0$, otherwise x_2 is a coincidence point of f and T. From inequalities (8), (9) and (10), we have

$$d(fx_2, fx_3) \leq \varphi^2(d(fx_0, fx_1)). \tag{11}$$

Note that $d(fx_2, fx_3) \in J$. Further, $fx_3 \in \overline{S}(fx_0, \rho_0)$, since

$$\begin{aligned} d(fx_0, fx_3) &\leq d(fx_0, fx_1) + d(fx_1, fx_2) + d(fx_2, fx_3) \\ &\leq d(fx_0, fx_1) + \varphi(d(fx_0, fx_1)) + \varphi^2(d(fx_0, fx_1))) \\ &\leq \sum_{j=0}^{\infty} \varphi^j(d(fx_0, fx_1)) \\ &= \sigma(d(fx_0, fx_1)) = \rho_0. \end{aligned}$$

Repeating the above argument, inductively we obtain a sequence $\{fx_n\}_{n\in\mathbb{N}}$ such that

$$fx_n \in Tx_{n-1} \cap D, \tag{12}$$

$$d(fx_n, fx_{n+1}) \leq \varphi^n(d(fx_0, fx_1)), \tag{13}$$

$$d(fx_{n-1}, fx_n) \in J, \text{ and } fx_n \in \overline{S}(fx_0, \rho_0). \tag{14}$$

We claim that $\{fx_n\}$ is a Cauchy sequence. For $n > p \in \mathbb{N}$, from (13), we have

$$\begin{aligned} d(fx_n, fx_p) &\leq d(fx_n, fx_{n+1}) + d(fx_{n+1}, fx_{n+2}) + \cdots + d(fx_{p-1}, x_p) \\ &\leq \varphi^n(d(fx_0, fx_1)) + \cdots + \varphi^{p-1}(d(fx_0, fx_1)) \\ &\leq \sum_{j=n}^{\infty} \varphi^j(d(fx_0, fx_1)). \end{aligned}$$

Using (1), it follows from the above inequality that $\{fx_n\}$ is a Cauchy sequence in $\overline{S}(fx_0, \rho_0)$. Thus there exists $f\xi \in \overline{S}(fx_0, \rho_0)$ with $fx_n \to f\xi$. Note that $f\xi \in D$, as well, since $fx_n \in Tx_{n-1} \cap D$. It follows from (13) that

$$d(fx_n, Tx_n \cap D) \leq d(fx_n, fx_{n+1}) \leq \varphi^n(d(fx_0, fx_1)). \tag{15}$$

Letting $n \to \infty$, from (15), we get

$$\lim_{n\to\infty} d(fx_n, Tx_n \cap D) = 0. \tag{16}$$

Suppose $g(x) = d(fx, Tx \cap D)$ is lower semi-continuous at ξ. Then

$$d(f\xi, T\xi \cap D) = g(\xi) \leq \liminf_{n\to\infty} g(x_n) = \liminf_{n\to\infty} d(fx_n, Tx_n \cap D) = 0.$$

Hence, $f\xi \in T\xi$, since $T\xi$ is closed. Conversely, if ξ is a coincidence point of f and T, then $g(\xi) = 0 \leq \liminf_n g(x_n)$. Suppose we have $f\xi = f(f\xi)$. Let $z = f\xi$, $z = f\xi = f(f\xi) = fz \in T\xi \cap D$. Then from (4), we have

$$d(fz, Tz \cap D) \leq \varphi(d(f\xi, fz)) = \varphi(0) = 0.$$

Thus $z = fz \in Tz$. □

Example 2.1. *Let* $X = \{0, 1\} \cup \left\{\frac{1}{2^n} : n \in \mathbb{N}\right\}$ *be endowed with the usual metric* d. *Let* $D = \{0\} \cup \left\{\frac{1}{2^n} : n \in \mathbb{N}\right\}$ *and* $J = [0, 1]$. *Define* $T : X \to CL(X)$ *by*

$$Tx = \begin{cases} \{0, 1\}, & \text{if } x = 0, 1 \\ \left\{0, \frac{1}{2}\right\}, & \text{if } x = \frac{1}{2} \\ \{\frac{1}{2^{n+1}}, \frac{1}{2^{n-1}}\}, & \text{if } x = \frac{1}{2^n} : n \geq 2, \end{cases}$$

and $f : D \to X$ *by*

$$fx = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x = \frac{1}{2} \\ \frac{1}{2^{n-1}}, & \text{if } x = \frac{1}{2^n} : n \geq 2. \end{cases}$$

