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SOME THEOREMS ON FUZZY HILBERT SPACES

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In this note, our aim is to present some properties of Felbin-type fuzzy inner product spaces and fuzzy bounded linear operators on the same spaces with some operator norm. At the first, fuzzy closed linear operators are considered briefly, latter the notion of fuzzy orthonormality is introduced. Finally, we establish Bessel's inequality in the sense of Felbin's norm.

Keywords: fuzzy normed linear spaces, fuzzy Hilbert spaces, Felbin-type fuzzy norm, fuzzy bounded linear operator.

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1. Introduction

An idea of fuzzy norm on a linear space introduced by Katsaras [11] in 1984. He studied fuzzy topological vector spaces. Following his pioneering work, Felbin [6] offered in 1992 an alternative definition of a fuzzy norm linear space (FNLS). With this definition of fuzzy normed linear space, it has been possible to introduce a notion of fuzzy bounded linear operator over fuzzy normed linear spaces to define "fuzzy norm" for such an operator. In [6], Felbin introduced an idea of fuzzy bounded operators and defined a fuzzy norm for such an operator which was erroneous as it shown in Example 3.1 of [3]. Xiao and Zhu ([14], [15]) studied various properties of Felbin-type fuzzy normed linear spaces. They gave a new definition for the norm of the bounded operators. A different definition of a fuzzy bounded linear operator and a "fuzzy norm" for such an operator was introduced by Bag and Samanta [3]. Finally, we note that the definition of the fuzzy norm of an operator was given in [3] and [14], does not satisfy the basic properties $\|Tx\| \leq \|T\|\|x\|$ and $\|TS\| \leq \|T\|\|S\|$ for operators T, S and vector x in general. The dual of a fuzzy normed space for fuzzy strongly bounded linear functional was introduced in [4]. Recently many authors studied Felbin-type fuzzy normed linear spaces and established some results (for references please see ([8], [9])). Actually after that, the research in fuzzy functional analysis has become a highly potential field of research in fuzzy mathematics. In this paper, we study the properties of fuzzy Hilbert spaces and fuzzy bounded linear operators on it with some operator norms. The organization of this paper is as follows:

In Section 1, unbounded linear operators are investigated and it is shown that a special fuzzy unbounded linear operator T can not be defined on all of H. In Section 2, we consider fuzzy closed linear operators using formulations which are convenient for fuzzy Hilbert spaces. In Section 3, we establish Bessel's inequality in the sense of Felbin's norm.

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2. Preliminaries

Let η be a fuzzy subset on $\mathbb{R}$, i.e. a mapping $\eta : \mathbb{R} \to [0,1]$ associating with each real number t its grade of membership $\eta(t)$.
In this paper, we consider the concept of fuzzy real numbers (fuzzy intervals) in the sense of Xiao and Zhu which is defined below:

Definition 2.1. [15] *A fuzzy subset η on $\mathbb{R}$ is called a fuzzy real number, whose α-level set is denoted by $[\eta]_\alpha$, i.e., $[\eta]_\alpha = \{t : \eta(t) \geq \alpha\}$, if it satisfies two axioms:*
(N1) *There exists $t_0 \in \mathbb{R}$ such that $\eta(t_0) = 1$.*
(N2) *For each $\alpha \in (0,1]$, there exist real numbers $-\infty < \eta_\alpha^- \leq \eta_\alpha^+ < +\infty$ such that $[\eta]_\alpha$ is equal to the closed interval $[\eta_\alpha^-, \eta_\alpha^+]$.*

The set of all fuzzy real numbers (fuzzy intervals) is denoted by $F(\mathbb{R})$. If $\eta \in F(\mathbb{R})$ and $\eta(t) = 0$ whenever $t < 0$, then η is called a non-negative fuzzy real number and $F^*(\mathbb{R})$ denotes the set of all non-negative fuzzy real numbers. The real number η_α^- is positive for all $\eta \in F^*(\mathbb{R})$ and all $\alpha \in (0,1]$ (see [15]).
Since each $r \in \mathbb{R}$ can be considered as the fuzzy real number $\tilde{r} \in F(\mathbb{R})$ defined by

$$\tilde{r}(t) = \begin{cases} 1, & t = r, \\ 0, & t \neq r, \end{cases}$$

it follows that $\mathbb{R}$ can be embedded in $F(\mathbb{R})$, that is if $r \in (-\infty, \infty)$, then $\tilde{r} \in F(\mathbb{R})$ satisfies $\tilde{r}(t) = \tilde{0}(t - r)$. Also, the α-level of $\tilde{r}$ is given by $[\tilde{r}]_\alpha = [r, r], 0 < \alpha \leq 1$ (see [3]).

