NEW TYPE OF MULTIVALUED CONTRACTIONS WITH RELATED RESULTS AND APPLICATIONS

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In this paper, we focus on achieving new results about the existence of fixed points for a new type of multivalued contractions. We furnish an example which demonstrate the supremacy of our results to the existing ones in the literature. We derive new fixed point results on a metric space endowed with a partial ordering/graph by using the results obtained herein. We also discuss sufficient conditions to ensure the existence of solutions of integral equations as an application of our results.

Keywords: θ-contraction, admissible mapping, α-ψ-contraction, partial order, graph, integral equation.

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1. Introduction

In 1937, Von Neumann [19] gave a start the fixed point theory for multivalued mappings in the study of game theory. In particular, the fixed point theorems for multivalued mappings are rather advantageous in optimal control theory and have been frequently used to solve many problems of economics and game theory. Consecutively, Nadler [9] initiated the development of the geometric fixed point theory for multivalued mappings by using the notion of the Hausdorff metric and extended Banach contraction principle to multivalued mappings, which is known as Nadler's multivalued contraction principle. Recently, new types of single valued week contractive mappings with control functions, called as θ-contraction and α-ψ-contraction respectively, are introduced in [7] and [11] along many others in the literature: for instance, please see [12, 13]. This approach allowed to establish existence and uniqueness results for fixed points, which improve Banach contraction principle [5, 6], and the development of numerical algorithms for suitable classes of problems with real world applications [14, 15, 16, 17]. Later on, by using these concepts, several researchers extended the results in [7] and [11] to multivalued mappings, see, for example, Ali *et al.* [1], Ali and Kiran [2], Asl *et al.* [3], Mohammadi *et al.* [8] and Vetro [18].

In this study, we introduce a new type of multivalued contractions to establish existence results for fixed points of this new type of contractions on complete metric spaces. Our results improve and extend the results in Asl *et al.* [3], Mohammadi *et al.* [8], Vetro [18] and many others in the literature. An example is constructed in order to illustrate the generality of our results. As applications of the obtained results, some new fixed point theorems are presented on a metric space endowed with a partial ordering/graph and sufficient conditions are discussed to ensure the existence of solutions of integral equations.

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2. **Preliminaries and Background**

Here, we recollect some basic definitions, lemmas, notations and some known theorems which are helpful for understanding of this paper. Let (X, d) be a metric space and denote the family of nonempty, closed and bounded subsets of X by $CB(X)$. For $A, B \in CB(X)$, define $H\colon CB(X) \times CB(X) \to [0, +\infty)$ by

$$H(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\}$$

where $d(a, B) = \inf\{d(a, x)\colon x \in B\}$. Such a function H is called the Pompeiu-Hausdorff metric induced by the metric d, for more details, see [4]. Also, denote the family of nonempty and closed subsets of X by $CL(X)$ and the family of nonempty and compact subsets of X by $K(X)$. Note that $H\colon CL(X) \times CL(X) \to [0, +\infty]$ is a generalized Pompeiu-Hausdorff metric, that is, $H(A, B) = +\infty$ if $\max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\}$ does not exist.

Lemma 2.1 ([18])**.** *Let (X, d) be a metric space and $A, B \in CL(X)$ with $H(A, B) > 0$. Then, for each $h > 1$ and for each $a \in A$, there exists $b = b(a) \in B$ such that $d(a, b) < hH(A, B)$.*

By the properties of closed sets, one deduces the following lemma.

Lemma 2.2. *Let (X, d) be a metric space. For $A \in CL(X)$ and $x \in X$, $d(x, A) = 0$ if and only if $x \in A$.*

Firstly, Asl *et al.* [3] adapted the notion of α-admissible to multivalued mappings as α_*-admissible. Afterwards, Mohammadi *et al.* [8] introduced the concept of α-admissible for multivalued mappings.