Note that $TD \subset fD$ *and* fD *is complete. Taking* $\varphi(t) = \frac{t}{2}$ *for each* $t \in J$*. Further, all other conditions of Theorem 2.1 hold. Thus* f *and* T *have a coincidence point.*

Theorem 2.2. *Let* (X, d) *be a metric space, let* D *be a nonempty closed subset of* X *and let* φ *be a Bianchini-Grandolfi gauge function on an interval* J*. Let* $T : X \to CL(X)$ *and* $f : X \to X$ *be two mappings such that* $TD \subset fX$*,* $Tx \cap D \neq \emptyset$ *and*

$$d(fy, Ty \cap D) \leq \varphi(d(fx, fy)),$$

for all $x \in D$ *and* $fy \in Tx \cap D$ *with* $d(fx, fy) \in J$*. Moreover, the strict inequality holds when* $d(fx, fy) \neq 0$*. Let* fX *be a complete metric subspace of* X*. Suppose that there exists* $x_0 \in D$ *such that* $d(fx_0, fz) \in J$ *for some* $fz \in Tx_0 \cap D$*. Then:*

- **:** *(i) there exists an* f*-orbit* $\{fx_n\}$ *of* T *in* D *and* $f\xi \in D$ *such that* $\lim_n fx_n = f\xi$*;*
- **:** *(ii)* ξ *is a coincidence point of* f *and* T *if and only if the function* $g(x) := d(fx, Tx \cap D)$ *is lower semi-continuous at* ξ*;*
- **:** *(iii) if* $ff\xi = f\xi$ *then* f *and* T *have a common fixed point.*

Proof. The proof of this theorem is similar to the proof of Theorem 2.1. □

Example 2.2. *Let* $X = \mathbb{R}$ *be endowed with the usual metric* d*. Let* $D = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ *and* $J = [0, 1]$*. Define* $T : X \to CL(X)$ *by*

$$Tx = \begin{cases} (-\infty, x], & \text{if } x < 0 \\ \{0\}, & \text{if } x = 0 \\ \left\{\frac{1}{n+2}, \frac{1}{n+3}\right\} \cup [n, \infty), & \text{if } x = \frac{1}{n}, : 1 \leq n \leq 6 \\ \left\{0, \frac{1}{n+1}\right\} \cup [n, \infty), & \text{if } x = \frac{1}{n} : n > 6 \\ [2x, \infty), & \text{otherwise}, \end{cases}$$

and $f : X \to X$ *by*

$$fx = \begin{cases} 0, & \text{if } x \leq 0 \\ \frac{1}{n+1}, & \text{if } x = \frac{1}{n} : n \in \mathbb{N} \\ \frac{2}{3}, & \text{if } x \in [0, 1] - \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}, \\ \frac{x}{2}, & \text{if } x > 1. \end{cases}$$

Note that $TD \subset fX$ *and* fX *is complete. Taking* $\varphi(t) = \frac{4t}{5}$ *for each* $t \in J$*. We see that all the conditions of Theorem 2.2 are satisfied. Thus* f *and* T *have a coincidence point. Moreover,* $f0 = ff0$*. Thus* 0 *is a common fixed point of* f *and* T*.*

