FRACTIONAL PROJECTED DYNAMICAL SYSTEM FOR QUASI VARIATIONAL INEQUALITIES

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In this paper, we introduce a dynamical system associated with quasi variational inequalities using projection operator technique. This dynamical system is called implicit fractional projected dynamical system. We show that the implicit fractional projected dynamical system is exponentially stable and converges to its unique equilibrium point under some suitable conditions. Some special cases are discussed, which can be obtained from our results. Results obtained in this paper continue to hold for these problems.

Keywords: Dynamical systems; Fractional derivative; Convergence; Quasi-variational inequalities.

1. Introduction

Quasi variational inequalities were introduced and studied by Bensoussan and Lions [2, 3] in impulse control system. It is well known that the set involved in the quasi variational inequalities depends upon the solution explicitly or implicitly. We remark that if the involved set does not depend upon the solution then quasi variational inequality reduces to the variational inequality, the origin of which can be traced back to Stampacchia [39]. Variational inequalities and quasi variational inequalities provide us a unifying and an efficient framework to study various related and unrelated problems which arise in different branches of pure and applied sciences, see [1-46] and references therein.

Dynamics is a concise term referring to the study of time evolving processes, and the corresponding system of equations, which describes this evolution, is called a dynamical system. Nonlinear systems are widely used as models to describe complex physical phenomena in various field of sciences, such as fluid dynamics, solid state physics, plasma physics, mathematical biology and chemical kinetics, vibrations, heat transfer and so on. It is well known that these problems can be studied via the quasi variational inequalities. Dynamical systems can be solved by using some analytical techniques such as Homotopy Perturbation Method, Variational Iteration Method, Neural Network techniques and their variant forms, see [19, 20, 24, 46] and references therein. For recent developments in nonlinear dynamical systems see [5, 6, 12, 21, 22].

In recent years, several dynamical systems associated with variational inequalities are being investigated using the projection operator methods and Wiener-Hopf equations. This can be traced back Dupuis and Nagurney [8], Friesz et al [10] and Noor [29]. The dynamical systems method is more attractive due to its wide applicability, flexibility and numerical efficiency. In this method, variational inequality problem is reformulated as an initial value

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problem. It enables us to study the stability properties of the unique solution of the variational inequality problem. There are two types of the projected dynamical systems. The first type is due to Dupuis and Nagurney [8], which is known as local projected dynamical systems where as, the second one which is due to Friesz et al [10] is called global projected dynamical systems. In projected dynamical systems the right-hand side of the ordinary differential equation is a projection operator and is discontinuous. The discontinuity arises from the constraints governing the applications in question. The novel and innovative feature of the projected dynamical systems is that the set of the stationary/equilibrium points of the solutions of the dynamical systems correspond to the set of the solutions of the variational inequalities. Consequently, the equilibrium problems which can be formulated in the setting of variational inequalities can now be studied in the more general setting of the dynamical systems. These dynamical systems enable us to describe the trajectories of real economics and physical process prior to reaching steady states. Noor [29] has introduced Wiener-Hopf dynamical systems for variational inequalities using fixed point formulation. Xia and Wang [41] have shown that the projected dynamical systems can be used effectively in designing neural network for solving variational inequalities, see [19, 20]. The neural network methods are used to find the approximate solutions of the global projected dynamical systems. Neural network methods are robust and efficient one.

Recently, Zeng-bao and Yun-zhi [46] investigated the fractional dynamical systems associated with linear variational inequalities. They have investigated the criteria for the asymptotically stability of the equilibrium points. In this paper, we consider the implicit fractional projected dynamical systems associated with quasi variational inequalities. We use this implicit fractional projected dynamical system to investigate the existence of the equilibrium points under some mild conditions. We also obtain the exponentially stability of the equilibrium point. Some special cases are discussed. Our results represent a significant extension of the results of Zeng-bao and Yun-zhi [46]. The ideas and techniques of this paper may inspire the interested readers for further research in this area.

2. **Formulation and Basic Results**

Let $\mathbb{R}^n$ be an n-dimensional Euclidean space, whose norm and inner product are denoted by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$, respectively. Let K be any closed and convex set in $\mathbb{R}^n$. Let $K : u \to K(u)$ be a point-to-set mapping which associates a closed and convex-valued set $K(u)$ of $\mathbb{R}^n$ with any element u of $\mathbb{R}^n$.

For given operator $T : \mathbb{R}^n \to \mathbb{R}^n$, consider the problem of finding $u \in K(u)$ such that

$$\langle Tu, v - u \rangle \geq 0, \quad \forall v \in K(u). \tag{1}$$

The inequality of type (1) is called the quasi variational inequality. The quasi variational inequalities were introduced and studied by Bensoussan and Lions [2, 3] in the study of impulse control system. For recent applications and numerical methods for solving quasi variational inequalities, see [9, 15, 32] and references therein.

If $K(u) = K$, then problem (1) is equivalent to finding $u \in K$ such that

$$\langle Tu, v - u \rangle \geq 0, \quad \forall v \in K, \tag{2}$$

which is called the classical variational inequality, introduced and studied by Stampacchia [39]. For the recent applications, numerical algorithms, sensitivity analysis, dynamical systems and other aspects of variational inequalities, see [1-28] and the references therein.

We also need the following well-known fundamental results and concepts.