Let (X, d) be a metric space and $\alpha\colon X \times X \to [0, +\infty)$ be a given mapping. A mapping $T\colon X \to CL(X)$ is an

(1) α_*-admissible, if $\alpha(x, y) \geq 1$ implies $\alpha_*(Tx, Ty) \geq 1$, where $\alpha_*(Tx, Ty) = \inf\{\alpha(a, b) : a \in Tx, b \in Ty\}$;

(2) α-admissible, if for each $x \in X$ and $y \in Tx$ with $\alpha(x, y) \geq 1$, we have $\alpha(y, z) \geq 1$ for all $z \in Ty$.

One can easily see that each α_*-admissible mapping is also α-admissible, but the converse is not true in general.

Let Ψ be the family of nondecreasing functions $\psi\colon [0, +\infty) \to [0, +\infty)$ such that $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$ for all $t > 0$. If $\psi \in \Psi$, then it is easy to see that $\psi(t) < t$ for all $t > 0$.

Let (X, d) be a metric space. A map $T\colon X \to CL(X)$ is a

(1) multivalued α_*-ψ-contraction, if

$$\alpha_*(Tx, Ty)H(Tx, Ty) \leq \psi(d(x, y));$$

(2) multivalued α-ψ-contraction, if

$$\alpha(x, y)H(Tx, Ty) \leq \psi(d(x, y)),$$

for all $x, y \in X$ where $\psi \in \Psi$ and $\alpha : X \times X \to [0, +\infty)$.

Theorem 2.1 ([3, 8])**.** *Let (X, d) be a complete metric space, $\psi \in \Psi$ be a strictly increasing function and $T\colon X \to CL(X)$ be a given mapping. Assume that the following conditions are satisfied:*

(i) T is α-admissible and multivalued α-ψ-contraction (or α_*-admissible and multivalued α_*-ψ-contraction);

(ii) There exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;

(iii) T is continuous or

(iv) *X is α-regular, that is, for every sequence $\{x_n\}$ in X such that $x_n \to x \in X$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*

Then T has a fixed point, that is, there exists $u \in X$ such that $u \in Tu$.

In 2014, Jleli and Samet [7] introduced a new type of contractive mappings, known as θ-contraction. Following the results in [7], Vetro [18] presented fixed point results for multivalued mappings.

Definition 2.1 ([7, 18])**.** Let (X, d) be a metric space. A map $T\colon X \to CL(X)$ is called a *weak θ-contraction*, if there exist $k \in (0,1)$ and $\theta \in \Theta$ such that

$$\theta(H(Tx, Ty)) \leq [\theta(d(x,y))]^k, \tag{1}$$

for all $x, y \in X$ with $H(Tx, Ty) > 0$, where Θ is the set of functions $\theta\colon (0,+\infty) \to (1,+\infty)$ satisfying the following conditions:

(θ1) θ is non-decreasing;

(θ2) for each sequence $\{t_n\} \subset (0,+\infty)$, $\lim_{n\to+\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to+\infty} t_n = 0$;

(θ3) there exist $r \in (0,1)$ and $\lambda \in (0,+\infty]$ such that $\lim_{t\to 0^+} \dfrac{\theta(t)-1}{t^r} = \lambda$.

The following functions $\theta_i\colon (0,+\infty) \to (1,+\infty)$ for $i \in \{1,2\}$, are the elements of Θ. Furthermore, substituting in (1) these functions, we obtain some contractions known in the literature: for all $x, y \in X$ with $H(Tx, Ty) > 0$,

$$\theta_1(t) = e^{\sqrt{t}}, \qquad H(Tx, Ty) \leq k^2 d(x,y),$$

$$\theta_2(t) = e^{\sqrt{te^t}}, \qquad \frac{H(Tx, Ty)}{d(x,y)} e^{H(Tx,Ty)-d(x,y)} \leq k^2.$$

Theorem 2.2 ([18])**.** *Let (X, d) be a complete metric space and $T\colon X \to K(X)$ be a weak θ-contraction. Then T has a fixed point.*

Note that Theorem 2.2 is invalid, if we take $CB(X)$ instead of $K(X)$. In the reference [18], Vetro showed that Theorem 2.2 is still true for $T\colon X \to CB(X)$, whenever $\theta \in \Theta$ is right continuous.

