COMMON SOLUTION OF SPLIT EQUILIBRIUM PROBLEM AND FIXED POINT PROBLEM WITH NO PRIOR KNOWLEDGE OF OPERATOR NORM

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In this paper, we introduce an iterative algorithm that does not require any knowledge of the operator norm for finding a common solution of split equilibrium problem and fixed point problem for infinite family of quasi-nonexpansive multi-valued mappings in real Hilbert spaces. Using our iterative algorithm, we state and prove a strong convergence result for approximating a common solution of split equilibrium problem and fixed point problem for infinite family of quasi-nonexpansive multi-valued mappings which also solves some variational inequality problem in real Hilbert spaces. An application and a numerical example were also given. Our result complement some related results in literature.

Keywords: split equilibrium problem; quasi-nonexpansive multi-valued mappings; iterative scheme; Fixed point problem

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1. INTRODUCTION

Let H be a real Hilbert space and C a nonempty, closed and convex subset of H. Let $CB(C), K(C)$ and $P(C)$ denote the families of nonempty closed and bounded subsets, nonempty and compact subsets and nonempty proximinal subset of C respectively. The Pompeiu Hausdorff metric on CB(C) is defined by

$$\mathcal{H}(A,B) = \max\big\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\big\},$$

for all $A, B \in CB(C)$ where $d(x,B) = \inf_{b\in B} ||x-b||$.
A point $p \in C$ is called a fixed point of a multi-valued mapping T, if $p \in Tp$. We denote the set of fixed point of T by F(T).

Definition 1.1. *A multivalued mapping $T : C \to CB(C)$ is said to be*
(i) a contraction if there exists a constant $k \in (0,1)$ such that

$$\mathcal{H}(Tx,Ty) \le k||x-y||, \ \forall\ x,y \in C; \tag{1}$$

(ii) nonexpansive if

$$\mathcal{H}(Tx,Ty) \le ||x-y||, \ \forall\ x,y \in C; \tag{2}$$

(iii) quasi-nonexpansive if

$$\mathcal{H}(Tx,Tp) \le ||x-p||, \forall\ x \in C,\ p \in F(T). \tag{3}$$

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It is well-known that every nonexpansive multi-valued mapping T with $F(T) \neq \emptyset$ is quasi-nonexpansive, but not all quasi-nonexpansive mapping are nonexpansive. (Check Example (4.1) in [20] to see that the inclusion is proper).

Definition 1.2. *A bounded linear operator D on H is called strongly positive if there exists a constant $\alpha > 0$ such that*

$$\langle Dx, x\rangle \geq \alpha||x||^2, \forall\ x \in C.$$

Definition 1.3. *A multi-valued mapping $T : H \to CB(H)$ is said to be demi-closed at the origin if for any sequence $\{x_n\} \subset H$ such that x_n converges weakly to x and $d(x_n, Tx_n) \to 0$, we have $x \in Tx$.*

Let C be a nonempty, closed and convex subset of a real Hilbert space H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$||x - P_C(x)|| \leq ||x - y||,\ \forall\ y \in C.$$

P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies:

$$||P_C(x) - P_C(y)|| \leq \langle x - y, P_C(x) - P_C(y)\rangle. \tag{4}$$

Moreover, $P_C(x)$ is characterized by the following properties:

$$\langle x - P_C(x), y - P_C(x)\rangle \leq 0, \tag{5}$$

and

$$||x - y||^2 \geq ||x - P_C(x)||^2 + ||y - P_C(x)||^2,\ \forall\ x \in H, y \in C. \tag{6}$$

For all $x, y \in H$, it is well known that every nonexpansive operator $T : H \to H$ satisfies the inequality below

$$\langle (x - T(x)) - (y - T(y)), T(y) - T(x)\rangle \leq \frac{1}{2}||(T(x) - x) - (T(y) - y)||^2, \tag{7}$$

and therefore, we have that for all $x \in H$ and $y \in F(T)$.

$$\langle x - T(x), y - T(x)\rangle \leq \frac{1}{2}||T(x) - x||^2. \tag{8}$$

Equilibrium problem was introduced by Blum and Oettli [1] and this problem have had a great impact and influence in the development of several branches of pure and applied sciences, (see [8],[14],[18],[27],[28], [29],[30],[31]).
Let H be a real Hilbert space and C a nonempty, closed and convex subsets of H . Let $F : C \times C \to \mathbb{R}$ be a nonlinear bifunction, then the Equilibrum Problem (EP) is to find $x^* \in C$ such that

$$F(x^*, y) \geq 0,\ \forall\ y \in C. \tag{9}$$

For solving EP, let C be a nonempty, closed and convex subset of Hilbert space H and $F : C \times C \to \mathbb{R}$ be a bifunction satisfying the following assumptions:
$(L1)$ $F(x, x) = 0, \forall\ x \in C$;
$(L2)$ F is monotone, i.e., $F(x, y) + F(y, x) \leq 0, \forall x \in C$;
$(L3)$ for each $x, y, z \in C, \limsup_{t\to 0} F(tz + (1 - t)x, y) \leq F(x, y)$;
$(L4)$ for each $x \in C, y \to F(x, y)$ is convex and lower semi-continuous.
Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall\ y \in C. \tag{10}$$

Assumptions (L1)-(L4) stated above was first used in [1].
In 2013, Kazmi and Rizvi [9] introduced and studied the following Split Equilibrium Problem

(SEP):
Let H_1 and H_2 be two real Hilbert spaces and C and Q be nonempty closed and convex subsets of H_1 and H_2 respectively. Let $F_1 : C \times C \to \mathbb{R}$, $F_2 : Q \times Q \to \mathbb{R}$ be two nonlinear bifunctions and $A : H_1 \to H_2$ be a bounded linear operator, then the SEP is to find $x^* \in C$ such that

$$F_1(x^*, x) \geq 0, \ \forall \ x \in C, \tag{11}$$

and such that

$$y^* = Ax^* \in Q \text{ solves } F_2(y^*, y) \geq 0, \ \forall y \in Q. \tag{12}$$

The inequalities (11) and (12) constitute a pair of equilibrium problems. The image $y^* = Ax^*$ of the solution of (11) in H_1 under a given bounded linear operator A, is also the solution of (12) in H_2. We denote the solution set of (11) and (12) by $EP(F_1)$ and $EP(F_2)$ respectively.
The solution set of SEP (11) and (12) is denoted by $\Theta := \{p \in EP(F_1) : Ap \in EP(F_2)\}$.
Recently, Kazmi and Rizvi [9] introduced the following iterative scheme to approximate a common solution of SEP, a variational inequality problem and a fixed point problem for nonexpansive mapping S in real Hilbert spaces.

$$\begin{cases} u_n = T_{r_n}^{F_1}(x_n + \gamma A^*(T_{r_n}^{F_2} - I)Ax_n); \\ y_n = P_c(u_n - \lambda_n Du_n); \\ x_{n+1} = \alpha_n v + \beta_n x_n + \gamma_n S y_n; \end{cases} \tag{13}$$

where $r_n \subset (0, \infty), \lambda_n \in (0, 2\tau), D : C \to H_1$ is a $\tau-$ inverse strongly monotone mapping and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $(0, 1)$. They proved a strong convergence result using iterative algorithm (13)
Very recently, Deepho et al.[6] considered an iterative scheme to approximate a common element of the set of solutions of split variational inclusion problem and the set of common fixed point problem of a finite family of k-strictly pseudo-contractive nonself mappings. A strong convergence theorem was established under suitable conditions, which also solves some variational inequality problem in real Hilbert spaces. They denote the solution set of the split variational inclusion problem by $\overline{\Gamma}$ and proved the following theorem.