Theorem 2.3. *Let* (X, d) *be a metric space, let* D *be a nonempty closed subset of* X*, let* φ *be a gauge function of order* $r \geq 1$ *on an interval* J *and let* $\phi : J \to \mathbb{R}^+$ *be a nondecreasing* *function defined by (3). Let $T : X \to CL(X)$ and $f : D \to X$ be two mappings such that $TD \subset fD$, $Tx \cap D \neq \emptyset$ and*

$$d(fy, Ty \cap D) \leq \varphi(d(fx, fy)), \tag{17}$$

for all $x \in D$ and $fy \in Tx \cap D$ with $d(fx, fy) \in J$. Moreover, the strict inequality holds when $d(fx, fy) \neq 0$. Let fD be a complete metric subspace of X. Suppose that there exists $x_0 \in D$ such that $d(fx_0, fz) \in J$ for some $fz \in Tx_0 \cap D$. Then:

- **:** *(i) there exists an f-orbit $\{fx_n\}$ of T in $\overline{S}(fx_0, \rho_0) = \{fx \in fD : d(fx_0, fx) \leq \rho_0\}$ that converges with rate of convergence at least r to a point $f\xi \in \overline{S}(fx_0, \rho_0)$, where $\rho_0 = \sigma(d(fx_0, fx_1))$ and σ is defined by (1);*
- **:** *(ii) for each $n \geq 0$, we have the following a priori estimate*

$$d(fx_n, f\xi) \leq \frac{\lambda^{S_n(r)} d(fx_0, fx_1)}{1 - \lambda^{r^n}}, \tag{18}$$

  where $\lambda = \phi(d(fx_0, fx_1))$;
- **:** *(iii) for each $n \geq 1$, we have the following a posteriori estimate*

$$d(fx_n, f\xi) \quad \leq \quad \frac{\varphi(d(fx_n, fx_{n-1}))}{1 - [\phi(d(fx_n, fx_{n-1}))]^r}; \tag{19}$$

- **:** *(iv) for each $n \geq 1$, we have*

$$d(fx_n, fx_{n+1}) \leq \lambda^{S_n(r)} d(fx_0, fx_1); \tag{20}$$

- **:** *(v) ξ is a coincidence point of f and T if and only if the function $g(x) := d(fx, Tx \cap D)$ is lower semi-continuous at ξ;*
- **:** *(vi) if $ff\xi = f\xi$ then f and T have a common fixed point.*

Proof. (i) Theorem 2.1 insures the existence of an f-orbit $\{fx_n\}$ of T in $\overline{S}(fx_0, \rho_0)$ that converges to $f\xi$ which belongs to $\overline{S}(fx_0, \rho_0)$.
(ii) For $m > n$, using (13) and Lemma 1.2-(iii) we have

$$\begin{aligned} d(fx_n, fx_m) &\leq \sum_{i=n}^{m-1} d(fx_i, fx_{i+1}) \\ &\leq \sum_{i=n}^{m-1} \varphi^i(d(fx_0, fx_1)) \\ &\leq d(fx_0, fx_1) \sum_{j=n}^{m-1} \lambda^{S_j(r)}. \end{aligned}$$

Keeping n fixed and letting $m \to \infty$, we get

$$d(fx_n, f\xi) \leq d(fx_0, fx_1) \sum_{j=n}^{\infty} \lambda^{S_j(r)}. \tag{21}$$

Note that,

$$\begin{aligned} \sum_{j=n}^{\infty} \lambda^{S_j(r)} &= \lambda^{S_n(r)} + \lambda^{S_{n+1}(r)} + \cdots \\ &= \lambda^{S_n(r)}[1 + \lambda^{r^n} + \lambda^{r^n + r^{n+1}} + \lambda^{r^n + r^{n+1} + r^{n+2}} + \cdots]. \end{aligned}$$

Since $r \geq 1$, therefore

$$r^n + r^{n+1} \geq 2r^n,\ r^n + r^{n+1} + r^{n+2} \geq 3r^n \cdots,$$

and hence,

$$\lambda^{r^n + r^{n+1}} \leq \lambda^{2r^n},\ \lambda^{r^n + r^{n+1} + r^{n+2}} \leq \lambda^{3r^n} \cdots,$$

since $0 < \lambda < 1$. Thus,

$$\sum_{j=n}^{\infty} \lambda^{S_j(r)} \leq \lambda^{S_n(r)}[1 + \lambda^{r^n} + \lambda^{2r^n} + \lambda^{3r^n} + \cdots] = \frac{\lambda^{S_n(r)}}{1 - \lambda^{r^n}}.$$