3. The Results

We begin this section with the following definition.

Definition 3.1. Let (X, d) be a metric space and $\alpha\colon X \times X \to [0,+\infty)$ be a given function. A mapping $T\colon X \to CL(X)$ is called a *multivalued (α-θ-ψ)-contraction*, if there exist $\theta \in \Theta$, $\psi \in \Psi$ and $k \in (0,1)$ such that

$$\theta(H(Tx, Ty)) \leq [\theta(\psi(d(x,y)))]^k, \tag{2}$$

for all $x, y \in X$ with $\alpha(x,y) \geq 1$ and $H(Tx, Ty) > 0$.

Remark 3.1. Let (X, d) be a metric space. If $T\colon X \to CL(X)$ is a multivalued (α-θ-ψ)-contraction, then by (θ1) and (2), we deduce that

$$H(Tx, Ty) < \psi(d(x,y)),$$

for all $x, y \in X$ with $\alpha(x,y) \geq 1$ and $H(Tx, Ty) > 0$. Hence, we have

$$H(Tx, Ty) \leq \psi(d(x,y)),$$

for all $x, y \in X$ with $\alpha(x,y) \geq 1$.

Now, we can state the first result of this paper.

Theorem 3.1. *Let (X,d) be a complete metric space and $T\colon X\to K(X)$ be a multivalued $(\alpha$-θ-$\psi)$-contraction. Assume that the following conditions are satisfied:*

(i) T is α-admissible;
(ii) There exist $x_0\in X$ and $x_1\in Tx_0$ such that $\alpha(x_0,x_1)\geq 1$;
(iii) T is continuous or X is α-regular.

Then T has a fixed point.

Proof. By the assumption (ii), there exist $x_0\in X$ and $x_1\in Tx_0$ such that $\alpha(x_0,x_1)\geq 1$. If $x_0=x_1$ or $x_1\in Tx_1$, then x_1 is a fixed point of T and so the proof is completed. Because of this, assume that $x_0\neq x_1$ and $x_1\notin Tx_1$, then $d(x_1,Tx_1)>0$ and hence $H(Tx_0,Tx_1)>0$. Since Tx_1 is compact, there exists $x_2\in Tx_1$ such that $d(x_1,x_2)=d(x_1,Tx_1)$. Now, considering (2) and $(\theta 1)$, we infer

$$\begin{aligned}1<\theta(d(x_1,x_2))&=\theta(d(x_1,Tx_1))\leq\theta(H(Tx_0,Tx_1))\\&\leq[\theta(\psi(d(x_0,x_1)))]^k\\&<[\theta(d(x_0,x_1))]^k.\end{aligned}$$

Following the previous procedures, we can assume that $x_1\neq x_2$ and $x_2\notin Tx_2$. Then $d(x_2,Tx_2)>0$, and so $H(Tx_1,Tx_2)>0$. Since, $\alpha(x_0,x_1)\geq 1$ and T is an α-admissible multivalued mapping, we derive that $\alpha(x_1,x_2)\geq 1$ for $x_2\in Tx_1$. Also, since Tx_2 is compact, there exists $x_3\in Tx_2$ such that $d(x_2,x_3)=d(x_2,Tx_2)$. Regarding $(\theta 1)$ and (2), we deduce

$$\begin{aligned}1<\theta(d(x_2,x_3))&=\theta(d(x_2,Tx_2))\leq\theta(H(Tx_1,Tx_2))\\&\leq[\theta(\psi(d(x_1,x_2)))]^k\\&<[\theta(d(x_1,x_2))]^k.\end{aligned}$$