Theorem 1.1. *Let H_1 and H_2 be two real Hilbert spaces and let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty, closed and convex subsets. Let $A : H_1 \to H_2$ be a bounded linear operator and D a strongly positive bounded linear operator on H_1 with a coefficient $\overline{\tau} > 0$. Assume that $\{T_i\}_{i=1}^N : C \to H_1$ is a finite family of k_i- strict pseudo-contraction mappings such that $\Upsilon := \cap_i^N F(T_i) \cap \overline{\Gamma} \neq \emptyset$. Let f be a contraction mapping with a coefficient $\rho \in (0, 1)$ and $\sum_{i=1}^N \eta_i^n = 1$ for all $n \geq 0$, for a given $x_0 \in C, \alpha_n, \ \beta_n \in (0, 1)$ and $0 < \tau < \frac{\overline{\tau}}{\rho}$. Let $\{x_n\}$ be a sequence generated as follows:*

$$\begin{cases} u_n = J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n), \\ y_n = \beta_n u_n + (1 - \beta_n) \sum_{i=1}^N \eta_{i=1}^n T_i u_n, \\ x_{n+1} = \alpha_n \tau f(x_n) + (I - \alpha_n D) y_n, n \geq 1, \end{cases} \tag{14}$$

where $\lambda > 0$ and $J_\lambda^{B_i}(i = 1, 2)$ is the resolvent of the maximal monotone mappings $B_i(i = 1, 2)$ respectively. Suppose the following conditions are satisfied;
$(C1)$ $\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^\infty \alpha_n = \infty$ and $\sum_{n=1}^\infty |\alpha_n - \alpha_{n-1}| < \infty$;
$(C2)$ $k_i \leq \beta_n \leq l < 1, \ \lim_{n=1}^\infty \beta_n = l$ and $\sum_{n=1}^\infty |\beta_n - \beta_{n-1}| < \infty$;
$(C3)$ $\sum_{n=1}^\infty \sum_{i=1}^N |\eta_i^{(n)} - \eta_i^{(n-1)}| < \infty$.
Then the sequence $\{x_n\}$ generated by the iterative scheme converges strongly to $q \in \Upsilon$ which solves the variational inequality $\langle (D - \tau f)q, q - p\rangle \leq 0 \ \forall \ p \ \in \ \Upsilon$.

Also, Suantai et al. [24] introduced an iterative scheme for solving SEP and fixed point problem of nonspreading multi-valued mappings in real Hilbert spaces and proved that a modified Mann iteration converges weakly to a common solution of the considered problems. Motivated by the works of Suantai et al. [24], Deepho et al. [6], Kazmi and Rizvi [9], we introduce an iterative method that does not require any knowledge of the operator norm for approximating a common solution of SEP (11)- (12) and fixed point problem of an infinite family of quasi-nonexpansive multi-valued mappings.
Furthermore, we obtain a strong convergence theorem for approximating the common solution of SEP and fixed point problem for infinite family of quasi-nonexpansive multi-valued mappings which also solves some variational inequality problem in real Hilbert spaces. The result presented in this paper improves and complements some recent corresponding known results in this research area (see [6]).

2. PRELIMINARIES

In this section, we state some well known results which will be used in the sequel. Throughout this paper, we denote the weak and strong convergence of a sequence $\{x_n\}$ to a point $x \in H$ by $x_n \rightharpoonup x$ and $x_n \to x$ respectively.
Let H be a real Hilbert space, then the following inequalities hold

$$||u - v||^2 = ||u||^2 - ||v||^2 - 2\langle u - v, v\rangle, \tag{15}$$

$$||u + v||^2 \leq ||u||^2 + 2\langle v, u + v\rangle, \tag{16}$$

and

$$||\lambda u + (1 - \lambda)v||^2 = \lambda||u||^2 + (1 - \lambda)||v||^2 - \lambda(1 - \lambda)||u - v||^2, \tag{17}$$

for all $u, v \in H$ and $\lambda \in [0, 1]$.

Lemma 2.1. [5] *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $F : C \times C \to \mathbb{R}$ be a bifunction satisfying $(L1) - (L4)$. For $r > 0$ and $x \in H$, define a mapping $T_r^F : H \to C$ as follows:*

$$T_r^F x = \big\{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\big\}.$$

Then the following hold:
(i) T_r^F is nonempty and single-valued;
(ii) T_r^F is firmly nonexpansive, that is $\forall\ x, y \in H$,

$$||T_r^F x - T_r^F y||^2 \leq \langle T_r^F x - T_r^F y, x - y\rangle;$$

(iii) $F(T_r^F) = EP(F)$;
(iv) $EP(F)$ is closed and convex.

Lemma 2.2. [13] *Assume D is a strongly positive bounded linear operator on a Hilbert space H with a coefficient $\overline{\tau} > 0$ and $0 < \mu < ||D||^{-1}$. Then $||I - \mu D|| \leq 1 - \mu\overline{\tau}$.*

Lemma 2.3. [19] *Every Hilbert space H satisfies the Opial condition that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality* $\liminf_{n\to\infty} ||x_n - x|| < \liminf_{n\to\infty} ||x_n - y||$, *holds for every $y \in H$ with $y \neq x$.*

Lemma 2.4. [26] *Assume $\{a_n\}$ is a sequence of nonnegative real sequence such that*

$$a_{n+1} \leq (1 - \sigma_n)a_n + \sigma_n\delta_n,\ n > 0,$$

where $\{\sigma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a real sequence such that
(i) $\sum_{n=1}^{\infty} \sigma_n = \infty$,

(ii) $\limsup_{n\to\infty}\delta_n \le 0$ *or* $\sum_{n=1}^{\infty}|\sigma_n\delta_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5. [13] *Let C be a nonempty, closed and convex subset of a Hilbert space H. Assume that $f : C \to C$ is a contraction with coefficient $\mu \in (0,1)$ and D is a strongly positive linear bounded operator with a coefficient $\overline{\sigma} > 0$. Then, for $0 < \sigma < \frac{\overline{\sigma}}{\mu}$,*

$$\langle x-y,(D-\sigma f)x-(D-\sigma f)y\rangle \ge (\overline{\sigma}-\sigma\mu)||x-y||^2, \quad x,y \in H.$$

That is, $D-\sigma f$ is strongly monotone with coefficient $\overline{\sigma}-\sigma\mu$.

Lemma 2.6. [3] *Let E be a uniformly convex real Banach space. For arbitrary $r > 0$, let $B_r(0) := \{x \in E : ||x|| \le r\}$. Then, for any given sequence $\{x_i\}_{i=1}^{\infty} \subset B_r(0)$ and for any given sequence $\{\lambda_i\}_{i=1}^{\infty}$ of positive numbers such that $\sum_{i=1}^{\infty}\lambda_i = 1$, there exists a continuous strictly increasing convex function*

$$g : [0,2r] \to \mathbb{R},\ g(0) = 0,$$

such that for any positive integers i,j with $i < j$, the following inequality holds:

$$||\sum_{i=1}^{\infty}\lambda_i x_i||^2 = \sum_{i=1}^{\infty}\lambda_i||x||^2 - \lambda_i\lambda_j g(||x_i-x_j||).$$

Lemma 2.7. [11]*(Demiclosedness principle) Let C be a nonempty, closed and convex subset of a real Hilbert space H and $T : C \to K(C)$ be a quasi-nonexpansive multi-valued mapping . Let $\{x_n\}$ be a sequence in C such that $x_n \rightharpoonup p$ and $\lim_{n\to\infty} d(x_n,Tx_n) = 0$, then $p \in Tp$.*