Substituting this in (21), we get

$$d(fx_n, f\xi) \leq d(fx_0, fx_1)\frac{\lambda^{S_n(r)}}{1 - \lambda^{r^n}}.$$

(iii) For each $n \geq 0$, from (21), we have

$$d(fx_n, f\xi) \leq d(fx_0, fx_1)\sum_{j=n}^{\infty}[\phi(d(fx_0, fx_1))]^{S_j(r)}.$$

Putting $n = 0, y_0 = fx_n$ and $y_1 = fx_1$, we have

$$d(y_0, f\xi) \ \leq \ d(y_0, y_1)\sum_{j=0}^{\infty}[\phi(d(y_0, y_1))]^{S_j(r)}.$$

Setting $y_0 = fx_n$, and $y_1 = fx_{n+1}$, we have

$$\begin{aligned} d(fx_n, f\xi) &\leq d(fx_n, fx_{n+1})\sum_{j=0}^{\infty}[\phi(d(fx_n, fx_{n+1}))]^{S_j(r)} \qquad (22) \\ &\leq \varphi(d(fx_n, fx_{n-1}))\sum_{j=0}^{\infty}[\phi(\varphi(d(fx_n, fx_{n-1})))]^{S_j(r)} \\ &\leq \varphi(d(fx_n, fx_{n-1}))\sum_{j=0}^{\infty}[\phi(\varphi(d(fx_n, fx_{n-1})))]^{j} \\ &= \frac{\varphi(d(fx_n, fx_{n-1}))}{1 - \phi(\varphi(d(fx_n, fx_{n-1})))}, \qquad (23) \end{aligned}$$

since $S_j(r) \geq j$. Now by Lemma 1.2-(iv), we have

$$\phi(\varphi(d(fx_n, fx_{n-1}))) \leq [\phi(d(fx_n, fx_{n-1}))]^r,$$

which means that,

$$\frac{1}{1 - \phi(\varphi(d(fx_n, fx_{n-1})))} \leq \frac{1}{1 - [\phi(d(fx_n, fx_{n-1}))]^r}. \qquad (24)$$

For each $n \geq 1$, from (23), we have

$$\begin{aligned} d(fx_n, f\xi) &\leq \frac{\varphi(d(fx_n, fx_{n-1}))}{1 - \phi(\varphi(d(fx_n, fx_{n-1})))} \\ &\leq \frac{\varphi(d(fx_n, fx_{n-1}))}{1 - [\phi(d(fx_n, fx_{n-1}))]^r} \text{ (using (24)).} \end{aligned}$$

(iv) For $n \geq 1$, using (13) and Lemma 1.2, we have

$$\begin{aligned} d(fx_n, fx_{n+1}) &\leq \varphi^n(d(fx_0, fx_1)) \\ &\leq d(fx_0, fx_1)\phi(d(fx_0, fx_1))^{S_n(r)} \\ &= d(fx_0, fx_1)\lambda^{S_n(r)}. \end{aligned}$$

The proofs of part (v) and (vi) are similar as in the proof of Theorem 2.1. □

Theorem 2.4. *Let (X, d) be a metric space, let D be a nonempty closed subset of X, let φ be a gauge function of order $r \geq 1$ on an interval J and let $\phi : J \to \mathbb{R}^+$ be a nondecreasing function defined by (3). Let $T : X \to CL(X)$ and $f : X \to X$ be two mappings such that $TD \subset fX$, $Tx \cap D \neq \emptyset$ and*

$$d(fy, Ty \cap D) \leq \varphi(d(fx, fy)),$$

for all $x \in D$ and $fy \in Tx \cap D$ with $d(fx, fy) \in J$. Moreover, the strict inequality holds when $d(fx, fy) \neq 0$. Let fX be a complete metric subspace of X. Suppose that there exists $x_0 \in D$ such that $d(fx_0, fz) \in J$ for some $fz \in Tx_0 \cap D$. Then:

- **:** *(i) there exists an orbit $\{fx_n\}$ of T in $\overline{S}(fx_0, \rho_0)$ that converges with rate of convergence at least r to a point $f\xi \in \overline{S}(fx_0, \rho_0)$, where $\rho_0 = \sigma(d(fx_0, fx_1))$ and σ is defined by (1);*
- **:** *(ii) for each $n \geq 0$, we have the following a priori estimate*

$$d(fx_n, f\xi) \leq \frac{\lambda^{S_n(r)} d(fx_0, fx_1)}{1 - \lambda^{r^n}},$$

  where $\lambda = \phi(d(fx_0, fx_1))$;
- **:** *(iii) for each $n \geq 1$, we have the following a posteriori estimate*

$$d(fx_n, f\xi) \leq \frac{\varphi(d(fx_n, fx_{n-1}))}{1 - [\phi(d(fx_n, fx_{n-1}))]^r};$$

- **:** *(iv) for each $n \geq 1$, we have*

$$d(fx_n, fx_{n+1}) \leq \lambda^{S_n(r)} d(fx_0, fx_1);$$

- **:** *(v) ξ is a coincidence point of f and T if and only if the function $g(x) := d(fx, Tx \cap D)$ is lower semi-continuous at ξ.*
- **:** *(vi) if $ff\xi = f\xi$ then f and T have a common fixed point.*

Proof. The proof of this theorem is similar to the proof of Theorem 2.3. □

Remark 2.1. *We can note the rate of convergence from the a priori estimate (18) as follows:*

$$\begin{aligned} \frac{d(fx_{n+1}, f\xi)}{(d(fx_n, f\xi))^r} &= \frac{\lambda^{S_{n+1}(r)} d(fx_0, fx_1)}{1 - \lambda^{r^{n+1}}} \Big(\frac{1 - \lambda^{r^n}}{\lambda^{S_n(r)} d(fx_0, fx_1)} \Big)^r \\ &= \frac{\lambda}{(d(fx_0, fx_1))^{r-1}} \frac{(1 - \lambda^{r^n})^r}{1 - \lambda^{r^{n+1}}}. \end{aligned}$$

Taking the limit when $n \to \infty$, we get

$$\lim_{n \to \infty} \frac{d(fx_{n+1}, f\xi)}{(d(fx_n, f\xi))^r} = \frac{\lambda}{(d(fx_0, fx_1))^{r-1}},$$

then by Definition 1.3 the rate of convergence of the sequence $\{fx_n\}$ is r with asymptotic error constant $\frac{\lambda}{(d(fx_0, fx_1))^{r-1}}$.

Remark 2.2. *In Haghi et al. [11] showed that some coincidence point theorem for hybrid maps (f, T) follows from the corresponding fixed point theorems for multivalued map T. Now the question is that can our theorems follow from the corresponding fixed point theorems for multivalued map T with f as an identity map, that is, Can Theorem 2.1 follows from the result obtain by considering $f = I$ in Theorem 2.1. By looking closely at the proof of [11, Theorem 2.16], the answer to this question in general is negative because our contractive condition holds for $x \in D$ and $fy \in Ty \cap D$.*

3. Consequences

By considering $f = I$, Theorem 2.2 and Theorem 2.4 generalize and extend: Theorem 5 of Mizoguchi and Takahashi [12]; Theorem 2.1 of Kamran [13]; Theorem 2.11 and Theorem 2.15 of Kiran and Kamran [14], respectively; Also by taking $f = I$ and $T : D \to X$, then from Theorem 2.2 and Theorem 2.4, we obtain Theorem 4.1 and Theorem 4.2 of Proinov [7], respectively.

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