According to Mizumoto and Tanaka [12], fuzzy arithmetic operations $\oplus, \ominus, \otimes$ and $\oslash$ on $F(\mathbb{R}) \times F(\mathbb{R})$ can be defined as:

$$\begin{aligned}(\eta \oplus \delta)(t) &= \vee_{t=x+y}(\min(\eta(x), \delta(y))) \\ &= \sup_{s \in \mathbb{R}}\{\eta(s) \wedge \delta(t-s)\}, t \in \mathbb{R}\end{aligned}$$

$$\begin{aligned}(\eta \ominus \delta)(t) &= \vee_{t=x-y}(\min(\eta(x), \delta(y))) \\ &= \sup_{s \in \mathbb{R}}\{\eta(s) \wedge \delta(s-t)\}, t \in \mathbb{R}\end{aligned}$$

$$\begin{aligned}(\eta \otimes \delta)(t) &= \vee_{t=xy}(\min(\eta(x), \delta(y))) \\ &= \sup_{s \in \mathbb{R}, s \neq 0}\{\eta(s) \wedge \delta(t/s)\}, t \in \mathbb{R}\end{aligned}$$

$$\begin{aligned}(\eta \oslash \delta)(t) &= \vee_{t=\frac{x}{y}}(\min(\eta(x), \delta(y))) \\ &= \sup_{s \in \mathbb{R}}\{\eta(st) \wedge \delta(s)\}, t \in \mathbb{R},\end{aligned}$$

which are special cases of Zadeh's extension principle. The additive and multiplicative identities in $F(\mathbb{R})$ are $\tilde{0}$ and $\tilde{1}$, respectively. Let $\ominus\eta$ be defined as $\tilde{0} \ominus \eta$. It is clear that $\eta \ominus \delta = \eta \oplus (\ominus\delta)$.

Lemma 2.1. [6] *Let $\eta, \delta \in F(\mathbb{R})$ and $[\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+]$, $[\delta]_\alpha = [\delta_\alpha^-, \delta_\alpha^+]$. Then for all $\alpha \in (0,1]$,*

$$\begin{aligned}[\eta \oplus \delta]_\alpha &= [\eta_\alpha^- + \delta_\alpha^-, \eta_\alpha^+ + \delta_\alpha^+] \\ [\eta \ominus \delta]_\alpha &= [\eta_\alpha^- - \delta_\alpha^+, \eta_\alpha^+ - \delta_\alpha^-] \\ [\eta \otimes \delta]_\alpha &= [\eta_\alpha^- \delta_\alpha^-, \eta_\alpha^+ \delta_\alpha^+] \\ [|\eta|]_\alpha &= [\max(0, \eta_\alpha^-, -\eta_\alpha^+), \max(|\eta_\alpha^-|, |\eta_\alpha^+|)].\end{aligned}$$

Theorem 2.1. [2] *Let $[a_\alpha, b_\alpha], 0 < \alpha \leq 1$, be a family of non-empty intervals. If*
a) *for all $0 < \alpha_1 \leq \alpha_2, [a_{\alpha_1}, b_{\alpha_1}] \supset [a_{\alpha_2}, b_{\alpha_2}]$,*

b) $[\lim_{k\to-\infty} a_{\alpha_k}, \lim_{k\to\infty} b_{\alpha_k}] = [a_\alpha, b_\alpha]$ *whenever* $\{\alpha_k\}$ *is an increasing sequence in* $(0,1]$ *converging to* α,

then the family $[a_\alpha, b_\alpha]$ *represents the* α*-level sets of a fuzzy real number* $\eta \in F(\mathbb{R})$ *such that* $\eta(t) = \sup\{\alpha \in (0,1] : t \in [a_\alpha, b_\alpha]\}$ *and* $[\eta]_\alpha = [\eta^-_\alpha, \eta^+_\alpha] = [a_\alpha, b_\alpha]$.

Definition 2.2. [15] *Let* $\eta, \gamma \in F(\mathbb{R})$ *and* $[\eta]_\alpha = [\eta^-_\alpha, \eta^+_\alpha]$, $[\gamma]_\alpha = [\gamma^-_\alpha, \gamma^+_\alpha]$, *for all* $\alpha \in (0,1]$. *Define a partial ordering by* $\eta \preceq \gamma$ *in* $F(\mathbb{R})$ *if and only if* $\eta^-_\alpha \le \gamma^-_\alpha$ *and* $\eta^+_\alpha \le \gamma^+_\alpha$, *for all* $\alpha \in (0,1]$. *The strict inequality in* $F(\mathbb{R})$ *is defined by* $\eta \prec \gamma$ *if and only if* $\eta^-_\alpha < \gamma^-_\alpha$ *and* $\eta^+_\alpha < \gamma^+_\alpha$, *for all* $\alpha \in (0,1]$.