3. MAIN RESULT

Theorem 3.1. *Let H_1 and H_2 be two real Hilbert spaces, C and Q be nonempty, closed and convex subsets of H_1 and H_2 respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and D be a strongly positive bounded linear operator on H_1 with coefficient $\overline{\tau} > 0$. Let $T_i : C \to K(C), i = 1,2,3,...$, be an infinite family of quasi-nonexpansive multi-valued mappings and $F_1 : C\times C \to \mathbb{R}$, $F_2 : Q\times Q \to \mathbb{R}$ be bifunctions satisfying assumptions $(L1)-(L4)$, where F_2 is upper semi-continuous in the first argument. Suppose $\Gamma := \cap_{i=1}^{\infty}F(T_i)\cap\Theta \ne \emptyset$ and f is a contraction mapping with coefficient $\mu \in (0,1)$. Let the sequences $\{u_n\},\{y_n\}$ and $\{x_n\}$ be generated by*

$$\begin{cases} u_n = T_{r_n}^{F_1}(x_n+\xi_n A^*(T_{r_n}^{F_2}-I)Ax_n); \\ y_n = \lambda_0 u_n + \sum_{i=1}^{\infty}\lambda_i z_n^i; \\ x_{n+1} = \gamma_n\tau f(x_n) + (I-\gamma_n D)y_n,\ n \ge 1; \end{cases} \tag{18}$$

where $z_n^i \in T_i u_n$, $r_n \subset (0,\infty)$ and the step size ξ_n be chosen in such a way that for some $\varepsilon > 0$,

$$\xi_n \in \left(\varepsilon, \frac{||(T_{r_n}^{F_2}-I)Ax_n||^2}{||A^*(T_{r_n}^{F_2}-I)Ax_n||^2} - \varepsilon\right),$$

for all $T_{r_n}^{F_2}Ax_n \ne Ax_n$ and $\xi_n = \xi$ otherwise (ξ being any nonnegative real number)with the sequences γ_n and r_n satisfying the following conditions;
(i) $\lim_{n\to\infty}\gamma_n = 0$ *and* $\sum_{n=1}^{\infty}\gamma_n = \infty$;
(ii) $\gamma_n \in (0,1)$, $0 < \tau < \frac{\overline{\tau}}{\mu}$ *and* $0 < \gamma_n < 2\mu$;
(iii) $\liminf_{n\to\infty} r_n > 0$;
(iv) $\lambda_0, \lambda_i \in (0,1)$ *such that* $\sum_{i=0}^{\infty}\lambda_i = 1$. *Then the sequence $\{x_n\}$ generated by (3.1) converges strongly to $q \in \Gamma$ which solves the variational inequality*

$$\langle (D-\tau f)q, q-p\rangle \le 0,\ \forall\ p \in \Gamma.$$

Proof. We first show that $\{x_n\}$ is bounded. For any $x, y \in C$, we need to show that $I - \gamma D$ is nonexpansive.
Now since $2\mu > \gamma_n$, we have

$$\begin{aligned}||(I-\gamma_n D)x-(I-\gamma_n D)y||^2 &= ||(x-y)-\gamma_n(Dx-Dy)||^2\\ &\le ||x-y||^2 - 2\gamma_n\langle x-y, Dx-Dy\rangle + \gamma_n^2||Dx-Dy||^2\\ &\le ||x-y||^2 - 2\mu\gamma_n||Dx-Dy||^2 + \gamma_n^2||Dx-Dy||^2\\ &= ||x-y||^2 - \gamma_n(2\mu-\gamma_n)||Dx-Dy||^2\\ &\le ||x-y||^2.\end{aligned}$$

Thus the mapping $I - \gamma_n D$ is nonexpansive.
Let $p \in \Gamma$, we have $T_{r_n}^{F_1}p = p$, $Ap = T_{r_n}^{F_2}Ap$, then

$$\begin{aligned}||u_n - p|| &= ||T_{r_n}^{F_1}(x_n + \xi_n A^*(T_{r_n}^{F_2} - I)Ax_n) - p||^2\\ &\le ||x_n + \xi_n A^*(T_{r_n}^{F_2} - I)Ax_n - p||^2\\ &\le ||x_n - p||^2 + \xi_n^2||A^*(T_{r_n}^{F_2} - I)Ax_n||^2 + 2\xi_n\langle x_n - p, A^*(T_{r_n}^{F_2} - I)Ax_n\rangle. \qquad (19)\end{aligned}$$

Where

$$\begin{aligned}&2\xi_n\langle x_n - p, A^*(T_{r_n}^{F_2} - I)Ax_n\rangle \qquad (20)\\ &= 2\xi_n\langle A(x_n - p), (T_{r_n}^{F_2} - I)Ax_n\rangle\\ &= 2\xi_n\langle A(x_n - p) + (T_{r_n}^{F_2} - I)Ax_n - (T_{r_n}^{F_2} - I)Ax_n, (T_{r_n}^{F_2} - I)Ax_n\rangle\\ &= 2\xi_n\big\{\langle T_{r_n}^{F_2}Ax_n - Ap, (T_{r_n}^{F_2} - I)Ax_n\rangle - ||(T_{r_n}^{F_2} - I)Ax_n||^2\big\}\\ &\le 2\xi_n\big\{\frac{1}{2}||(T_{r_n}^{F_2} - I)Ax_n||^2 - ||(T_{r_n}^{F_2} - I)Ax_n||^2\big\}\\ &\le -\xi_n||(T_{r_n}^{F_2} - I)Ax_n||^2. \qquad (21)\end{aligned}$$

Hence,

$$\begin{aligned}||u_n - p||^2 &\le ||x_n - p||^2 + \xi_n^2||A^*(T_{r_n}^{F_2} - I)Ax_n||^2 - \xi_n||(T_{r_n}^{F_2} - I)Ax_n||^2\\ &= ||x_n - p||^2 - \xi_n[||(T_{r_n}^{F_2} - I)Ax_n||^2 - \xi_n||A^*(T_{r_n}^{F_2} - I)Ax_n||^2]. \qquad (22)\end{aligned}$$

Since $\xi_n \in \left(\varepsilon, \frac{||(T_{r_n}^{F_2} - I)Ax_n||^2}{||A^*(T_{r_n}^{F_2} - I)Ax_n||^2} - \varepsilon\right)$, we obtain

$$||u_n - p||^2 \le ||x_n - p||^2. \qquad (23)$$

Since $T_i : C \to K(C)$ is an infinite family of a quasi-nonexpansive multi-valued mapping, we have that

$$
\begin{aligned}
||y_n - p|| &= ||\lambda_0(u_n - p) + \sum_{i=1}^{\infty} \lambda_i(z_n^i - p)|| \\
&\leq \lambda_0||u_n - p|| + \sum_{i=1}^{\infty} \lambda_i||z_n^i - p|| \\
&\leq \lambda_0||u_n - p|| + \sum_{i=1}^{\infty} \lambda_i d(z_n^i, T_i p) \\
&\leq \lambda_0||u_n - p|| + \sum_{i=1}^{\infty} \lambda_i H(T_i u_n, T_i p) \\
&\leq \lambda_0||u_n - p|| + \sum_{i=1}^{\infty} \lambda_i||u_n - p|| \\
&= ||u_n - p|| \\
&\leq ||x_n - p||. \qquad (24)
\end{aligned}
$$

Moreover, by Lemma 2.2, we have

$$
\begin{aligned}
||x_{n+1} - p|| &= ||\gamma_n[\tau f(x_n) - Dp] + (I - \gamma_n D)(y_n - p)|| \\
&\leq (1 - \gamma_n\overline{\tau})||y_n - p|| + \gamma_n||\tau f(x_n) - Dp|| \\
&\leq (1 - \gamma_n\overline{\tau})||y_n - p|| + \gamma_n[||\tau f(x_n) - \tau f(p)|| + ||\tau f(p) - Dp||] \\
&\leq [1 - (\overline{\tau} - \tau\mu)\gamma_n]||x_n - p|| + \gamma_n||\tau f(p) - Dp||.
\end{aligned}
$$