Definition 2.1. *A nonlinear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be strongly monotone, if there exists a constant $\alpha > 0$ such that*

$$\langle Tu - Tv, u - v \rangle \geq \alpha \|u - v\|^2, \quad \forall u, v \in \mathbb{R}^n.$$

Definition 2.2. *A nonlinear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be Lipschitz continuous, if there exists a constant $\beta > 0$ such that*

$$\|Tu - Tv\| \leq \beta \|u - v\|, \quad \forall u, v \in \mathbb{R}^n.$$

If the operator T is strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$, then $\alpha \leq \beta$.

Lemma 2.1. [31] *Let K be a closed and convex set in $\mathbb{R}^n$. Then for a given $z \in \mathbb{R}^n$, $u \in K$ satisfies*

$$\langle u - z, v - u \rangle \geq 0, \quad \forall v \in K,$$

if and only if

$$u = P_K [z],$$

where P_K is the projection of $\mathbb{R}^n$ onto the closed and convex set K in $\mathbb{R}^n$.

It is well known that the projection operator P_K is nonexpansive, that is

$$\|P_K [u] - P_K [v]\| \leq \|u - v\|, \quad \forall u, v \in \mathbb{R}^n.$$

Using Lemma 2.1, one can show that problem (1) is equivalent to the fixed point problem.

Lemma 2.2. [31] *Let $K(u)$ be a closed and convex valued set in $\mathbb{R}^n$. The function $u \in K(u)$ is a solution of problem (1), if and only if, $u \in K(u)$ satisfies the relation*

$$u = P_{K(u)} [u - \rho T u], \tag{3}$$

where $P_{K(u)}$ is the implicit projection operator from $\mathbb{R}^n$ onto the closed and convex valued set $K(u)$ and $\rho > 0$ is a constant.

From Lemma 2.2, it follows that problem (1) is equivalent to a fixed point problem (3). This equivalent formulation plays a crucial part in developing several iterative methods, see [30, 31, 32].

We would like to mention that the implicit projection operator $P_{K(u)}$ is not non-expansive. We assume that the implicit projection operator $P_{K(u)}$ satisfies the following condition, which has played an important role in the analysis of the convergence criteria of quasi variational inequalities, see [4].

Assumption 2.1. [31] *The implicit projection operator $P_{K(u)}$ satisfies the condition*

$$\|P_{K(u)} [w] - P_{K(v)} [w]\| \leq \nu \|u - v\|, \quad \text{for all } u, v, w \in \mathbb{R}^n, \tag{4}$$

where $\nu > 0$ is a constant.

Remark 2.1. [4, 26] It is known that Assumption 2.1 holds true, if

$$K(u) = m(u) + K, \tag{5}$$

where $m(u)$ is a point-to-point mapping and K is a closed convex set in $\mathbb{R}^n$. It is well known that

$$P_{K(u)} [w] = P_{m(u)+K} [w] = m(u) + P_K [w - m(u)], \quad \forall u, v \in \mathbb{R}^n. \tag{6}$$

We note that if $K(u)$ is as defined by (5) and $m(u)$ is a Lipschitz continuous mapping with constant $\gamma > 0$, then using (6), we have

$$\begin{aligned}
\left\|P_{K(u)} [w] - P_{K(v)} [w]\right\| &= \|m(u) - m(v) + P_K [w - m(u)] - P_K [w - m(v)]\| \\
&\leq \|m(u) - m(v)\| + \|P_K [w - m(u)] - P_K [w - m(v)]\| \\
&\leq 2 \|m(u) - m(v)\| \leq 2\gamma \|u - v\|, \quad \forall u, v \in \mathbb{R}^n,
\end{aligned}$$

which shows that Assumption 2.1 holds with $\nu = 2\gamma > 0$.

We note that if the point-to-point mapping $m(u)$ is zero, that is $m(u) = 0$, then from (5), we have $K(u) = K$, the closed convex set. It is obvious that, Assumption 2.1 is not required. In fact, we have $P_{K(u)} = P_K$, the projection of $\mathbb{R}^n$ onto the closed convex set K, which is nonexpansive mapping.

We now define the residue vector $\mathcal{R}(u)$ as:

$$\mathcal{R}(u) = u - P_{K(u)}[u - \rho Tu]. \tag{7}$$

It is clear from Lemma 2.2 that problem (1) has a solution $u \in K(u)$, if and only if, $u \in K(u)$ is a zero of the equation

$$\mathcal{R}(u) = 0. \tag{8}$$

3. **Projected Dynamical System**

We use the residue vector, defined in (7), to consider the following dynamical system

$$\begin{aligned} \frac{du}{dt} &= -\gamma \mathcal{R}(u) \\ &= \gamma \left\{ P_{K(u)}[u - \rho Tu] - u \right\}, \quad u(t_0) = u_0 \in K(u), \end{aligned} \tag{9}$$

where γ is any constant, associated with problem (1). The dynamical system (9) is called the implicit projected dynamical system. It was introduced and studied by Noor [30]. Since right hand side is related to the projection operator, and thus, is discontinuous on the boundary of $\mathbb{R}^n$. It is clear from the definition that the solution of dynamical system (9) belongs to the constraint set. From this it is clear that the results such as the existence, uniqueness and continuous dependence on the given data can be studied.

During recent years the interest of mathematicians in the field of fractional calculus has been steadily increasing. The main reason for this development in the expectation that the use of these techniques will lead to a much more elegant and effective way of treating problems in different field of mathematics. Using these fractional techniques Zeng-bao and Yun-zhi [46] introduced the fractional dynamical systems for linear variational inequalities, Li et al [18] considered the stability of fractional order nonlinear dynamic systems and Yu et al [45] studied fractional ordered neural networks. Inspired and motivated by these developments, we suggest a dynamical system for linear quasi variational inequalities.