Lemma 2.2. [13] *Let* η, δ *be fuzzy real numbers. Then*

$$\eta(t) = \delta(t), \quad \forall t \in \mathbb{R} \Leftrightarrow [\eta]_\alpha = [\delta]_\alpha, \quad \forall \alpha \in (0,1].$$

Definition 2.3. [10] *The absolute value* $|\eta|$ *of* $\eta \in F(\mathbb{R})$ *is defined by*

$$|\eta|(t) = \begin{cases} \max(\eta(t), \eta(-t)), & t \ge 0, \\ 0, & t < 0. \end{cases}$$

Proposition 2.1. [3] *Let* $\{[a_\alpha, b_\alpha] : \alpha \in (0,1]\}$, *be a family of nested bounded closed intervals and* $\eta : \mathbb{R} \to [0,1]$ *be a function defined by* $\eta(t) = \bigvee\{\alpha \in (0,1] : t \in [a_\alpha, b_\beta]\}$. *Then* η *is a fuzzy real number (fuzzy interval).*

Definition 2.4. *(Bag and Samanta* [3] *) Let* X *be a real linear space,* L *and* R *(respectively, left norm and right norm) be symmetric and non-decreasing mappings from* $[0,1] \times [0,1]$ *into* $[0,1]$ *satisfying* $L(0,0) = 0, R(1,1) = 1$. *Then* $\|\cdot\|$ *is called a fuzzy norm and* $(X, \|\cdot\|, L, R)$ *is a fuzzy normed linear space (abbreviated to FNLS) if the mapping* $\|\cdot\|$ *from* X *into* $F^*(\mathbb{R})$ *satisfies the following axioms, where* $[\|x\|]_\alpha = [\|x\|^-_\alpha, \|x\|^+_\alpha]$ *for* $x \in X$ *and* $\alpha \in (0,1]$:

$(A1)$ $x = 0$ *if and only if* $\|x\| = \overline{0}$,

$(A2)$ $\|rx\| = \tilde{r} \odot \|x\|$ *for all* $x \in X$ *and* $r \in (-\infty, \infty)$,

$(A3)$ *for all* $x, y \in X$:

$(A3R)$ *if* $s \ge \|x\|^-_1, t \ge \|y\|^-_1$ *and* $s + t \ge \|x + y\|^-_1$, *then* $\|x + y\|(s + t) \le R(\|x\|(s), \|y\|(t))$,

$(A3L)$ *if* $s \le \|x\|^-_1, t \le \|y\|^-_1$ *and* $s + t \le \|x + y\|^-_1$, *then* $\|x + y\|(s + t) \ge L(\|x\|(s), \|y\|(t))$.

In the rest of this paper, we use this definition of fuzzy norm. We note that $\|x\|^s_\alpha, s = -, +$, are crisp norms on X where $[\|x\|]_\alpha = [\|x\|^-_\alpha, \|x\|^+_\alpha]$.

Proposition 2.2. *(Bag and Samanta* [3] *) Let* $\{a_\alpha\}$ *and* $\{b_\alpha\}$ *be two, respectively, non-decreasing and non-increasing families of real numbers such that* $-\infty < a_\alpha \le b_\alpha < +\infty, 0 < \alpha \le 1$ *and* η *be the fuzzy real number (fuzzy interval) generated by the families of closed intervals* $\{[a_\alpha, b_\alpha]; \alpha \in (0,1]\}$. *Then*

(A) $\sup_{\beta<\alpha} \eta^-_\beta = \eta^-_\alpha$,

(B) $\inf_{\beta<\alpha} \eta^+_\beta = \eta^+_\alpha$, *where* $[\eta]_\alpha = [\eta^-_\alpha, \eta^+_\alpha] = [a_\alpha, b_\alpha]$.

Corollary 2.1. *(Bag and Samanta* [3] *) Let* $(X, \|\cdot\|)$ *be a fuzzy normed linear space. If for* $x \in X, [\|x\|]_\alpha = [\|x\|^-_\alpha, \|x\|^+_\alpha], 0 < \alpha \le 1$, *then*

(A) $\|x\|^-_\alpha = \sup_{\beta<\alpha} \|x\|^-_\beta$,

(B) $\|x\|^+_\alpha = \inf_{\beta<\alpha} \|x\|^+_\beta$.

Definition 2.5. [7] *Two fuzzy normed linear spaces* $(X, \|\cdot\|)$ *and* $(X^*, \|\cdot\|^*)$ *are called congruent if there exists an isometry of* $(X, \|\cdot\|)$ *onto* $(X^*, \|\cdot\|^*)$.

Definition 2.6. [7] *A complete fuzzy normed linear space* $(X^*, \|\cdot\|^*)$ *is a completion of a fuzzy normed linear space* $(X, \|\cdot\|)$ *if*

(i) $(X, \|\cdot\|)$ *is congruent to a subspace* $(X_0, \|\cdot\|^*)$ *of* $(X^*, \|\cdot\|^*)$ *and*
(ii) *the closure* $\overline{X_0}$ *of* X_0*, is all of* X^**; i.e.,* $\overline{X_0} = X^*$.

Definition 2.7. [15] *Let* $(X, \|\cdot\|)$ *be a fuzzy normed linear space.*
(i) *A sequence* $\{x_n\} \subseteq X$ *converge to* $x \in X$ *(*$\lim_{n\to\infty} x_n = x$*), if* $\lim_{n\to\infty} \|x_n - x\|^+_\alpha = 0$, *for all* $\alpha \in (0, 1]$.
(ii) *A sequence* $\{x_n\} \subseteq X$ *is said to be a Cauchy sequence if* $\lim_{m,n\to\infty} \|x_n - x_m\|^+_\alpha = 0$, *for all* $\alpha \in (0, 1]$.