Repeating this process, we can constitute a sequence $\{x_n\}\subset X$ such that $x_n\neq x_{n+1}\in Tx_n$, $\alpha(x_n,x_{n+1})\geq 1$ and

$$1<\theta(d(x_n,x_{n+1}))<[\theta(d(x_{n-1},x_n))]^k, \tag{3}$$

for all $n\in\mathbb{N}$. Letting $\rho_n:=d(x_n,x_{n+1})$ for all $n\in\mathbb{N}\cup\{0\}$, from (3), we get

$$1<\theta(\rho_n)<[\theta(\rho_0)]^{k^n},\qquad\text{for all } n\in\mathbb{N}. \tag{4}$$

This implies that

$$\lim_{n\to+\infty}\theta(\rho_n)=1,$$

and by $(\theta 2)$, we have

$$\lim_{n\to+\infty}\rho_n=0. \tag{5}$$

To prove that $\{x_n\}$ is a Cauchy sequence, let us consider condition $(\theta 3)$. Then there exist $r\in(0,1)$ and $\lambda\in(0,+\infty]$ such that

$$\lim_{n\to+\infty}\frac{\theta(\rho_n)-1}{(\rho_n)^r}=\lambda. \tag{6}$$

Take $\delta\in(0,\lambda)$. By the definition of limit, there exists $n_0\in\mathbb{N}$ such that

$$[\rho_n]^r\leq\delta^{-1}[\theta(\rho_n)-1],\qquad\text{for all } n>n_0.$$

Using (4) and the above inequality, we deduce

$$n[\rho_n]^r\leq\delta^{-1}n([\theta(\rho_0)]^{k^n}-1),\qquad\text{for all } n>n_0.$$

This implies that

$$\lim_{n\to+\infty}n[\rho_n]^r=\lim_{n\to+\infty}n[d(x_n,x_{n+1})]^r=0.$$

Thence, there exists $n_1 \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{1/r}}, \quad \text{for all } n > n_1. \tag{7}$$

Let $m > n > n_1$. Then, using the triangular inequality and (7), we have

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \leq \sum_{k=n}^{m-1} \frac{1}{k^{1/r}} \leq \sum_{k=n}^{\infty} \frac{1}{k^{1/r}}$$

and hence $\{x_n\}$ is a Cauchy sequence in X. From the completeness of (X, d), there exists $u \in X$ such that $x_n \to u$ as $n \to +\infty$.

If T is continuous, then

$$\lim_{n\to+\infty} H(Tx_n, Tu) = 0,$$

which gives that

$$d(u, Tu) = \lim_{n\to+\infty} d(x_{n+1}, Tu) \leq \lim_{n\to+\infty} H(Tx_n, Tu) = 0,$$

and so $d(u, Tu) = 0$. Since Tu is closed, we obtain that $u \in Tu$, that is, u is a fixed point of T.

If X is α-regular, then $\alpha(x_n, u) \geq 1$ for all $n \in \mathbb{N}$. If there exists $k \in \mathbb{N}$ such that $d(x_{k+1}, Tu) = 0$, then from the uniqueness of limit, $d(u, Tu) = 0$. So the proof is finished. Hence, there exists $n_2 \in \mathbb{N}$ such that $d(x_{n+1}, Tu) > 0$ and so $H(Tx_n, Tu) > 0$ for all $n > n_2$. Considering Remark 3.1, we have

$$d(x_{n+1}, Tu) \leq H(Tx_n, Tu) < \psi(d(x_n, u)) < d(x_n, u),$$

and so

$$0 < d(x_{n+1}, Tu) < d(x_n, u), \quad \text{for all } n > n_2.$$

Passing to limit as $n \to +\infty$ in the above inequality, we obtain $d(u, Tu) = 0$ and so $u \in Tu$. □

In the next theorem, we replace $K(X)$ with $CB(X)$ by considering an additional condition for the function θ.

Theorem 3.2. *Let (X, d) be a complete metric space and $T\colon X \to CB(X)$ be a multivalued (α-θ-ψ)-contraction with right continuous function $\theta \in \Theta$. Assume that the following conditions are satisfied:*

- *(i) T is α-admissible;*
- *(ii) There exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;*
- *(iii) T is continuous or X is α-regular.*

Then T has a fixed point.