We now introduce an implicit fractional projected dynamical system associated with problem (1) as:

$${}_0^C D_t^{\alpha} u(t) = \gamma \left\{ P_{K(u)}[u - \rho Tu] - u \right\}, \quad u(t_0) = u_0 \in K(u), \tag{10}$$

where $0 < \alpha < 1$, $\rho > 0$, and γ is any parameter. For the sake of simplicity, we take $\gamma = 1$.

If we take $\alpha = 1$, then implicit fractional projected dynamical system (10) reduces to the implicit projected dynamical system (9), which was introduced and studied by [30].

In the affine case, that is, if $Tu = Au + b$, where $A = a_{ij}$ is a real $n \times n$ matrix and $b = b_j$ is an n-dimensional vector, then the proposed dynamical system (10) can be written as:

$$\left. \begin{aligned} {}_0^C D_t^{\alpha} u(t) &= P_{K(u(t))}[u(t) - \rho A u(t) - \rho b] - u(t), \quad t \geq 0 \\ &= P_{K(u_i(t))}\left[u_i(t) - \rho \sum_{j=1}^{n} a_{ij} u_j(t) - \rho b_i \right] - u_i(t), \quad t \geq 0 \\ u_i(0) &= u_{i0}, \quad i = 1, 2, \ldots, n \end{aligned} \right\}. \tag{11}$$

It is important to note that, if $K(u) = K$, a closed and convex set in $\mathbb{R}^n$, then implicit fractional projected dynamical system (11) reduces to the following dynamical system, which was introduced and studied by [46].

$$\left.\begin{array}{c} {}_{0}^{C}D_{t}^{\alpha}u\left(t\right)=P_{K}\left[u\left(t\right)-\rho Au\left(t\right)-\rho b\right]-u\left(t\right), \quad t\geq 0 \\ u_{i}\left(0\right)=u_{i0}, \quad i=1,2,\ldots,n \end{array}\right\}, \tag{12}$$

where $0<\alpha<1$, $\rho>0$ is a constant, K is a closed and convex set in $\mathbb{R}^n$, $A=a_{ij}$ is a real $n\times n$ matrix and $b=b_i$ is an n-dimensional vector.

Definition 3.1. [16] *The fractional integral (or, Riemann-Liouville integral) with order $\alpha\in\mathbb{R}_+$ of continuous function $u(t)$ is defined as:*

$$I^{\alpha}u\left(t\right)=\frac{1}{\Gamma\left(\alpha\right)}\int_{t_0}^{t}\left(t-\tau\right)^{\alpha-1}u\left(\tau\right)d\tau, \quad t>t_0.$$

Definition 3.2. [16] *The Caputo derivative with order $\alpha\in\mathbb{R}_+$ of continuous function $u(t)\in C^n\left([t_0,+\infty],\mathbb{R}\right)$ is defined as:*

$${}_{t_0}^{C}D_{t}^{\alpha}u\left(t\right)=I^{n-\alpha}u^{(n)}\left(t\right)=\frac{1}{\Gamma\left(n-\alpha\right)}\int_{t_0}^{t}\left(t-\tau\right)^{n-\alpha-1}u^{(n)}\left(\tau\right)d\tau, \quad t>t_0,$$

where n is positive integer such that $n-1<\alpha<n$.

Definition 3.3. [46] *The point $u^*=\left(u_1^*,u_2^*,\ldots,u_n^*\right)$ is said to be an equilibrium point of the fractional order implicit dynamical system* (11)*, if and only if, $u^*=\left(u_1^*,u_2^*,\ldots,u_n^*\right)$ satisfies the following*

$$P_{K\left(u_i^*(t)\right)}\left[u_i^*\left(t\right)-\rho\sum_{j=1}^{n}a_{ij}u_j^*\left(t\right)-\rho b_i\right]-u_i^*\left(t\right)=0, \quad i=1,2,\ldots,n.$$

Definition 3.4. [45] *The dynamical system* (11) *is said to be α-exponentially stable with degree λ, if for any two solutions $u(t)$ and $v(t)$ of* (11) *with different initial values by u_0 and v_0 satisfies*

$$\left\|u\left(t\right)-v\left(t\right)\right\|\leq\eta\left\|u_0-v_0\right\|e^{-\lambda t^{\alpha}}, \quad \forall t\geq t_0,$$

where $\eta>0$ is a constant.

Lemma 3.1. *(Gronwall's Lemma* [30]*) Let u and v be real valued non-negative continuous functions with domain $\{t:t\geq t_0\}$ and let $\alpha(t)=\alpha_0|t-t_0|$, where α_0 is a monotone increasing function. If, for $t\geq t_0$,*

$$u\left(t\right)\leq\alpha\left(t\right)+\int_{t_0}^{t}u\left(s\right)v\left(s\right)ds,$$

then

$$u\left(t\right)\leq\alpha\left(t\right)\exp\left(\int_{t_0}^{t}v\left(s\right)ds\right).$$

Lemma 3.2. [16] *Let n be a positive integer such that $n-1<\alpha<n$. If $u(t)\in C^n[a,b]$, then*

$$I^{\alpha}\,{}_{\alpha}^{C}D_{t}^{\alpha}u\left(t\right)=u\left(t\right)-\sum_{k=0}^{n-1}\frac{u^{(k)}\left(a\right)}{k!}\left(t-a\right)^{k}.$$

In particular, if $0<\alpha\leq 1$ and $u(t)\in C^1[a,b]$, then

$$I^{\alpha}\,{}_{\alpha}^{C}D_{t}^{\alpha}u\left(t\right)=u\left(t\right)-u\left(a\right). \tag{13}$$

Using the technique of [45] we have the following result. For the sake of completeness and to convey an idea, we include its proof.