Definition 2.8. [15] *Let* $(X, \|\cdot\|)$ *be a fuzzy normed linear space. A subset* A *of* X *is said to be complete, if every Cauchy sequence in* A *converges in* A.

In [8], the authors introduced a definition of fuzzy inner product space.

Definition 2.9. [8] *Let* X *be a vector space over* $\mathbb{R}$. *A fuzzy inner product on* X *is a mapping* $\langle\cdot,\cdot\rangle : X \times X \to F(\mathbb{R})$ *such that for all vectors* $x, y, z \in X$ *and* $r \in \mathbb{R}$*, we have*
(IP_1) $\langle x + y, z\rangle = \langle x, z\rangle \oplus \langle y, z\rangle$,
(IP_2) $\langle rx, y\rangle = \widetilde{r}\langle x, y\rangle$,
(IP_3) $\langle x, y\rangle = \langle y, x\rangle$,
(IP_4) $\langle x, x\rangle \succeq \widetilde{0}$,
(IP_5) $\inf_{\alpha\in(0,1]}\langle x, x\rangle^-_\alpha > 0$ *if* $x \neq 0$,
(IP_6) $\langle x, x\rangle = \widetilde{0}$ *if and only if* $x = 0$.

The vector space X equipped with a fuzzy inner product is called a fuzzy inner product space. A fuzzy inner product on X defines a fuzzy number

$$\|x\| = \sqrt{\langle x, x\rangle}, \quad \forall x \in X. \tag{1}$$

A fuzzy Hilbert space is a complete fuzzy inner product space with the fuzzy norm defined by (1). Therefore by the above definition, any fuzzy inner product space with origin 0 is a subspace of a fuzzy normed linear space.

In [8] the authors, clarified the fuzzy inner product spaces by Definition 2.10 and Lemma 2.3.

Definition 2.10. [8] *Let* $(X^*, \|\cdot\|^*)$ *be a completion of a fuzzy normed linear space* $(X, \|\cdot\|)$ *and* $x^*, y^* \in X^*$ *with representatives* $\{x_n\}$ *and* $\{y_n\}$*, respectively. Suppose* $\alpha \in (0, 1]$ *and* $\{\alpha_k\}$ *is a strictly increasing sequence converging to* α*. Define*

$$[\langle x^*, y^*\rangle]_\alpha = [\lim_{n,k\to\infty} \langle x_n, y_n\rangle^-_{\alpha_k}, \lim_{n,k\to\infty} \langle x_n, y_n\rangle^+_{\alpha_k}].$$

Lemma 2.3. [8] *The function* $\langle\cdot,\cdot\rangle$ *defined in Definition 2.10, is a fuzzy inner product on* X^*.

Example 2.1. [5] *The linear space* $\mathbf{C}(\Omega)$ *(the vector space of all complex valued continuous functions on* Ω*), is a fuzzy Hilbert space, for any open subset* $\Omega \subseteq R^n$ *with the fuzzy inner product*

$$\langle f, g\rangle(t) = \sup\{\alpha \in (0, 1] | t \in [f^-_\alpha g^-_\alpha, f^+_\alpha g^+_\alpha]\} \tag{2}$$

for $f, g \in \mathbf{C}(\Omega)$. *For* $\alpha \in (0, 1]$, *there exists an* $n \in \mathbb{N}$ *such that* $\frac{1}{n+1} < \alpha \leq \frac{1}{n}$. *Also* Ω *is a countable union of sets* $K_n \neq \emptyset$ *which can be chosen so that* K_n *lies in the interior of* $K_{n+1} (n = 1, 2, 3, \cdots)$. *Let for an arbitrary function* $f \in \mathbf{C}(\Omega)$ *have*

$$f^-_\alpha = \sup\{|f(x)|; x \in K_1\} \quad \textit{and} \quad f^+_\alpha = \sup\{|f(x)|; x \in K_{n+1}\},$$

so $\{[f^-_\alpha, f^+_\alpha]; \alpha \in (0, 1]\}$ *is a family of nested bounded closed intervals. Similarly for* g *and so on holds. With the above discussion,* $(\mathbf{C}(\Omega), \langle\cdot,\cdot\rangle)$ *is a fuzzy Hilbert space.*

Definition 2.11. [6] *Let* $(X, \|\cdot\|)$ *and* $(Y, \|\cdot\|^{\sim})$ *be fuzzy normed linear spaces. Furthermore, let* $T : X \to Y$ *be a linear operator. The operator* T *is said to be fuzzy bounded (* $\boldsymbol{F}$*-bounded), if there is a fuzzy real number* $\eta \succeq \tilde{0}$*, such that*

$$\|Tx\|^{\sim} \preceq \eta \odot \|x\|, \quad \forall x \in X.$$

The set of all fuzzy bounded linear operators $T : X \to Y$ is denoted by $\mathbf{B}(X, Y)$. Now we introduce the notion of fuzzy norm of linear operators. At the first, we prove a memorable result for $\mathbf{B}(X, Y)$, which has a famous analogous in functional analysis. A similar result on $\mathbf{B}(X, C)$ is defined in [3].