Proof. Starting with (*ii*), there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. Arguing similar lines in Theorem 3.1, we can assume that $x_0 \neq x_1$ and $x_1 \notin Tx_1$. By ($\theta 1$) and (2), we get

$$\theta(H(Tx_0, Tx_1)) \leq [\theta(\psi(d(x_0, x_1)))]^k < [\theta(d(x_0, x_1))]^k.$$

By the property of right continuity of $\theta \in \Theta$, there exists a real number $h_1 > 1$ such that

$$\theta(h_1 H(Tx_0, Tx_1)) \leq [\theta(d(x_0, x_1))]^k. \tag{8}$$

From $d(x_1, Tx_1) < h_1 H(Tx_0, Tx_1)$, by Lemma 2.1, there exists $x_2 \in Tx_1$ such that $d(x_1, x_2) \leq h_1 H(Tx_0, Tx_1)$. Then, using ($\theta 1$), (8) and last inequality, we infer that

$$\theta(d(x_1, x_2)) \leq \theta(h_1 H(Tx_0, Tx_1)) \leq [\theta(d(x_0, x_1))]^k.$$

In view of the fact that T is α-admissible and $\alpha(x_0, x_1) \geq 1$, we have $\alpha(x_1, x_2) \geq 1$ for $x_2 \in Tx_1$. Assume that $x_2 \notin Tx_2$. Since θ is right continuous, there exists $h_2 > 1$ such that

$$\theta(h_2 H(Tx_1, Tx_2)) \leq [\theta(d(x_1, x_2))]^k. \tag{9}$$

From $d(x_2, Tx_2) < h_2 H(Tx_1, Tx_2)$, by Lemma 2.1, there exists $x_3 \in Tx_2$ such that $d(x_2, x_3) \leq h_2 H(Tx_1, Tx_2)$. Then, using ($\theta$1), (9) and last inequality, we deduce that

$$\theta(d(x_2, x_3)) \leq \theta(h_2 H(Tx_1, Tx_2)) \leq [\theta(d(x_1, x_2))]^k \leq [\theta(d(x_0, x_1))]^{k^2}.$$

Continuing in this manner, we build two sequences $\{x_n\} \subset X$ and $\{h_n\} \subset (1, +\infty)$ such that $x_n \neq x_{n+1} \in Tx_n$, $\alpha(x_n, x_{n+1}) \geq 1$ and

$$1 < \theta(d(x_n, x_{n+1})) \leq \theta(h_n H(Tx_{n-1}, Tx_n)) \leq [\theta(d(x_{n-1}, x_n))]^k, \text{ for all } n \in \mathbb{N}.$$

Hence,

$$1 < \theta(d(x_n, x_{n+1})) \leq [\theta(d(x_0, x_1))]^{k^n}, \quad \text{for all } n \in \mathbb{N}.$$

which gives that

$$\lim_{n \to +\infty} \theta(d(x_n, x_{n+1})) = 1.$$

From (θ2), we obtain

$$\lim_{n \to +\infty} d(x_n, x_{n+1}) = 0.$$

The rest of the proof is like in the proof of Theorem 3.1. □

The following example illustrates Theorem 3.2 (resp. Theorem 3.1) where Theorems 2.1 and 2.2 are not applicable.

Example 3.1. Let $X = [0, +\infty)$ with the usual metric $d(x, y) = |x - y|$ for all $x, y \in X$. Define $T\colon X \to CB(X)$ and $\alpha\colon X \times X \to [0, +\infty)$ by

$$Tx = \begin{cases} \left[0, \dfrac{x}{8}\right], & \text{if } x \in [0, 3], \\ [0, x], & \text{if } x > 3, \end{cases} \quad \text{and} \quad \alpha(x, y) = \begin{cases} \dfrac{9}{2}, & \text{if } x, y \in [0, 3], \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, $\alpha(x, y) \geq 1$ and $H(Tx, Ty) > 0$ for each $x, y \in [0, 3]$ with $x \neq y$.