Lemma 3.3. [45] *Let $u(t)$ be a continuous function on $[0,+\infty)$ and satisfies*

$$ {}_{\alpha}^{C}D_{t}^{\alpha}u(t) \leq \theta u(t), \tag{14}$$

where $0<\alpha\leq 1$ and θ is a constant. Then

$$u(t) \leq u(0)\exp\left(\frac{\theta t^{\alpha}}{\Gamma(\alpha+1)}\right).$$

Proof. For a nonnegative continuous function $h(t)$, relation (14) can be written as:

$$ {}_{\alpha}^{C}D_{t}^{\alpha}u(t) + h(t) = \theta u(t). \tag{15}$$

By taking the fractional integral of order α of (15), we have

$$I^{\alpha}\, {}_{\alpha}^{C}D_{t}^{\alpha}u(t) + I^{\alpha}h(t) = I^{\alpha}\theta u(t). \tag{16}$$

Using Lemma 3.2, we have

$$I^{\alpha}\, {}_{\alpha}^{C}D_{t}^{\alpha}u(t) = u(t) - u(0). \tag{17}$$

Since $h(t)$ is a nonnegative continuous function, therefore by the definition of fractional integral we have

$$I^{\alpha}h(t) = \frac{1}{\Gamma(\alpha)}\int_{0}^{t}(t-\tau)^{\alpha-1}h(\tau)\,d\tau \geq 0, \quad t>0, \tag{18}$$

and

$$I^{\alpha}\theta u(t) = \frac{\theta}{\Gamma(\alpha)}\int_{0}^{t}(t-\tau)^{\alpha-1}u(\tau)\,d\tau, \quad t>0. \tag{19}$$

Combining $(16)-(19)$, we have

$$u(t) - u(0) \leq \frac{\theta}{\Gamma(\alpha)}\int_{0}^{t}(t-\tau)^{\alpha-1}u(\tau)\,d\tau, \quad t>0,$$

which implies

$$\begin{aligned} u(t) &\leq u(0) + \frac{\theta}{\Gamma(\alpha)}\int_{0}^{t}(t-\tau)^{\alpha-1}u(\tau)\,d\tau \\ &\leq u(0)\exp\left(\frac{\theta}{\Gamma(\alpha)}\int_{0}^{t}(t-\tau)^{\alpha-1}\,d\tau\right) \\ &= u(0)\exp\left(\frac{\theta t^{\alpha}}{\Gamma(\alpha+1)}\right), \quad t>0 \end{aligned}$$

where we have used Lemma 3.1. This is the desired result. □

Lemma 3.4. [18, 46] *Consider the system*

$$ {}_{t_0}^{C}D_{t}^{\alpha}u(t) = \tau(t,x), \quad t>t_0, \tag{20}$$

with initial condition $u(t_0)$, where $0<\alpha\leq 1$ and $\tau : [t_0,\infty)\times\Omega \to \mathbb{R}^n$, $\Omega\subset\mathbb{R}^n$. If $\tau(t,x)$ satisfies the locally Lipschitz condition with respect to x, then there exists a unique solution of (20) on $[t_0,\infty)\times\Omega$.

We now discuss the existence and uniqueness of the solution of problem (10).

Theorem 3.1. *If operator T is Lipschitz continuous with constant $\beta > 0$, then for each $u_0 = (u_{10}, u_{20}, \ldots, u_{n0}) \in \mathbb{R}^n$, there exists a unique solution $u(t) \in \mathbb{R}^n$ of problem* (10) *with $u(0) = u_0$, that is defined for all $t \geq 0$.*

Proof. Let

$$\tau(t,u) = P_{K(u(t))}[u(t) - \rho T u(t)] - u(t).$$

To prove that $\tau(t,u)$ is Lipschitz continuous for all $u(t), v(t) \in \mathbb{R}^n$, we have to consider

$$\begin{aligned}
&\|\tau(t,u) - \tau(t,v)\| \\
&= \left\| P_{K(u(t))}[u(t) - \rho T u(t)] - u(t) - P_{K(v(t))}[v(t) - \rho T v(t)] + v(t) \right\| \\
&\leq \left\| P_{K(u(t))}[u(t) - \rho T u(t)] - P_{K(v(t))}[v(t) - \rho T v(t)] \right\| \\
&\quad + \|u(t) - v(t)\| \\
&\leq \left\| P_{K(u(t))}[u(t) - \rho T u(t)] - P_{K(v(t))}[u(t) - \rho T u(t)] \right\| \\
&\quad + \left\| P_{K(v(t))}[u(t) - \rho T u(t)] - P_{K(v(t))}[v(t) - \rho T v(t)] \right\| \\
&\quad + \|u(t) - v(t)\| \\
&\leq \|u(t) - v(t)\| + \nu \|u(t) - v(t)\| + \|u(t) - \rho T u(t) - [v(t) - \rho T v(t)]\| \\
&\leq 2\|u(t) - v(t)\| + \nu \|u(t) - v(t)\| + \rho \|T u(t) - T v(t)\| \\
&\leq (2 + \nu + \rho\beta) \|u(t) - v(t)\|,
\end{aligned}$$

where we have used Lipschitz continuity of operator T with constant $\beta > 0$ and Assumption 2.1. This implies that operator $\tau(t,u)$ is a Lipschitz continuous in $\mathbb{R}^n$. Thus from Lemma 3.4, it is clear that there exists a unique solution $u(t)$ of problem (10). □

In the affine case, that is for $Tu = Au + b$, where $A = a_{ij}$ is a real $n \times n$ matrix and $b = b_j$ is an n-dimensional vector, then the Theorem 3.1 reduces to the following result.