It follows by induction that

$$
||x_n - p|| \leq \max\{||x_0 - p||, \frac{||\tau f(p) - Dp||}{\overline{\tau} - \tau\mu}\}, n \geq 1. \qquad (25)
$$

Hence $\{x_n\}$ is bounded and consequently, we deduce that $\{u_n\}$ and $\{y_n\}$ are bounded. Applying Lemma 2.2 and (22), we have that

$$
\begin{aligned}
||x_{n+1} - p||^2 &= ||\gamma_n[\tau f(x_n) - Dp] + (I - \gamma_n D)(y_n - p)||^2 \\
&\leq (1 - \gamma_n\overline{\tau})^2||y_n - p||^2 + \gamma_n^2||\tau f(x_n) - Dp||^2 \\
&+ 2\gamma_n(1 - \gamma_n\overline{\tau})||\tau f(x_n) - Dp||\ ||y_n - p|| \\
&\leq (1 - \gamma_n\overline{\tau})^2||u_n - p||^2 + \gamma_n^2||\tau f(x_n) - Dp||^2 \\
&+ 2\gamma_n(1 - \gamma_n\overline{\tau})||\tau f(x_n) - Dp||\ ||y_n - p|| \\
&\leq (1 - \gamma_n\overline{\tau})^2[||x_n - p||^2 + \xi_n^2||A^*(T_{r_n}^{F_2} - I)Ax_n||^2 \\
&- \xi_n||(T_{r_n}^{F_2} - I)Ax_n||^2] + \gamma_n^2||\tau f(x_n) - Dp|| \\
&+ 2\gamma_n(1 - \gamma_n\overline{\tau})||\tau f(x_n) - Dp||\ ||y_n - p|| \\
&\leq (1 - \gamma_n\overline{\tau})^2||x_n - p||^2 + \xi_n[\xi_n||A^*(T_{r_n}^{F_2} - I)Ax_n||^2 \\
&- ||(T_{r_n}^{F_2} - I)Ax_n||^2] + \gamma_n^2||\tau f(x_n) - Dp||^2 \\
&+ 2\gamma_n(1 - \gamma_n\overline{\tau})||\tau f(x_n) - Dp||\ ||y_n - p||. \qquad (26)
\end{aligned}
$$

It follows from (26) and the condition $\xi_n \in \left(\varepsilon, \frac{||(T_{r_n}^{F_2}-I)Ax_n||^2}{||A^*(T_{r_n}^{F_2}-I)Ax_n||^2} - \varepsilon\right)$ that

$$\begin{aligned} ||x_{n+1}-p||^2 &\leq (1-\gamma_n\overline{\tau})^2||x_n-p||^2 - \varepsilon||A^*(T_{r_n}^{F_2}-I)Ax_n||^2 \\ &+ \gamma_n^2||\tau f(x_n) - Dp||^2 + 2\gamma_n(1-\gamma_n\overline{\tau})||\tau f(x_n) - Dp||\,||y_n - p||. \end{aligned} \tag{27}$$

We now consider two cases.

CASE A: Assume that $\{||x_n-p||\}$ is a monotonically decreasing sequence. Then $\{||x_n-p||\}$ is convergent and clearly,

$$\lim_{n\to\infty}||x_n-p|| = \lim_{n\to\infty}||x_{n+1}-p||.$$

Since $\{x_n\}$ is bounded and $\xi_n \in \left(\varepsilon, \frac{||(T_{r_n}^{F_2}-I)Ax_n||^2}{||A^*(T_{r_n}^{F_2}-I)Ax_n||^2} - \varepsilon\right)$, then we deduce from (27) that

$$\begin{aligned} \varepsilon||A^*(T_{r_n}^{F_2}-I)Ax_n||^2 &\leq (1-\gamma_n\overline{\tau})^2||x_n-p||^2 - ||x_{n+1}-p||^2 + \gamma_n^2||\tau f(x_n) - Dp||^2 \\ &+ 2\gamma_n(1-\gamma_n\overline{\tau})||\tau f(x_n) - Dp||\,||y_n-p||. \end{aligned}$$

Hence,

$$\lim_{n\to\infty}||A^*(T_{r_n}^{F_2}-I)Ax_n|| = 0. \tag{28}$$

Furthermore, from (27), we have

$$\begin{aligned} \xi_n||(T_{r_n}^{F_2}-I)Ax_n||^2 &\leq (1-\gamma_n\overline{\tau})^2||x_n-p||^2 - ||x_{n+1}-p||^2 \\ &+ \xi_n^2||A^*(T_{r_n}^{F_2}-I)Ax_n||^2 + \gamma_n^2||\tau f(x_n) - Dp||^2 \\ &+ 2\gamma_n(1-\gamma_n\overline{\tau})||\tau f(x_n) - Dp||\,||y_n-p||. \end{aligned} \tag{29}$$

Therefore, since $\lim_{n\to\infty}\gamma_n = 0$, from (28) and the condition
$\xi_n \in \left(\varepsilon, \frac{||(T_{r_n}^{F_2}-I)Ax_n||^2}{||A^*(T_{r_n}^{F_2}-I)Ax_n||^2} - \varepsilon\right)$, we have that

$$\lim_{n\to\infty}||(T_{r_n}^{F_2}-I)Ax_n||^2 = 0. \tag{30}$$

Next, we show that $||u_n - x_n|| \to 0$ as $n \to \infty$. Since $p \in \Gamma$, we obtain

$$\begin{aligned} ||u_n-p||^2 &= ||T_{r_n}^{F_1}(x_n + \xi A^*(T_{r_n}^{F_2}-I)Ax_n - p||^2 \\ &\leq \langle u_n - p, x_n + \xi A^*(T_{r_n}^{F_2}-I)Ax_n - p\rangle \\ &= \frac{1}{2}\big\{||u_n-p||^2 + ||x_n + \xi A^*(T_{r_n}^{F_2}-I)Ax_n - p||^2 \\ &- ||(u_n-p) - [x_n + \xi A^*(T_{r_n}^{F_2}-I)Ax_n - p]||^2\big\} \\ &= \frac{1}{2}\big\{||u_n-p||^2 + ||x_n-p||^2 + \xi(L\xi-1)||(T_{r_n}^{F_2}-I)Ax_n||^2 \\ &- ||u_n - x_n - \xi A^*(T_{r_n}^{F_2}-I)Ax_n||^2\big\} \\ &\leq \frac{1}{2}\big\{||u_n-p||^2 + ||x_n-p||^2 - [||u_n-x_n||^2 \\ &+ \xi^2||A^*(T_{r_n}^{F_2}-I)Ax_n||^2 - 2\xi\langle u_n - x_n, A^*(T_{r_n}^{F_2}-I)Ax_n\rangle]\big\} \\ &\leq \frac{1}{2}\big\{||u_n-p||^2 + ||x_n-p||^2 - ||u_n-x_n||^2 \\ &+ 2\xi||A(u_n-x_n)||\,||(T_{r_n}^{F_2}-I)Ax_n||\big\}. \end{aligned}$$

Hence, we obtain

$$||u_n-p||^2 \leq ||x_n-p||^2 - ||u_n-x_n||^2 + 2\xi||A(u_n-x_n)||\,||(T_{r_n}^{F_2}-I)Ax_n||. \tag{31}$$