Firstly, we claim that T is a multivalued (α-θ-ψ)-contraction with $k = \frac{1}{2}$, $\psi(t) = \frac{t}{2}$ and $\theta(t) = e^{\sqrt{te^t}}$. For all $x, y \in [0, 3]$ with $x \neq y$,

$$\begin{aligned} \theta(H(Tx, Ty)) &= \theta\left(\frac{|x - y|}{8}\right) \\ &= e^{\sqrt{\frac{|x-y|}{8} e^{\frac{|x-y|}{8}}}} \\ &\leq e^{\frac{1}{2}\sqrt{\frac{|x-y|}{2} e^{\frac{|x-y|}{2}}}} \\ &= e^{\frac{1}{2}\sqrt{\psi(d(x,y)) e^{\psi(d(x,y))}}} \\ &= [\theta(\psi(d(x, y)))]^k, \end{aligned}$$

that is, the condition (2) is satisfied. Moreover, it is easy to see that T is an α-admissible multivalued mapping and there exist $x_0 = 3$ and $x_1 = 3/8 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$.

For each sequence $\{x_n\}$ in X with $x_n \to x \in X$ as $n \to +\infty$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all n, we have $x, x_n \in [0, 3]$ for all n. Hence, $\alpha(x_n, x) \geq 1$ for all n, that is, X is α-regular. Consequently, all conditions of Theorem 3.2 (resp. Theorem 3.1) are satisfied. Then T has a fixed point in X. Note that the set of fixed points of T is not finite.

On the other side, for $x = 0$ and $y = 4$, we have

$$\theta\left(H\left(Tx, Ty\right)\right) = \theta\left(H\left(T0, T4\right)\right) = \theta\left(4\right) > \left[\theta\left(4\right)\right]^k = \left[\theta\left(d(x, y)\right)\right]^k,$$

for all $\theta \in \Theta$ and $k \in (0, 1)$. Therefore, T is not weak θ-contraction and hence Theorem 2.2 can not applied to this example.

Also, for $x = 0$ and $y = 3$, we get

$$\alpha(x, y)H\left(Tx, Ty\right) = \alpha(0, 3)H\left(T0, T3\right) = \frac{9}{2} \cdot \frac{3}{8} = \frac{27}{16} > \frac{3}{2} = \psi(d(x, y)),$$

for $\psi(t) = \frac{t}{2}$. Thus, T is not multivalued α-ψ-contraction and so Theorem 2.1 can not applied to this example.

Since each α_*-admissible mapping is also α-admissible, we obtain following result.

Corollary 3.1. *Let (X, d) be a complete metric space, $\alpha\colon X \times X \to [0, +\infty)$ be a function and $T\colon X \to CB(X)$ (resp. $K(X)$) be a multivalued mapping. Assume that the following assertions hold:*

(*i*) *T is an α_*-admissible;*
(*ii*) *There exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha\left(x_0, x_1\right) \geq 1$;*
(*iii*) *T is continuous or X is α-regular;*
(*iv*) *There exist $k \in (0, 1)$, $\psi \in \Psi$ and $\theta \in \Theta$ such that*

$$\theta(H(Tx, Ty)) \leq [\theta(\psi(d(x, y)))]^k,$$

for all $x, y \in X$ with $\alpha_*(Tx, Ty) \geq 1$ and $H(Tx, Ty) > 0$.

Then T has a fixed point.

Corollary 3.2. *Let (X, d) be a complete metric space, $\alpha\colon X \times X \to [0, +\infty)$ be a function and $T\colon X \to CB(X)$ (resp. $K(X)$) be an α-admissible multivalued mapping and the following assertions hold:*

(*i*) *There exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha\left(x_0, x_1\right) \geq 1$;*
(*ii*) *T is continuous or X is α-regular;*
(*iii*) *there exist $k \in (0, 1)$, $\psi \in \Psi$ and $\theta \in \Theta$ such that*

$$x, y \in X, \quad H(Tx, Ty) > 0 \Rightarrow \theta(\alpha(x, y)H(Tx, Ty)) \leq [\theta(\psi(d(x, y)))]^k. \tag{10}$$

Then T has a fixed point.