Corollary 3.1. *If operator $Tu = Au + b$ and A is bounded with constant $\beta > 0$, then for each $u_0 = (u_{10}, u_{20}, \ldots, u_{n0}) \in \mathbb{R}^n$, there exists a unique solution $u(t) \in \mathbb{R}^n$ of problem* (11) *with $u(0) = u_0$, that is defined for all $t \geq 0$.*

We now discuss the stability and existence of the equilibrium point for the dynamical system (11) under some suitable conditions, essentially using the technique of Zeng-bao and Yun-zhi [46]. This is the main motivation of our next result.

Theorem 3.2. *If*

$$\lambda_i = 1 - \nu - \sum_{j=1,\ j\neq i}^{n} \left\{ \frac{\mu_j}{\mu_i} \rho |a_{ji}| - |1 - \rho a_{ii}| \right\} > 0, \quad i = 1, 2, \ldots, n,$$

then implicit fractional projected dynamical system (11) *is α-exponentially stable.*

Proof. Let $u(t) = (u_1(t), u_2(t), \ldots, u_n(t))^T$ and $v(t) = (v_1(t), v_2(t), \ldots, v_n(t))^T$ be any two solutions of dynamical system (11) with initial values $u(0) = (u_1(0), u_2(0), \ldots, u_n(0))^T$ and $v(0) = (v_1(0), v_2(0), \ldots, v_n(0))^T$ respectively.

Let

$$e_i(t) = u_i(t) - v_i(t), \quad i = 1, 2, \ldots, n,$$

then $e_i(0) \neq 0$ and taking the fractional derivative of above equation, we have

$${}_0^C D_t^{\alpha} e_i(t) = {}_0^C D_t^{\alpha} u_i(t) - {}_0^C D_t^{\alpha} v_i(t), \quad i = 1, 2, \ldots, n,$$

using (11), we have obtain error system

$$\begin{aligned} {}_0^C D_t^\alpha e_i(t) &= P_{K(u_i(t))}\left[u_i(t) - \rho\sum_{j=1}^{n} a_{ij}u_j(t) - \rho b_i\right] \\ &\quad -P_{K(v_i(t))}\left[v_i(t) - \rho\sum_{j=1}^{n} a_{ij}v_j(t) - \rho b_i\right] - e_i(t), \qquad (21) \end{aligned}$$

where $0 < \alpha < 1$, $t \geq 0$. From Theorem 3.1, $u_i(t)$ and $v_i(t)$ are uniquely determined solutions. Therefore $e_i(t)$ is the uniquely determined solution of error system (21) with initial value $e_i(0)$.

We claim that if $e_i(0) > 0$, then $e_i(t) \geq 0$ for $t \geq 0$, if $e_i(0) < 0$, then $e_i(t) \leq 0$ for $t \geq 0$. In fact, if $e_i(0) > 0$, there exists t_1, such that $e_i(t) < 0$ for $t \geq t_1$, so there must be $0 < t_0 < t_1$ such that $e_i(t_0) = 0$. It means that dynamical system (11) has two different solutions with initial value t_0 to $e_i(t_0) \leq 0$ for $t \geq t_0$, which contradicts to Theorem 3.1. In a similar way, we can prove that if $e_i(0) < 0$, then $e_i(t_0) \leq 0$ for $t \geq 0$.

So, we can have

$${}_0^C D_t^\alpha \|e_i(t)\| = sgn(e_i(t))\left({}_0^C D_t^\alpha e_i(t)\right).$$

Let

$$G(t) = \sum_{i=1}^{n} \mu_i \|e_i(t)\|, \quad \mu_i > 0,\ i = 1, 2, \ldots, n. \qquad (22)$$

Now by evaluating the fractional order derivative of (22), we have

$$\begin{aligned} {}_0^C D_t^\alpha G(t) &= \sum_{i=1}^{n} \mu_i\ {}_0^C D_t^\alpha \|e_i(t)\| \\ &= \sum_{i=1}^{n} \mu_i\ sgn(e_i(t))\left({}_0^C D_t^\alpha e_i(t)\right) \\ &= \sum_{i=1}^{n} \mu_i\ sgn(e_i(t))\left\{P_{K(u_i(t))}\left[u_i(t) - \rho\sum_{j=1}^{n} a_{ij}u_j(t) - \rho b_i\right]\right. \\ &\quad \left. -P_{K(v_i(t))}\left[v_i(t) - \rho\sum_{j=1}^{n} a_{ij}v_j(t) - \rho b_i\right] - e_i(t)\right\} \end{aligned}$$