From (26) and (31), we have

$$\begin{aligned}||x_{n+1}-p||^2 &\le (1-\gamma\overline{\tau})^2||u_n-p||^2+\gamma_n^2||\tau f(x_n)-Dp||^2\\ &+2\gamma_n(1-\gamma_n\overline{\tau})||\tau f(x_n)-Dp||\;||y_n-p||\\ &\le (1-\gamma_n\overline{\tau})^2[||x_n-p||^2-||u_n-x_n||^2\\ &+2\xi||A(u_n-x_n)||\;||(T_{r_n}^{F_2}-I)Ax_n||]\\ &+\gamma_n^2||\tau f(x_n)-Dp||^2+2\gamma_n(1-\gamma_n\overline{\tau})||\tau f(x_n)-Dp||\;||y_n-p||\\ &=(1-2\gamma_n\overline{\tau}+(\gamma_n\overline{\tau})^2)||x_n-p||^2-(1-\gamma_n\overline{\tau})^2||u_n-x_n||^2\\ &+2\xi(1-\gamma_n\overline{\tau})^2||A(u_n-x_n)||\;||(T_{r_n}^{F_2}-I)Ax_n||\\ &+\gamma_n^2||\tau f(x_n)-Dp||^2+2\gamma_n(1-\gamma_n\overline{\tau})||\tau f(x_n)-Dp||\;||y_n-p||\\ &\le ||x_n-p||^2+(\gamma_n\overline{\tau})^2||x_n-p||^2-(1-\gamma_n\overline{\tau})^2||u_n-x_n||^2\\ &+2\xi(1-\gamma_n\overline{\tau})^2||A(u_n-x_n)||\;||(T_{r_n}^{F_2}-I)Ax_n||\\ &+\gamma_n^2||\tau f(x_n)-Dp||^2+2\gamma_n(1-\gamma_n\overline{\tau})||\tau f(x_n)-Dp||\;||y_n-p||,\end{aligned}$$

which gives

$$\begin{aligned}(1-\gamma_n\overline{\tau})^2||u_n-x_n||^2 &\le ||x_n-p||^2-||x_{n+1}-p||^2+(\gamma_n\overline{\tau})^2||x_n-p||^2\\ &+2\xi(1-\gamma_n\overline{\tau})^2||A(u_n-x_n)||\;||(T_{r_n}^{F_2}-I)Ax_n||\\ &+\gamma_n^2||\tau f(x_n)-Dp||^2\\ &+2\gamma_n(1-\gamma_n\overline{\tau})||\tau f(x_n)-Dp||\;||y_n-p||. \qquad (32)\end{aligned}$$

Since $\{x_n\},\{y_n\}$ are bounded and from condition (i) of (3.1), (30), we have that

$$\lim_{n\to\infty}||u_n-x_n||=0. \qquad (33)$$

Since T_i is an infinite family of a quasi-nonexpansive multi-valued mapping, then applying Lemma 2.6, we have

$$\begin{aligned}||y_n-p||^2 &= ||\lambda_0u_n+\sum_{i=1}^{\infty}\lambda_iz_n^i-p||^2\\ &\le \lambda_0||u_n-p||^2+\sum_{i=1}^{\infty}\lambda_i(d(z_n^i,T_ip))^2-\lambda_0\lambda_ig(||u_n-z_n^i||)\\ &\le \lambda_0||u_n-p||^2+\sum_{i=1}^{\infty}\lambda_i(\mathcal{H}(T_iu_n,T_ip))^2-\lambda_0\lambda_ig(||u_n-z_n^i||)\\ &\le \lambda_0||u_n-p||^2+\sum_{i=1}^{\infty}\lambda_i||u_n-p||^2-\lambda_o\lambda_ig(||u_n-z_n^i||)\\ &= ||u_n-p||^2-\lambda_0\lambda_ig(||u_n-z_n^i||)^2\\ &\le ||x_n-p||^2-\lambda_0\lambda_ig(||u_n-z_n^i||)^2.\end{aligned}$$

This implies that

$$0<\lambda_0\lambda_ig(||u_n-z_n^i||)\le ||x_n-p||^2-||y_n-p||^2,$$

hence $\lim_{n\to\infty}g(||u_n-z_n^i||)=0$. By property of g (see Lemma 2.6), we have $\lim_{n\to\infty}||u_n-z_n^i||=0$. Since $\{x_n\}$ and $\{y_n\}$ are bounded, we have that

$$\lim_{n\to\infty}d(u_n,T_iu_n)\le \lim_{n\to\infty}||u_n-z_n^i||=0. \qquad (34)$$

From (3.1), we have that

$$\begin{aligned}||y_n - u_n|| &= ||\lambda_0 u_n - \sum_{i=1}^{\infty} \lambda_i z_n^i - u_n|| \\ &= ||\lambda_0 (u_n - u_n) + \sum_{i=1}^{\infty} \lambda_i (z_n^i - u_n)|| \\ &\leq \sum_{i=1}^{\infty} \lambda_i ||z_n^i - u_n||. \end{aligned} \tag{35}$$

From (34), we have that

$$\lim_{n\to\infty} ||y_n - u_n|| = 0. \tag{36}$$

Also, we have

$$||y_n - x_n|| \leq ||y_n - u_n|| + ||u_n - x_n||. \tag{37}$$

From (33) and (36), we have that

$$\lim_{n\to\infty} ||y_n - x_n|| = 0. \tag{38}$$

From (3.1), we have

$$\begin{aligned}||x_{n+1} - x_n|| &= ||x_{n+1} - y_n|| + ||y_n - x_n|| \\ &||\gamma_n \tau f(x_n) + (I - \gamma_n D)y_n - y_n|| + ||y_n - x_n|| \\ &\leq \gamma_n ||\tau f(x_n) - Dy_n|| + ||y_n - x_n|| \end{aligned} \tag{39}$$

From condition (i) of (3.1) and (38), we have that

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{40}$$

Now, we need to show that $\omega(x_n) \subset \Gamma$, where
$\omega(x_n) := \{x \in H_1 : x_{n_k} \rightharpoonup x, \{x_{n_k}\} \subset \{x_n\}\}$.
Since $\{x_n\}$ is bounded and H_1 is reflexive, $\omega(x_n)$ is nonempty. Let $q^* \in \omega(x_n)$ be an arbitrary element, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to q^*. From (33), we have that $u_{n_k} \rightharpoonup q^*$ as $k \to \infty$. By the demiclosedness principle and (34), we obtain $q^* \in \cap_{i=1}^{\infty} F(T_i)$.
Let us show that $q^* \in EP(F_1)$. Since $u_n = T_{r_n}^{F_1}(x_n + \xi A^*(T_{r_n}^{F_2} - I)Ax_n$, we have

$$F_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n - \xi A^*(T_{r_n}^{F_2} - I)Ax_n\rangle \geq 0,$$

for all $y \in C$, which implies that

$$F_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle - \frac{1}{r_n}\langle y - u_n, \xi A^*(T_{r_n}^{F_2} - I)Ax_n\rangle \geq 0,$$

for all $y \in C$. From $(L2)$, we have:

$$\frac{1}{r_{n_k}}\langle y - u_{n_k}, u_{n_k} - x_{n_k}\rangle - \frac{1}{r_{n_k}}\langle y - u_{n_k}, \xi A^*(T_{r_{n_k}}^{F_1} - I)Ax_{n_k}\rangle \geq F_1(y, u_{n_k}),$$

for all $y \in C$. From $\liminf_{n\to\infty} r_n > 0$, (30), (33) and $(L4)$, we have that
$F_1(y, q^*) \leq 0, \forall\ q^* \in C$. For any $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)q^*$. Since $y \in C, q^* \in C$, we get $y_t \in C$ and hence $F_1(y_t, q^*) \leq 0$. Therefore from $(L1)$ and $(L4)$, we have that

$$0 = F_1(y_t, y_t) \leq tF_1(y_t, y) + (1-t)F_1(y_t, q^*) \leq tF_1(y_t, y).$$

Hence $0 \leq F_1(y_t, y)$. Applying $(L3)$, we have that
$0 \leq F_1(q^*, y)$. This implies that $q^* \in EP(F_1)$. Since A is a bounded linear operator, $Ax_{n_k} \rightharpoonup Aq^*$. From (30), we have that