Proof. Let $x, y \in X$ such that $\alpha\left(x, y\right) \geq 1$ and $H(Tx, Ty) > 0$. Using (θ1) and (10), we have

$$\theta(H(Tx, Ty)) \leq \theta(\alpha(x, y)d(Tx, Ty)) \leq [\theta(\psi(d(x, y)))]^k,$$

and hence

$$\theta(H(Tx, Ty)) \leq [\theta(\psi(d(x, y)))]^k,$$

for all $x, y \in X$ with $\alpha\left(x, y\right) \geq 1$ and $H(Tx, Ty) > 0$. This implies that the inequality (2) holds. Thus, the rest of proof follows from Theorem 3.2 (resp. Theorem 3.1). □

4. Some Consequences

In this section we give new fixed point results on a metric space endowed with a partial ordering/graph, by using the results provided in previous section. Define

$$\alpha\colon X \times X \to [0, +\infty), \quad \alpha\left(x, y\right) = \begin{cases} 1, & \text{if } x \preceq y, \\ 0, & \text{otherwise.} \end{cases}$$

Then the following result is a direct consequence of our results.

Theorem 4.1. *Let $(X, \preceq, d)$ be a complete ordered metric space and $T\colon X \to CB(X)$ (resp. $K(X)$) be a multivalued mapping. Assume that the following assertions hold:*

(i) For each $x \in X$ and $y \in Tx$ with $x \preceq y$, we have $y \preceq z$ for all $z \in Ty$;
(ii) There exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $x_0 \preceq x_1$;
(iii) T is continuous or, for every sequence $\{x_n\}$ in X such that $x_n \to x \in X$ and $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$, we have $x_n \preceq x$ for all $n \in \mathbb{N}$;
(iv) There exist $k \in (0,1)$, $\psi \in \Psi$ and $\theta \in \Theta$ such that

$$\theta(H(Tx, Ty)) \leq [\theta(\psi(d(x, y)))]^k,$$

for all $x, y \in X$ with $x \preceq y$ and $H(Tx, Ty) > 0$.

Then T has a fixed point.

Now, we present the existence of fixed point for multivalued mappings from a metric space X, endowed with a graph, into the space of nonempty closed and bounded subsets of the metric space. Consider a graph G such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains all loops; that is, $E(G) \supseteq \Delta$, where $\Delta = \{(x, x) : x \in X\}$. We assume G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$.

If we define the function

$$\alpha\colon X \times X \to [0, +\infty), \quad \alpha(x, y) = \begin{cases} 1, & \text{if } (x, y) \in E(G), \\ 0, & \text{otherwise}, \end{cases}$$

then the following result is a direct consequence of our results.

Theorem 4.2. *Let (X, d) be a complete metric space endowed with a graph G and $T\colon X \to CB(X)$ (resp. $K(X)$) be a multivalued mapping. Assume that the following conditions are satisfied:*

(i) For each $x \in X$ and $y \in Tx$ with $(x, y) \in E(G)$, we have $(y, z) \in E(G)$ for all $z \in Ty$;
(ii) There exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E(G)$;
(iii) T is continuous or, for every sequence $\{x_n\}$ in X such that $x_n \to x \in X$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, we have $(x_n, x) \in E(G)$ for all $n \in \mathbb{N}$;
(iv) There exist $k \in (0,1)$, $\psi \in \Psi$ and $\theta \in \Theta$ such that

$$\theta(H(Tx, Ty)) \leq [\theta(\psi(d(x, y)))]^k,$$

for all $x, y \in X$ with $(x, y) \in E(G)$ and $H(Tx, Ty) > 0$.

Then T has a fixed point.