$$\begin{aligned}
&\leq -\sum_{i=1}^{n}\mu_i\ \|e_i(t)\| + \sum_{i=1}^{n}\mu_i \left\|\left\{P_{K(u_i(t))}\left[u_i(t)-\rho\sum_{j=1}^{n}a_{ij}u_j(t)-\rho b_i\right]\right.\right. \\
&\quad \left.\left. -P_{K(v_i(t))}\left[v_i(t)-\rho\sum_{j=1}^{n}a_{ij}v_j(t)-\rho b_i\right]\right\}\right\| \\
&\leq -\sum_{i=1}^{n}\mu_i\ \|e_i(t)\| + \sum_{i=1}^{n}\mu_i\ \nu\|u_i(t)-v_i(t)\| \\
&\quad + \sum_{i=1}^{n}\mu_i\left\|u_i(t)-\rho\sum_{j=1}^{n}a_{ij}u_j(t)-\rho b_i - v_i(t)+\rho\sum_{j=1}^{n}a_{ij}v_j(t)+\rho b_i\right\| \\
&\leq -\sum_{i=1}^{n}\mu_i\ \|e_i(t)\| + \sum_{i=1}^{n}\mu_i\ \nu\|e_i(t)\| \\
&\quad + \sum_{i=1}^{n}\mu_i\left\|u_i(t)-v_i(t)-\rho\sum_{j=1}^{n}a_{ij}(u_j(t)-v_j(t))\right\| \\
&= -\sum_{i=1}^{n}\mu_i\ \|e_i(t)\| + \sum_{i=1}^{n}\mu_i\ \nu\|e_i(t)\| \\
&\quad + \sum_{i=1}^{n}\mu_i\left\|e_i(t)-\rho\sum_{j=1}^{n}a_{ij}e_j(t)\right\|. \qquad (23)
\end{aligned}$$

We now evaluate,

$$\begin{aligned}
&\sum_{i=1}^{n}\mu_i\left\|e_i(t)-\rho\sum_{j=1}^{n}a_{ij}e_j(t)\right\| \\
&= \sum_{i=1}^{n}\mu_i\|e_i(t)-\rho(a_{i1}e_1(t)+a_{i2}e_2(t)+\ldots+a_{ii}e_i(t)+\ldots+a_{in}e_n(t))\| \\
&\leq \sum_{i=1}^{n}\mu_i\{\rho|a_{i1}|\|e_1(t)\|+\rho|a_{i2}|\|e_2(t)\|+\ldots+|1-\rho a_{ii}|\|e_i(t)\| \\
&\quad +\ldots+\rho|a_{in}|\|e_n(t)\|\} \\
&= \sum_{i=1}^{n}\left\{\sum_{j=1,\ j\neq i}^{n}\mu_i\frac{\mu_j}{\mu_i}\rho|a_{ji}|\|e_i(t)\|\right\}+\sum_{i=1}^{n}\mu_i|1-\rho a_{ii}|\|e_i(t)\| \\
&= \sum_{i=1}^{n}\mu_i\left\{\sum_{j=1,\ j\neq i}^{n}\frac{\mu_j}{\mu_i}\rho|a_{ji}|+|1-\rho a_{ii}|\right\}\|e_i(t)\|. \qquad (24)
\end{aligned}$$

Combining (23) and (24), we have

$$\begin{aligned}
{}_0^C D_t^{\alpha} G(t) &= -\sum_{i=1}^{n}\mu_i\ \|e_i(t)\| + \sum_{i=1}^{n}\mu_i\ \nu\|e_i(t)\| \\
&\quad + \sum_{i=1}^{n}\mu_i\left\{\sum_{j=1,\ j\neq i}^{n}\frac{\mu_j}{\mu_i}\rho|a_{ji}|+|1-\rho a_{ii}|\right\}\|e_i(t)\|
\end{aligned}$$

$$\begin{aligned}
&= -\sum_{i=1}^{n}\mu_i\left\{1-\nu-\sum_{j=1,\ j\neq i}^{n}\frac{\mu_j}{\mu_i}\rho\left|a_{ji}\right|-\left|1-\rho a_{ii}\right|\right\}\left\|e_i\left(t\right)\right\| \\
&= -\sum_{i=1}^{n}\mu_i\lambda_i\left\|e_i\left(t\right)\right\|\leq-\lambda G\left(t\right), \qquad (25)
\end{aligned}$$

where $\lambda=\min\limits_{1\leq i\leq n}\lambda_i$.

Using Lemma 3.3, in (25), we have

$$G\left(t\right)\leq G\left(0\right).\exp\left(\frac{-\lambda.t^{\alpha}}{\Gamma\left(\alpha+1\right)}\right),$$

which shows that the dynamical system (11) is α-exponentially stable. □

Theorem 3.3. *If*

$$\theta_i=\mu_i\nu+\sum_{j=1,\ j\neq i}^{n}\left\{\rho\frac{\mu_j}{\mu_i}\left|a_{ji}\right|+\left|1-\rho a_{ii}\right|\right\}<1,\quad i=1,2,\ldots,n,$$

then implicit fractional projected dynamical system (11) *has an equilibrium point.*

Proof. Let $T_i:\mathbb{R}^n\to\mathbb{R}^n$ be a mapping, such that

$$T_i=\mu_iP_{K(u_i)}\left[\frac{u_i}{\mu_i}-\rho\sum_{j=1}^{n}a_{ij}\frac{u_j}{\mu_j}-\rho b_i\right], \qquad (26)$$

and

$$T\left(u\right)=\left(T_1\left(u\right),T_2\left(u\right),\ldots,T_n\left(u\right)\right),\quad\forall u\in\mathbb{R}^n. \qquad (27)$$