$$T^{F_2}_{r_{n_k}} Ax_{n_k} \rightharpoonup Aq^*, \tag{41}$$

as $k \to \infty$. By the definition of $T^{F_2}_{r_{n_k}} Ax_{n_k}$, we have

$$F_2(T^{F_2}_{r_{n_k}} Ax_{n_k}, y) + \frac{1}{r_{n_k}} \langle y - T^{F_2}_{r_{n_k}} Ax_{n_k} - Ax_{n_k} \rangle \geq 0,$$

for all $y \in C$. Since F_2 is upper semi-continuous in the first argument and from (41), it follows that

$$F_2(Aq^*, y) \geq 0, \forall y \in C.$$

This implies that $Aq^* \in EP(F_2)$ and hence $q^* \in \Theta$.
We now show that $\limsup_{k\to\infty} \langle (D - \tau f)q, q - x_n \rangle \leq 0$, where
$q = P_\Gamma(I - \tau f + D)q$.
Indeed, we can choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle (D - \tau f)q, x_n - q \rangle = \lim_{n\to\infty} \langle (D - \tau f)q, x_{n_k} - q \rangle. \tag{42}$$

We also assume that $x_{n_k} \rightharpoonup q^*$. Therefore

$$\begin{aligned} \limsup_{n\to\infty} \langle (D - \tau f)q, x_n - q \rangle &= \lim_{n_k\to\infty} \langle (D - \tau f)q, x_{n_k} - q \rangle \\ &= \langle Dq - \tau f(q), q^* - q \rangle \\ &= \langle (I - \tau f + D)q - q, q^* - q \rangle \\ &= \langle (I - \tau f + D)q - P_\Gamma(I - \tau f + D)q, q^* - P_\Gamma(I - \tau f + D)q \rangle \\ &\leq 0. \end{aligned}$$

Furthermore, we show the uniqueness of a solution of the variational inequality

$$\langle (D - \tau f)x, x - q \rangle \leq 0, \quad q \in \Gamma. \tag{43}$$

Suppose $q \in \Gamma$ and $q^* \in \Gamma$, both are solutions of (43), then

$$\langle (D - \tau f)q,\ q - q^* \rangle \leq 0, \tag{44}$$

and

$$\langle (D - \tau f)q^*,\ q^* - q \rangle \leq 0. \tag{45}$$

Adding (44) and (45), we have

$$\langle (D - \tau f)q - (D - \tau f)q^*,\ q - q^* \rangle \leq 0.$$

By Lemma 2.5, the strong monotonicity of $D-\tau f$, we have that $q = q^*$. Hence the uniqueness is proved. Lastly, we prove that $x_n \to q$ as $n \to \infty$. From (3.1) and (24), we have that

$$\begin{aligned}
||x_{n+1} - q||^2 &= \langle \gamma_n \tau f(x_n) + (I - \gamma_n D)y_n - q, x_{n+1} - q\rangle \\
&= \gamma_n \langle \tau f(x_n) - f(q), x_{n+1} - q\rangle + \langle (I - \gamma_n D)(y_n - q), x_{n+1} - q\rangle \\
&\leq \gamma_n \tau \langle f(x_n) - f(q), x_{n+1} - q\rangle + \gamma_n \langle \tau f(q) - Dq, x_{n+1} - q\rangle \\
&+ (1 - \gamma_n \overline{\tau})||y_n - q||\, ||x_{n+1} - q|| \\
&\leq \gamma_n \tau \mu ||x_n - q||\, ||x_{n+1} - q|| + \gamma_n \langle \tau f(q) - Dq, x_{n+1} - q\rangle \\
&+ (1 - \gamma_n \overline{\tau})||x_n - q||\, ||x_{n+1} - q|| \\
&= [1 - (\overline{\tau} - \tau\mu)\gamma_n]||x_n - q||||x_{n+1} - q|| + \gamma_n \langle \tau f(q) - Dq, x_{n+1} - q\rangle \\
&\leq \frac{1 - (\overline{\tau} - \tau\mu)\gamma_n}{2}(||x_n - q||^2 + ||x_{n+1} - q||^2) + \gamma_n \langle \tau f(q) - Dq, x_{n+1} - q\rangle \\
&\leq \frac{1 - (\overline{\tau} - \tau\mu)\gamma_n}{2}||x_n - q||^2 + \frac{1}{2}||x_{n+1} - q||^2 + \gamma_n \langle \tau f(q) - Dq, x_{n+1} - q\rangle.
\end{aligned}$$

Then, it follows that

$$||x_{n+1} - q||^2 \leq [1 - (\overline{\tau} - \tau\mu)\gamma_n]||x_n - q||^2 + \gamma_n(\overline{\tau} - \tau\mu)\frac{2\langle \tau f(q) - Dq, x_{n+1} - q\rangle}{(\overline{\tau} - \tau\mu)}. \tag{46}$$

From $0 < \tau < \frac{\overline{\tau}}{\mu}$, condition (i) of (3.1), then we conclude that $\lim_{n\to\infty} ||x_n - q|| = 0$ using Lemma 2.4.

CASE B: Assume that $\{||x_n - p||\}$ is not a monotonically decreasing sequence. Then, we define an integer sequence $\{\sigma(n)\}$ for all $n \geq n_0$ (for some n_0 large enough) by

$$\sigma(n) := \max\{k \in \mathbb{N}; k \leq n : ||x_k - p|| < ||x_{k+1} - p||\}.$$

Clearly, σ is a nondecreasing sequence such that $\sigma(n) \to \infty$ as $n \to \infty$ and for all $n \geq n_0$. From (27), we have

$$\begin{aligned}
&\xi_{\sigma(n)}||(T^{F_2}_{r_{\sigma(n)}} - I)Ax_{\sigma(n)}||^2 \\
&\leq (1 - \gamma_{\sigma(n)}\overline{\tau})^2||x_{\sigma(n)} - p||^2 - ||x_{\sigma(n+1)} - p||^2 \\
&+ \xi^2_{\sigma(n)}||A^*(T^{F_2}_{r_{\sigma(n)}} - I)Ax_{\sigma(n)}||^2 + \gamma^2_{\sigma(n)}||\tau f(x_{\sigma(n)}) - Dp||^2 \\
&+ 2\gamma_{\sigma(n)}(1 - \gamma_{\sigma(n)}\overline{\tau})||\tau f(x_{\sigma(n)}) - Dp||\, ||y_{\sigma(n)} - p||.
\end{aligned} \tag{47}$$

Therefore, since $\lim_{n\to\infty} \gamma_{\sigma(n)} = 0$, from (28) and the condition $\xi_{\sigma(n)} \in \left(\varepsilon, \frac{||(T^{F_2}_{r_{\sigma(n)}} - I)Ax_{\sigma_(n)}||^2}{||A^*(T^{F_2}_{r_{\sigma(n)}} - I)Ax_n||^2} - \varepsilon\right)$, we have that

$$\lim_{n\to\infty} ||(T^{F_2}_{r_{\sigma(n)}} - I)Ax_{\sigma(n)}||^2 = 0. \tag{48}$$