Definition 2.12. [3] *Let* $(X, \|\cdot\|)$ *and* $(Y, \|\cdot\|^{\sim})$ *be fuzzy normed linear spaces. The linear operator* $T : X \to Y$ *is said to be strongly fuzzy bounded if there exists a real number* $k > 0$ *such that*

$$\|Tx\|^{\sim} \oslash \|x\| \preceq \tilde{k}, \quad \forall x \in X, x \neq \underline{0}$$

i.e. by [3],

$$\|Tx\|_{\alpha}^{\sim -} \leq k\|x\|_{\alpha}^{+} \quad \textit{and} \quad \|Tx\|_{\alpha}^{\sim +} \leq k\|x\|_{\alpha}^{-}.$$

Definition 2.13. [3] *Let* $(X, \|\cdot\|)$ *and* $(Y, \|\cdot\|^{\sim})$ *be fuzzy normed linear spaces. A linear operator* $T : X \to Y$ *is said to be weakly fuzzy bounded if there exists a fuzzy interval* $\tilde{0} \prec \eta \in F^*$ *such that*

$$\|Tx\|^{\sim} \oslash \|x\| \preceq \eta, \quad \forall x (\neq \underline{0}) \in X.$$

The following result of Bag and Samanta [3] is essential in this paper.

Proposition 2.3. [3] *Let* $(X, \|\cdot\|)$ *and* $(Y, \|\cdot\|^{\sim})$ *be fuzzy normed linear spaces and* $T \in \boldsymbol{B}(X, Y)$ *(i.e.,* $T : X \to Y$ *is a fuzzy bounded linear operator). There exists* $\tilde{0} \prec \eta \in F^*$ *such that for all* $x(\neq \underline{0}) \in X$*,* $\|Tx\|^{\sim} \oslash \|x\| \preceq \eta$*. If* $[\eta]_{\alpha} = [\eta_{\alpha}^{-}, \eta_{\alpha}^{+}], 0 < \alpha \leq 1$*, then we get*

$$\|Tx\|_{\alpha}^{\sim -} \leq \eta_{\alpha}^{-} \cdot \|x\|_{\alpha}^{+} \quad \textit{and} \quad \|Tx\|_{\alpha}^{\sim +} \leq \eta_{\alpha}^{+} \cdot \|x\|_{\alpha}^{-}.$$

Define

$$\|T\|_{\alpha}^{*-} = \sup_{0 \neq x \in X} \frac{\|Tx\|_{\alpha}^{\sim -}}{\|x\|_{\alpha}^{+}} \leq \eta_{\alpha}^{-}, \tag{3}$$

$$\|T\|_{\alpha}^{*+} = \sup_{0 \neq x \in X} \frac{\|Tx\|_{\alpha}^{\sim +}}{\|x\|_{\alpha}^{-}} \leq \eta_{\alpha}^{+}. \tag{4}$$

Then $\{\|\cdot\|_{\alpha}^{*+}, \alpha \in (0,1]\}$ *and* $\{\|\cdot\|_{\alpha}^{*-}, \alpha \in (0,1]\}$ *are descending and ascending family of norms, respectively. Thus* $\{[\|T\|_{\alpha}^{*-}, \|T\|_{\alpha}^{*+}] : \alpha \in (0,1]\}$ *is a family of nested bounded closed intervals in* $\mathbb{R}$*. Define the function* $\|T\|^* : \mathbb{R} \to [0,1]$ *by*

$$\|T\|^*(t) = \bigvee \{\alpha \in (0,1] : t \in [\|T\|_{\alpha}^{*-}, \|T\|_{\alpha}^{*+}]\}.$$

$(\boldsymbol{B}(X, Y), \|\cdot\|^*)$ *is a fuzzy normed space.*

In spite of our expectation, Example 3.2 in [9] shows that for $S, T \in \mathbf{B}(X)$ ($\mathbf{B}(X, X)$), the relations $\|SoT\|^* \preceq \|S\|^* \odot \|T\|^*, \|SoT\|^* \oslash \|S\|^* \preceq \|T\|^*$ and $\|SoT\|^* \oslash \|T\|^* \preceq \|S\|^*$ are not valid.