Let

$$\left\|u\right\|=\sum_{i=1}^{n}\left\|u_i\right\|=\left\|u_1\right\|+\left\|u_2\right\|+\ldots+\left\|u_n\right\|,\quad\forall u\in\mathbb{R}^n.$$

We now prove that T has a unique point in $\mathbb{R}^n$. Thus, for any $u,v\in\mathbb{R}^n$, consider

$$\begin{aligned}
&\left\|T\left(u\right)-T\left(v\right)\right\| \\
\leq\ & \sum_{i=1}^{n}\left\|T_i\left(u\right)-T_i\left(v\right)\right\| \\
=\ & \sum_{i=1}^{n}\left\|\mu_iP_{K(u_i)}\left[\frac{u_i}{\mu_i}-\rho\sum_{j=1}^{n}a_{ij}\frac{u_j}{\mu_j}-\rho b_i\right]-\mu_iP_{K(v_i)}\left[\frac{v_i}{\mu_i}-\rho\sum_{j=1}^{n}a_{ij}\frac{v_j}{\mu_j}-\rho b_i\right]\right\| \\
=\ & \sum_{i=1}^{n}\mu_i\left\|P_{K(u_i)}\left[\frac{u_i}{\mu_i}-\rho\sum_{j=1}^{n}a_{ij}\frac{u_j}{\mu_j}-\rho b_i\right]-P_{K(v_i)}\left[\frac{u_i}{\mu_i}-\rho\sum_{j=1}^{n}a_{ij}\frac{u_j}{\mu_j}-\rho b_i\right]\right. \\
&\left.+P_{K(v_i)}\left[\frac{u_i}{\mu_i}-\rho\sum_{j=1}^{n}a_{ij}\frac{u_j}{\mu_j}-\rho b_i\right]-P_{K(v_i)}\left[\frac{v_i}{\mu_i}-\rho\sum_{j=1}^{n}a_{ij}\frac{v_j}{\mu_j}-\rho b_i\right]\right\| \\
\leq\ & \sum_{i=1}^{n}\mu_i\left\{\nu\left\|u_i-v_i\right\|+\left\|\frac{u_i}{\mu_i}-\rho\sum_{j=1}^{n}a_{ij}\frac{u_j}{\mu_j}-\rho b_i-\frac{v_i}{\mu_i}+\rho\sum_{j=1}^{n}a_{ij}\frac{v_j}{\mu_j}+\rho b_i\right\|\right\} \\
=\ & \sum_{i=1}^{n}\mu_i\ \nu\left\|u_i-v_i\right\|+\sum_{i=1}^{n}\mu_i\left\|\frac{u_i-v_i}{\mu_i}-\rho\sum_{j=1}^{n}a_{ij}\frac{u_j-v_j}{\mu_j}\right\|. \qquad (28)
\end{aligned}$$

We now evaluate,

$$
\begin{aligned}
&\sum_{i=1}^{n}\mu_i\left\|\tfrac{u_i-v_i}{\mu_i}-\rho\sum_{j=1}^{n}a_{ij}\tfrac{u_j-v_j}{\mu_j}\right\| \\
=\;& \mu_1\left\|\frac{u_1-v_1}{\mu_1}-\rho\sum_{j=1}^{n}a_{1j}\frac{u_j-v_j}{\mu_j}\right\|+\mu_2\left\|\frac{u_2-v_2}{\mu_2}-\rho\sum_{j=1}^{n}a_{2j}\frac{u_j-v_j}{\mu_j}\right\|+\ldots \\
& +\mu_i\left\|\frac{u_i-v_i}{\mu_i}-\rho\sum_{j=1}^{n}a_{ij}\frac{u_j-v_j}{\mu_j}\right\|+\ldots+\mu_n\left\|\frac{u_n-v_n}{\mu_n}-\rho\sum_{j=1}^{n}a_{nj}\frac{u_j-v_j}{\mu_j}\right\| \\
\leq\;& \left\{|1-\rho a_{11}|\,\|u_1-v_1\|+\rho\,|a_{12}|\,\frac{\mu_1}{\mu_2}\,\|u_2-v_2\|+\ldots+\rho\,|a_{1n}|\,\frac{\mu_1}{\mu_n}\,\|u_n-v_n\|\right\} \\
& +\left\{\rho\,|a_{21}|\,\frac{\mu_2}{\mu_1}\,\|u_1-v_1\|+|1-\rho a_{22}|\,\|u_2-v_2\|+\ldots+\rho\,|a_{2n}|\,\frac{\mu_2}{\mu_n}\,\|u_n-v_n\|\right\}+\ldots \\
& +\left\{\rho\,|a_{i1}|\,\frac{\mu_i}{\mu_1}\,\|u_1-v_1\|+\ldots+|1-\rho a_{ii}|\,(u_i-v_i)+\ldots+\rho\,|a_{in}|\,\frac{\mu_i}{\mu_n}\,\|u_n-v_n\|\right\}+\ldots \\
& +\left\{\rho\,|a_{n1}|\,\frac{\mu_n}{\mu_1}\,\|u_1-v_1\|+\ldots+\rho\,|a_{ni}|\,\frac{\mu_n}{\mu_i}\,(u_i-v_i)+\ldots+|1-\rho a_{nn}|\,\|u_n-v_n\|\right\} \\
=\;& \sum_{i=1}^{n}|1-\rho a_{ii}|\,\|u_i-v_i\|+\sum_{i=2}^{n}\rho\,|a_{1i}|\,\frac{\mu_1}{\mu_i}\,\|u_i-v_i\|+\sum_{i=1,i\neq 2}^{n}\rho\,|a_{2i}|\,\frac{\mu_2}{\mu_i}\,\|u_i-v_i\| \\
& +\ldots+\sum_{i=1}^{n-1}\rho\,|a_{ni}|\,\frac{\mu_n}{\mu_i}\,\|u_i-v_i\|
\end{aligned}
$$