Following the same argument as in CASE A, we conclude that there exist a subsequence $\{x_\sigma(n)\}$ which converges weakly to $p \in \Gamma$. Now for all $n \geq n_0$, we have

$$\begin{aligned}
0 &\leq ||x_{\sigma(n+1)} - q||^2 - ||x_{\sigma(n)} - q||^2 \\
&\leq (1 - \gamma_{\sigma(n)}\overline{\tau})||x_{\sigma(n)} - q||^2 + \gamma^2_{\sigma(n)}||\tau f(x_{\sigma(n)}) - Dq||^2 \\
&+ 2\gamma_{\sigma(n)}(1 - \gamma_{\sigma(n)}\overline{\tau})||\tau f(x_{\sigma(n)}) - Dq||\, ||x_{\sigma(n)} - q|| - ||x_{\sigma(n)} - q||^2 \\
&= -\gamma_{\sigma(n)}\overline{\tau}||x_{\sigma(n)} - q||^2 + \gamma^2_{\sigma(n)}||\tau f(x_{\sigma(n)}) - Dq||^2 \\
&+ 2\gamma_{\sigma(n)}(1 - \gamma_{\sigma(n)}\overline{\tau})\langle \tau f(x_{\sigma(n)} - Dq, x_{\sigma(n+1)} - q\rangle.
\end{aligned}$$

Thus

$$||x_{\sigma(n)} - q||^2 \leq \frac{\gamma_{\sigma(n)}}{\overline{\tau}}||\tau f(x_{\sigma(n)}) - Dq||^2 + \frac{2(1-\gamma_{\sigma(n)}\overline{\tau})}{\overline{\tau}}\langle \tau f(x_{\sigma(n)} - Dq, x_{n+1} - q\rangle.$$

Since $\lim_{n\to\infty}\gamma_n \to 0$ as $n \to \infty$ and $\limsup\langle \tau f(x_{\sigma(n)} - Dp, x_{n+1} - q\rangle \leq 0$, then we conclude that $\{x_n\}$ converges strongly to q. This complete the proof. □

Corollary 3.1. *In Theorem 3.1, if we let $T_i : C \to K(C), i = 1, 2\cdots$ be an infinite family of multivalued nonexpansive mappings, we obtain a strong convergence theorem for approximating the common solution of SEP and fixed point problem for infinite family of nonexpansive multi-valued mappings which also solves some variational inequality problem in real Hilbert spaces.*

4. APPLICATIONS AND NUMERICAL EXAMPLE

4.1. Application to Optimization Problem

Let H_1, H_2 be two real Hilbert spaces, C and Q be nonempty, closed and convex subsets of H_1 and H_2 respectively. Let $f : C \to \mathbb{R}$, $g : Q \to \mathbb{R}$ be two operators and $A : H_1 \to H_2$ be a bounded linear operator, then the optimization problem is to find:

$$\begin{aligned} x^* \in C \quad &\text{such that } f(x^*) \leq f(x), \quad \forall\, x \in C,\\ \text{and} \quad y^* = Ax^* \quad &\text{such that} \quad g(y^*) \leq g(y), \quad \forall\, y \in Q. \end{aligned} \tag{49}$$

We denote the set of solutions of (49) by Ω and assume that $\Omega \neq \emptyset$. Let $F_1(x, y) := f(y) - f(x)$ for all $x, y \in C$ and $F_2(x, y) := g(y) - g(x)$ for all $x, y \in Q$ respectively. Then $F_1(x, y)$ and $F_2(x, y)$ satisfy conditions $(L1)-(L4)$ provided f and g are convex and lower semi-continuous on C and Q respectively, Clearly, $\Theta = \Omega$. Thus from Theorem 3.1, we obtain a strong convergence theorem for approximating the common solution of split minimization problem and fixed point problem for infinite family of quasi-nonexpansive multi-valued mappings which also solves some variational inequality problem in real Hilbert spaces.

4.2. Numerical Example

Let $H_1 = H_2 = \mathbb{R}$ and $C = Q = \mathbb{R}$. Let $F_1(u, v) = -11u^2 + uv + 10v^2$, then we derive our resolvent function $T_r^{F_1}$ using Lemma 2.1 as follows:

$$\begin{aligned} F_1(u, v) + \frac{1}{r}(v - u)(u - x) \geq 0 &\Longleftrightarrow -11ru^2 + ruv + 10rv^2 + uv - vx - u^2 + ux \geq 0\\ &\Longleftrightarrow 10rv^2 + ruv + uv - vx - 11ru^2 - u^2 + ux \geq 0\\ &\Longleftrightarrow 10rv^2 + (ru + u - x)v - (11ru^2 + u^2 - ux) \geq 0 \end{aligned}$$

Let $Q(v) = 10rv^2 + (ru + u - x)v - (11ru^2 + u^2 - ux)$. Then Q is a quadratic function of v with coefficients $a = 10r, b = ru + u - x, c = -11ru^2 - u^2 + ux$. We compute the discriminant of $Q(v)$ as follows:

$$\begin{aligned}\Delta = b^2 - 4ac &= (ru+u-x)(ru+u-x) - 4(10r)(-11ru^2 - u^2 + ux)\\ &= r^2u^2 + ru^2 - rux + ru^2 + u^2 - ux - rux - ux\\ &+ x^2 + 440r^2u^2 + 40ru^2 - 40rux\\ &= 441r^2u^2 + 42ru^2 - 42rux - 2ux + u^2 + x^2\\ &= x^2 - 42rux - 2ux + 441r^2u^2 + 42ru^2 + u^2\\ &= x^2 - 2((21r+1)u)x + u^2 + 441r^2u^2 + 42ru^2\\ &= x^2 - 2((21r+1)u)x + ((21r+1)u)^2\\ &= (x-(21r+1)u)^2 \geq 0.\end{aligned}$$

Thus, $\Delta \geq 0\ \forall\ y \in \mathbb{R}$ and it has at most one solution in $\mathbb{R}$, then $\Delta \leq 0$, $T_{r_n}^{F_1}(x) = \frac{x}{21r_n+1}$.
Let $F_2(u,v) = -15u^2 + uv + 14v^2, Ax = x$ and $A^*x = x$. Following the same process used in deriving $T_r^{F_1}$, we have $T_{r_n}^{F_2}(x) = \frac{x}{29r_n+1}$.
Furthermore, define $T_i : \mathbb{R} \to K(\mathbb{R})$ $(i = 1,2,3,\cdots)$ by:

$$T_i x = \begin{cases} [0, \frac{x}{2i}] & x \in [0,\infty),\\ [\frac{x}{2i}, 0] & x \in (-\infty, 0],\end{cases}$$

where $K(\mathbb{R})$ is the family of nonempty, closed and bounded subsets of $\mathbb{R}$. Clearly, T_i for each i is a multivalued quasi-nonexpansive mapping. Let $f : \mathbb{R} \to \mathbb{R}$ be given as; $f(x) = \frac{1}{8}x$, then $\mu = \frac{1}{6}$ is a contraction constant for f. Take $D(x) = 2x$ with constant $\overline{\tau} = 1$. On the other hand, we take $\tau = 2$ which satisfies $0 < \tau < \frac{\overline{\tau}}{\mu}$.
Furthermore, we take $\gamma_n = \frac{n+1}{8n}, r_n = \frac{n}{n+1}, \lambda_0 = \frac{1}{2}, \lambda_i = \frac{1}{2^{i+1}}, z_n^i \in T_i u_n$ and let the step size ξ_n be chosen in such a way that for some $\varepsilon > 0$, $\xi_n \in \left(\varepsilon, \frac{||(T_{r_n}^{F_2} - I)Ax_n||^2}{||A^*(T_{r_n}^{F_2} - I)Ax_n||^2} - \varepsilon\right)$ for all $T_{r_n}^{F_2}Ax_n \neq Ax_n$ and ξ_n be any positive real number otherwise, in iterative scheme (3.1) we obtain

$$\begin{cases} u_n = \frac{(1-\xi_n)x_n}{21r_n+1} + \frac{\xi_n x_n}{(21r_n+1)(29r_n+1)},\\ y_n = \frac{1}{2}u_n + \sum_{i=1}^{\infty} \frac{1}{2^{i+1}} z_n^i,\\ x_{n+1} = (\frac{n+1}{8n})(\frac{x_n}{4}) + (1 - \frac{(n+1)}{4n})y_n.\end{cases}$$

Case 1: $x_0 = 1$ and $\xi_n \in \left(\varepsilon, \frac{||(T_{r_n}^{F_2} - I)Ax_n||^2}{||A^*(T_{r_n}^{F_2} - I)Ax_n||^2} - \varepsilon\right)$ for all $T_{r_n}^{F_2}Ax_n \neq Ax_n$ and $\xi_n = 0.0003$ otherwise.
Case 2: $x_0 = 2$ and $\xi_n \in \left(\varepsilon, \frac{||(T_{r_n}^{F_2} - I)Ax_n||^2}{||A^*(T_{r_n}^{F_2} - I)Ax_n||^2} - \varepsilon\right)$ for all $T_{r_n}^{F_2}Ax_n \neq Ax_n$ and $\xi_n = 0.0000021$ otherwise.