Definition 2.14. [8] *Let* $(X, \|\cdot\|)$ *and* $(Y, \|\cdot\|)$ *be fuzzy normed linear spaces and let* $T : X \to Y$ *be a fuzzy bounded linear operator. We define* $\|T\|$ *by,*

$$[\|T\|]_{\alpha} = [\sup_{\beta < \alpha} \sup_{\|x\|_{\beta}^{-} \leq 1} \|Tx\|_{\beta}^{-}, \inf\{\eta_{\alpha}^{+} : \|Tx\| \preceq \eta \|x\|\}], \quad \forall \alpha \in (0,1].$$

Then $\|T\|$ *is called the fuzzy norm of the operator* T*.*

Remark 2.1. $\|T\|_\alpha^- = \sup_{\beta<\alpha}\sup_{\|x\|_\beta^-\le 1}\|Tx\|_\beta^-$ *and* $\|T\|_\alpha^+ = \inf\{\eta_\alpha^+ : \|Tx\| \preceq \eta\|x\|\}$, *i.e.* $[\|T\|]_\alpha = [\|T\|_\alpha^-, \|T\|_\alpha^+], \quad \forall\alpha \in (0,1]$.

Lemma 2.4. *(Schwarz inequality)*[8] *A fuzzy inner product space X together with its corresponding norm $\|\cdot\|$ satisfy the Schwarz inequality*

$$|\langle x,y\rangle| \preceq \|x\|\|y\|, \quad \forall x,y \in X.$$

Theorem 2.2. [8] *Let $T : X \to Y$ be a fuzzy bounded linear operator. If $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ are fuzzy normed linear spaces. Then $\|T\|$ is a fuzzy real number.*

Lemma 2.5. [8] *Let $T : X \to Y$ be a fuzzy bounded linear operator. If $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ are fuzzy normed linear spaces, then $\|Tx\| \preceq \|T\|\|x\|, \quad \forall x \in X$.*

Theorem 2.3. [8] *Let $(X, \|\cdot\|), (Y, \|\cdot\|)$ and $(Z, \|\cdot\|)$ be fuzzy normed linear spaces. Furthermore, let $T \in \boldsymbol{B}(X,Y)$ and $S \in \boldsymbol{B}(Y,Z)$. Then $ST \in \boldsymbol{B}(X,Z)$ and $\|ST\| \preceq \|S\|\|T\|$.*

Theorem 2.4. [8] *Let X be a fuzzy inner product space. For all $x, y \in X$, if $x_n \to x$ and $y_n \to y$, then $\langle x_n, y_n\rangle \to \langle x, y\rangle$.*

Definition 2.15. *(Orthogonality)*[8] *An element x of an inner product space X is said to be orthogonal to an element $y \in X$ if $\langle x,y\rangle = 0$, we also say that x and y are orthogonal and we write $x \perp y$. Similarly, for subsets $A, B \subseteq X$ we write $x \perp A$ if $x \perp a$ for all $a \in A$ and $A \perp B$ if $a \perp b$ for all $a \in A$ and all $b \in B$.*

3. Fuzzy unbounded operator

Now we study the case of fuzzy unbounded operators. The theory of fuzzy unbounded operators is more complicated than that of bounded operators.

Example 3.1. *Let $X = \mathbb{R}$ (the linear space of real numbers). Define two functions $\|\cdot\|_1$ and $\|\cdot\|_2$ by*

$$\|x\|_1(t) = \begin{cases} 1, & t = |x| \\ 0, & t \neq |x| \end{cases} \quad \text{and} \quad \|x\|_2(t) = \begin{cases} \frac{|x|}{t}, & |x| \le t, x \neq \underline{0} \\ 1, & t = |x| = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Then $\|\cdot\|_1$ and $\|\cdot\|_2$ are fuzzy norms on $\mathbb{R}$ and the α-level sets of $\|\cdot\|_1$ and $\|\cdot\|_2$ are given by

$$[\|x\|_1]_\alpha = [|x|, |x|] \quad \text{and} \quad [\|x\|_2]_\alpha = [|x|, \frac{|x|}{\alpha}], \quad x \in \mathbb{R}.$$

We define $T : (\mathbb{R}, \|\cdot\|_1) \to (\mathbb{R}, \|\cdot\|_2)$ by $T(x) = nx$, for each $n \in \mathbb{N}$, for all $x \in \mathbb{R}$. Clearly T is linear.
Now $\frac{\|Tx\|_{2,\alpha}^+}{\|x\|_{1,\alpha}^-} = \frac{\frac{|nx|}{\alpha}}{|x|} = \frac{n}{\alpha}$, for all $x(\neq 0) \in \mathbb{R}, 0 < \alpha \le 1$. If T is bounded, then $n \le \alpha\eta$ for each $n \in \mathbb{N}$. Hence $\eta = \infty$, which contradicts the definition of fuzzy real number. This shows that T is fuzzy unbounded. That such a fuzzy operator T may be unbounded, that is, T may not be bounded. Bag in [1] *established that the uniform boundedness theorem for weakly fuzzy bounded holds and the authors in* [9] *presented that a version of uniform boundedness theorem (the point-wise and uniformly boundedness) with new definitions of different from it in* [1]*. Using the elementary functional analysis, we can prove the fuzzy version of uniform boundedness theorem. Using of the uniform boundedness theorem with Felbin norm, one can verify the main result of this section.*