$$
\begin{aligned}
=\;& \sum_{i=1}^{n}|1-\rho a_{ii}|\,\|u_i-v_i\| \\
& +\left(\sum_{i=2,i\neq 1}^{n}\rho\,|a_{1i}|\,\frac{\mu_1}{\mu_i}+\sum_{i=1,i\neq 2}^{n}\rho\,|a_{2i}|\,\frac{\mu_2}{\mu_i}+\ldots+\sum_{i=1,i\neq n}^{n}\rho\,|a_{ni}|\right)\|u_i-v_i\| \\
=\;& \sum_{i=1}^{n}|1-\rho a_{ii}|\,\|u_i-v_i\|+\sum_{i=1}^{n}\left(\sum_{j=1,j\neq i}^{n}\rho\,|a_{ji}|\,\frac{\mu_j}{\mu_i}\right)\|u_i-v_i\| \\
=\;& \sum_{i=1}^{n}\left(\sum_{j=1,j\neq i}^{n}\rho\,|a_{ji}|\,\frac{\mu_j}{\mu_i}+|1-\rho a_{ii}|\right)\|u_i-v_i\|. \qquad (29)
\end{aligned}
$$

Combining (28) and (29), we have

$$
\begin{aligned}
\|T(u)-T(v)\| \;&\leq\; \sum_{i=1}^{n}\mu_i\;\nu\,\|u_i-v_i\|+\sum_{i=1}^{n}\left(\sum_{j=1,j\neq i}^{n}\rho\frac{\mu_j}{\mu_i}\,|a_{ji}|+|1-\rho a_{ii}|\right)\|u_i-v_i\| \\
&=\; \sum_{i=1}^{n}\left(\mu_i\nu+\sum_{j=1,j\neq i}^{n}\rho\frac{\mu_j}{\mu_i}\,|a_{ji}|+|1-\rho a_{ii}|\right)\|u_i-v_i\| \\
&\leq\; \theta\sum_{i=1}^{n}\|u_i-v_i\|, \quad \text{where } \theta=\max_{1\leq i\leq n}\theta_i<1.
\end{aligned}
$$

Hence, T is a contraction mapping. Therefore, there is a unique fixed point $u = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n$ such that $T(u^*) = u^*$, that is,

$$\mu_i P_{K(u_i^*)}\left[\frac{u_i^*}{\mu_i} - \rho \sum_{j=1}^{n} a_{ij}\frac{u_j^*}{\mu_j} - \rho b_i\right] = u_i^*, \quad i = 1, 2, \ldots, n.$$

Let $x_i^* = \frac{u_i^*}{\mu_i}$, $\quad i = 1, 2, \ldots, n$, then

$$P_{K(\mu_i\ x_i^*)}\left[x_i^* - \rho \sum_{j=1}^{n} a_{ij}x_j^* - \rho b_i\right] - x_i^* = 0, \quad i = 1, 2, \ldots, n,$$

this together with the uniqueness of u^*, we know that the system (11) has a unique equilibrium point. □

If $K(u) = K$, a closed and convex set in $\mathbb{R}^n$, then Theorem 3.2 and Theorem 3.3 reduces to the following results of [46], respectively.

Corollary 3.2. *If*

$$\lambda_i = 1 - \sum_{j=1,\ j\neq i}^{n} \left\{\frac{\mu_j}{\mu_i}\rho\left|a_{ji}\right| - \left|1 - \rho a_{ii}\right|\right\} > 0, \quad i = 1, 2, \ldots, n,$$

then dynamical system (12) *is α-exponentially stable.*

Corollary 3.3. *If*

$$\theta_i = \sum_{j=1,\ j\neq i}^{n} \left\{\rho\frac{\mu_j}{\mu_i}\left|a_{ji}\right| + \left|1 - \rho a_{ii}\right|\right\} < 1, \quad i = 1, 2, \ldots, n,$$

then dynamical system (12) *has a unique equilibrium point.*

Remark 3.1. We would like to mention that, for $\alpha = 1$, Theorem 3.2 and Theorem 3.3 are also valid under the same conditions. This gives us a new technique to study dynamical systems associated with quasi variational inequalities.

Remark 3.2. In this paper, our main emphasis is to consider and investigate the fractional dynamical systems associated with quasi variational inequalities. We have shown that these dynamical systems are exponentially stable under suitable conditions. Convergence analysis of these dynamical systems is also discussed. The implementation of these dynamical systems needs further research. However, we would like to point out the special case of these new dynamical systems, that is, if the closed convex valued set $K(u) = K$, the closed convex set, then Zeng-bao and Yun-zhi [46] have developed neural network systems for solving the dynamical systems associated with variational inequalities. These methods are efficient and robust one. It is expected that the technique of Zeng-bao and Yun-zhi [46] can be extended for solving fractional dynamical systems associated with quasi variational inequalities. This is an interesting topic for future research.

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