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REFERENCES

[1] E. Blum, W. Oettli, From Optimization and variational inequalities to equilibrum problems, *Math. Stud.* **63**, (1994); 123-145.

[2] A. Bnouhachem, Y. Chen, An iterative method for a common solution of generalized mixed equilibrum problems, variational inequalities and hierarchical fixed point problems, *Fixed Point Theory Appl.*, **155**, (2014).

[3] S. Chang, J. K. Kim, X. R. Wang, Modified Block iterative algorithm for solving convex feasibility problems in Banach spaces, *J. Ineq. Appl.*, (2010); Article ID 869684, 14 pages, doi.10.1155/2010/869684.

[4] F. Cianciaruso, G. Marino, L. Muglia, Y. Yao, A hybrid projection algorithm for finding solutions of mixed equilibrium problem and variational inequality problem, *Fixed Point Theory Appl.*, (2010), 19pp. Article ID 383740.

[5] P. L. Combettes, S. A. Histoaga, Equilibrium programming in Hilbert spaces, *J. Nonlinear Convex Anal.*, **6**, (2005), 117-136.

[6] J. Deepho et al., A new general iterative scheme for split variational inclusion and fixed point problems of k-strict pseudo-contraction mappings with convergence analysis, *J. Comput. Appl. Math.*, **318**, (2017), 293-306.

[7] K. Geobel, W. A. Kirk, Topics in Metric Fixed Point Theory, *Cambridge Studies in Advanced Mathematics*, **28**, Cambridge University press, 1990.

[8] C. Izuchukwu, C.C. Okeke, O.T. Mewomo, Systems of variational inequality problem and Multiple-sets Split equality fixed point problem for countable families of multi-valued type-one demicontractive-type mappings, *Ukr. Mat. Zh.*, (2018), (to appear).

[9] K. R. Kazmi, S. H. Rizvi, Iterative approximation of a common solution of a split equilibrum problem, a variational inequality problem and a fixed point problem, *J. Egypt. Math. Soc.*, **21**, (2013), 44-51.

[10] F. Kohsaka, W. Takahashi, Fixed point theorems for a class of non linear mappings related to maximal monotone operators in Banach space, *Arch. Math. (Basel)*, **91**, (2008), 166-177.

[11] S. Lemoto, W. Takahashi, Approximating common fixed points of nonexpansive mappings and nonspreading mappings in a Hilbert space, *Nonlinear Anal.*, **71**, (2009), 2080-2089.

[12] H.B. Liu, Convergence theorems for a finite family of nonspreading and nonexpansive multivalued mappings and equilibrum problems with application, *Theor. Math. Appl.*, **3**, (2013), 49-61.

[13] G. Marino, H.K. Xu, A general iterative method for nonexpansive mappings in Hilbert spaces, *J. Math. Anal. Appl.*, **318**, (2006), 43-52.

[14] O.T. Mewomo, F.U. Ogbuisi, Convergence analysis of iterative method for multiple set split feasibility problems in certain Banach spaces, Quaest. Math., (2018), (to appear). DOI:10.2989/16073606.2015.1070376.

[15] F. U. Ogbuisi, O. T. Mewomo, Iterative solution of split variational inclusion problem in real Banach space, *Afrika Mathematika* **28**, (2017), 295-309. DOI: 10.1007/s13370-016-0450-z.

[16] F. U. Ogbusi, O. T. Mewomo, On split generalized mixed equilibrum problem and fixed point problem with no prior knowledge of operator norm, *J. Fixed Point Theory Appl.*, (2017), 1-20. DOI:10.1007./s11784-016-0397-6.

[17] F. U. Ogbuisi, O. T. Mewomo, Convergence analysis of common solution of certain nonlinear problems, *Fixed Point Theory*, **19**(1) (2018), (to appear).

[18] C.C. Okeke, O.T. Mewomo, On split equilibrium problem, variational inequality problem and fixed point problem for multi-valued mappings, *Ann. Acad. Rom. Sci. Ser. Math. Appl.*, **9** (2), (2017), 255-280.

[19] Z. Opial; Weak convergence of successive approximations for nonexpansive mappings, *Bull. Amer. Math. Soc.*; **73**; (1967); 591-597.

[20] K. Samanmit, B. Panyanak, Remarks on multi-valued quasi-nonexpansive mappings in $\mathbb{R}$-Trees, *J. Nonlinear Convex Anal.*, (2014).

[21] Y. Shehu, Iterative approximation for split equality fixed point problem for family of multi-valued mappings, *RACSAM* DOI 10.1007/s13398-014-0207-1, (2015).

[22] Y. Shehu and O.T. Mewomo, Further investigation into split common fixed point problem for demicontractive operators, *Acta Math. Sinica (Engl. Ser.),* **32** (11), (2016), 1357-1376.

[23] Y. Shehu, O. T. Mewomo, F. U. Ogbuisi, Further investigation into approximation of a common solution of fixed point problems and split feasibility problems, *Acta. Math. Sci. Ser. B,* Engl. Ed., **36**, (3), (2016), 913-930

[24] Suantai et al, On solving split equilibrum problems and fixed point problems of nonspreading multivalued mappings in real Hilbert spaces, *Fixed Point Theory Appl.*, **35**; (2016), DOI 10.1186/s13663-016-0509-4.

[25] Wang et al., Iterative algorithm for a family of split equilibrum problems and fixed point problems in Hilbert spaces with applications, *Fixed Point Theory and Appl.*, **4**, (2016), DO1 10.11861s13663-015-0475-2

[26] H.K. Xu, An iterative approach to quadratic optimization; *J. Optim. Theory Appl.*; **116**; (2003); 659-678.

[27] Y. Yao, M. Postolache, Iterative methods for pseudomonotone variational inequalities and fixed point problems , *J. Optim. Theory Appl.* 155(2012), No. 1, 273-287.

[28] Y. Yao, M. Postolache, Y.C. Liou, Strong convergence of a self-adaptive method for the split feasibility problem, *Fixed Point Theory Appl.* 2013, Art. No. 201 (2013).

[29] Y. Yao, R.P. Agarwal, M. Postolache, Y.C. Liu, Algorithms with strong convergence for the split common solution of the feasibility problem and fixed point problem, *Fixed Point Theory Appl.* 2014, Art. No. 183 (2014).

[30] Y. Yao, L. Leng, M. Postolache, X. Zheng, A unified framework for the two-sets split common fixed point problem in Hilbert spaces, *J. Nonlinear Sci. Appl.* 9(2016), No. 12, 6113-6125.

[31] Y. Yao, L. Leng, M. Postolache, X. Zheng , Mann-type iteration method for solving the split common fixed point problem, *J. Nonlinear Convex Anal.* 18(2017), No. 5, 875-882.