Definition 3.1. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|^\sim)$ be two fuzzy normed linear spaces. A family $\{T_n\} \subseteq \boldsymbol{B}(X,Y)$ is called point-wise bounded if for every $x(\neq 0) \in X$, there exists a fuzzy number $\tilde{0} \prec \delta_x \in F^*$ such that for all $n > 0$,*

$$\|T_n(x)\|^\sim \preceq \delta_x,$$

and is said uniformly bounded if there exists a fuzzy number $\tilde{0} \prec \delta \in F^*$ *such that for each* $n > 0$ *and* $x(\neq 0) \in X$,

$$\|T_n\|^{\sim} \preceq \delta.$$

Theorem 3.1. *(Uniform Boundedness Theorem) Let* $\{T_n\} \subset \boldsymbol{B}(H, H)$ *such that for each* $x \in H, \{T_n\}$ *is bounded in* H*, i.e. there exists a fuzzy real number* η_x *such that* $\|T_n x\| \preceq \eta_x$*, for all* n*. Then there exists a fuzzy real number* δ *such that* $\|T_n\| \preceq \delta$*, for all* n*, where* $(H, \|\cdot\|)$ *is a complete fuzzy normed linear space for each* $\alpha \in (0, 1]$.

Proof. Let $[\|T_n x\|]_\alpha = [\|T_n x\|_\alpha^-, \|T_n x\|_\alpha^+]$, $[\|x\|]_\alpha = [\|x\|_\alpha^-, \|x\|_\alpha^+]$ and $[\eta_x]_\alpha = [\eta_{x,\alpha}^-, \eta_{x,\alpha}^+]$, $\alpha \in (0, 1]$. Since $\|T_n x\| \preceq \eta_x, \forall n$, we have

$$\|T_n x\|_\alpha^- \leq \eta_{x,\alpha}^- \quad \textit{and} \quad \|T_n x\|_\alpha^+ \leq \eta_{x,\alpha}^+.$$

Again, since H is a complete fuzzy normed linear space for each $\alpha \in (0, 1]$ $(H, \|\cdot\|_\alpha^-)$ and $(H, \|\cdot\|_\alpha^+)$ are complete spaces for each $\alpha \in (0, 1]$. We know that $\{T_n\} \subset \mathbf{B}(H, H)$. So,

$$T_n : (H, \|\cdot\|_\alpha^-) \to (H, \|\cdot\|_\alpha^+) \quad \textit{and} \quad T_n : (H, \|\cdot\|_\alpha^+) \to (H, \|\cdot\|_\alpha^-)$$

are sequences of bounded linear operators for each $\alpha \in (0, 1]$. Now by the uniform boundedness theorem, it follows that for each $\alpha \in (0, 1]$ there exist constants C_α^-, C_α^+ such that

$$\sup_n \|T_n\|_\alpha^- = C_\alpha^- \quad \textit{and} \quad \sup_n \|T_n\|_\alpha^+ = C_\alpha^+,$$

where $\|T\|_\alpha^- = \sup_{\beta<\alpha} \sup_{\|x\|_\beta^- \leq 1} \|Tx\|_\beta^-$ and $\|T\|_\alpha^+ = \inf\{\eta_\alpha^+ : \|Tx\| \leq \eta\|x\|\}$. We have

$$\|T_n\|_\alpha^- \leq C_\alpha^- \tag{5}$$

and

$$\|T_n\|_\alpha^+ \leq C_\alpha^+. \tag{6}$$

It can be easily verified that $\{[C_\alpha^-, C_\alpha^+], \alpha \in (0, 1]\}$ is a family of nested, bounded and closed intervals of real numbers and thus it generates a fuzzy real number say δ (by Proposition 2.1). Hence by Proposition 2.1, from (5) and (6) we have $\|T_n\| \preceq \delta, \forall n$. □

Remark 3.1. *If* T *is weakly fuzzy bounded, then the above theorem is also true.*

Theorem 3.2. *If a fuzzy linear operator* T *is defined on a fuzzy Hilbert space* H *and satisfies:* $\langle Tx, y\rangle = \langle x, Ty\rangle$*, for all* $x, y \in H$*, then* T *is bounded.*

Proof. If T is not bounded, then H shall contain a sequence $\{y_n\}$ such that $\|y_n\| = 1$ and $\|Ty_n\| \to \infty$. We consider the functional f_n defined by

$$f_n(x) = \langle Tx, y_n\rangle$$

where for $n = 1, 2, ...$, any f_n is defined on H and is linear for any fixed n, the functional f_n is fuzzy bounded since

$$\begin{aligned} \|f_n(x)\| &= \|\langle x, Ty_n\rangle\| \\ &\preceq \|Ty_n\|\|x\|. \end{aligned}$$