5. An Application

Consider the following integral equation:

$$p(r) = q(r) + \lambda \int_a^b H(r, z) f(z, p(z)) dz, \quad r \in I = [a, b], \tag{11}$$

where $q\colon I \to \mathbb{R}$, $H\colon I \times I \to \mathbb{R}$, $f\colon I \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

In this section, we establish the existence of solutions for the integral equation (11) that belongs to the space $X := C(I, \mathbb{R})$ of the continuous functions defined on I and with real values. Let X be endowed with the metric d defined by

$$d(x, y) = \|x - y\|_\infty \quad \text{for all } x, y \in X.$$

Then (X, d) is a complete metric space.

We define an operator

$$T\colon X\to X,\quad Tp(r):=q(r)+\lambda\int_a^b H\,(r,z)\,f(z,p(z))dz,\quad r\in I,$$

then the existence of solutions of (11) is equivalent to the existence of fixed points of T.

We will analyze (11) under the following assumptions:

(A) $|\lambda|\le 1$;

(B) For each $z\in I$ and for all $x,y\in X$ with $(x,y)\in E(G)$ and $x\neq y$, there exists $\beta\in(0,+\infty)$ such that

$$|f(z,x(z))-f(z,y(z))|\le \xi(x,y)(|x(z)-y(z)|)$$

and

$$\left\|\int_a^b H\,(r,z)\,\xi(x,y)dz\right\|_\infty\le\frac{e^{-\beta}}{2};$$

(C) $x,y\in X$, $(x,y)\in E(G)$ implies $(Tx,Ty)\in E(G)$;

(D) There exists $x_0\in X$ such that $\xi\,(x_0,Tx_0)\in E(G)$;

(E) if $\{x_n\}$ is a sequence in X such that $x_n\to x\in X$ and $(x_n,x_{n+1})\in E(G)$ for all $n\in\mathbb{N}$, then $(x_n,x)\in E(G)$ for all $n\in\mathbb{N}$.

Theorem 5.1. *Under the assumptions* (*A*)-(*E*)*, the integral equation* (11) *has at least one solution in* X*.*

Proof. Let $(x,y)\in E(G)$ and $Tx\neq Ty$. On account of (A), for all $r\in I$

$$\begin{aligned}|Tx(r)-Ty(r)|&=\left|\lambda\left(\int_a^b H\,(r,z)\,f(z,x(z))dz-\int_a^b H\,(r,z)\,f(z,y(z))dz\right)\right|\\&=|\lambda|\left|\int_a^b H\,(r,z)\,[f(z,x(z))-f(z,y(z))]\,dz\right|\\&\le|\lambda|\int_a^b H\,(r,z)\,|f(z,x(z))-f(z,y(z))|\,dz\\&\le|\lambda|\int_a^b H\,(r,z)\,\xi(x,y)|x(z)-y(z)|dz\\&\le|\lambda|\,\|x-y\|_\infty\int_a^b H\,(r,z)\,\xi(x,y)dz\\&\le\|x-y\|_\infty\int_a^b H\,(r,z)\,\xi(x,y)dz.\end{aligned}$$

Thus, we have

$$\|Tx-Ty\|_\infty\le\|x-y\|_\infty\left\|\int_a^b H\,(r,z)\,\xi(x,y)dz\right\|_\infty,$$

and hence

$$d(Tx,Ty)\le\frac{e^{-\beta}}{2}d(x,y).$$

Since $\theta(t)=e^{\sqrt{t}}\in\Theta$ for all $t>0$, by the last inequality, we infer

$$e^{\sqrt{d(T(x,y))}}\le e^{\sqrt{e^{-\beta}\frac{d(x,y)}{2}}}\le\left[e^{\sqrt{\frac{d(x,y)}{2}}}\right]^k$$

which implies that, for all $x, y \in X$ with $(x, y) \in E(G)$ and $Tx \neq Ty$

$$\theta(d(T(x,y)) \leq [\theta(\psi(d(x,y))]^k ,$$

where $k = \sqrt{e^{-\beta}}$ and $\psi(t) = t/2$ for all $t \geq 0$. Consequently, all conditions of Theorem 4.2 are fulfilled and so T has a fixed point, that is, the integral equation (11) has at least one solution in X. □

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