Moreover, for any fixed $x \in H$, the sequence $\{f_n(x)\}$ is fuzzy bounded. Indeed, since $\|y_n\| = 1$, we have

$$\|f_n(x)\| \preceq \|T\|\|x\|,$$

we know that $\|T\|$ is fuzzy real number that is independent of x and $\|T\|\|x\|$ is depend of x. Also $f_n(x)$ is fuzzy bounded. From this and the uniform boundedness theorem, we conclude that there exists $k \in F(\mathbb{R})$ such that $\|f_n\| \preceq k$ for all n. This implies that for any $x \in H$ we have by Lemma 2.5,

$$\begin{aligned} \|f_n(x)\| &\leq \|f_n\|\|x\| \\ &\leq k\|x\| \end{aligned}$$

and taking $x = Ty_n$, we arrive at

$$\begin{aligned}\|Ty_n\|^2 &= \langle Ty_n, Ty_n\rangle \\ &= \|f_n(Ty_n)\| \\ &\leq k\|Ty_n\|.\end{aligned}$$

Hence $\|Ty_n\| \preceq k$, which contradicts our initial assumption $\|Ty_n\| \to \infty$ and proves the theorem. □

4. **Closed linear operators**

We review the definition of a closed linear operator, using formulations which are convenient for fuzzy Hilbert spaces. Let $(H, \|\cdot\|)$ and $(H, \|\cdot\|^*)$ be two α-complete fuzzy normed linear spaces for each $\alpha \in (0,1]$. Define $\langle x, y\rangle + \langle x', y'\rangle = \langle x + x', y + y'\rangle$ and $c\langle x, y\rangle = \langle cx, cy\rangle$ where $\langle x, y\rangle, \langle x', y'\rangle \in H \times H$ and c is a scalar, $\langle x, y\rangle = x + y$.
Since $(H, \|\cdot\|)$ and $(H, \|\cdot\|^*)$ are α-complete fuzzy normed linear spaces for each $\alpha \in (0,1]$, thus $(H, \|\cdot\|_\alpha^s)$ and $(H, \|\cdot\|_\alpha^{*\,s})$ are fuzzy Hilbert spaces for $s = -, +$ and for $\alpha \in (0,1]$. Now we define the functions $\|\cdot\|'_\alpha$ and $\|\cdot\|''_\alpha$ from $H \times H$ to $\mathbb{R}^+$ by

$$\|\langle x, y\rangle\|'_\alpha = \|x\|^-_\alpha + \|y\|_\alpha^{*\,+} \quad and \quad \|\langle x, y\rangle\|''_\alpha = \|x\|^+_\alpha + \|y\|_\alpha^{*\,-}$$

where $[\|x\|]_\alpha = [\|x\|^-_\alpha, \|x\|^+_\alpha]$ and $[\|y\|^*]_\alpha = [\|y\|_\alpha^{*\,-}, \|y\|_\alpha^{*\,+}], \alpha \in (0,1]$.
Then it can be verified that $(H \times H, \|\cdot\|'_\alpha)$ and $(H \times H, \|\cdot\|''_\alpha)$ are fuzzy normed linear spaces for each $\alpha \in (0,1]$. If H be fuzzy Hilbert space and $T : H \to H$ is a linear operator then the set given by $G(T) = \{(x, Tx) : x \in H\} \subset H \times H$ is called the graph of T.

Definition 4.1. *Let $(H, \|\cdot\|)$ and $(H, \|\cdot\|^*)$ be two fuzzy Hilbert spaces and $T : (H, \|\cdot\|) \to (H, \|\cdot\|^*)$ be a linear operator and $\alpha \in (0,1]$. Then $G(T) = \{(x, Tx) : x \in H\} \subset H \times H$ is said to be α-closed if for every sequence $\{x_n\}$ in H, where $x_n = (x'_n, y'_n), \|x_n - x\|^+_\alpha \to 0$ and $\|Tx_n - y\|_\alpha^{*\,+} \to 0$ as $n \to \infty$, implies $x \in H$ and $y = Tx$ where $[\|x\|]_\alpha = [\|x\|^-_\alpha, \|x\|^+_\alpha]$ and $[\|y\|^*]_\alpha = [\|y\|_\alpha^{*\,-}, \|y\|_\alpha^{*\,+}], \alpha \in (0,1]$.*

If $G(T)$ is α-closed, then T is called an α-closed linear operator. The same discussion in closed graph theorem holds for the above definition.

5. **Bessel's inequality**

Definition 5.1. *Let X be a fuzzy inner product space. A fuzzy orthogonal set M in X is said to be fuzzy orthonormal if $\langle x, y\rangle = \begin{cases}\tilde{1}, & x = y\\ \tilde{0}, & x \neq y\end{cases}$, for all $x, y \in M$.*

Theorem 5.1. *Let X be a fuzzy inner product space and assume that for any $x \in X$ there exists $\{x_n\} \subset X, x_n \to x$. Let $\{e_k\}$ be a fuzzy orthonormal sequence in X. Then*

$$\sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 \leq \|x\|^2, \quad x \in X.$$

Proof. We cautiously note that the classic form of Bessel's inequality can not be directly applied in this case because $\|\cdot\|$ is Felbin's fuzzy norm. Due to Bessel's inequality in crisp inner product space we have

$$\sum_k |\langle x, e_k\rangle|^2 \leq \|x\|^2, \quad x \in X. \tag{7}$$

Using x_n, for $n = 1, 2, \dots$, substitute for x in (7), we obtain $\sum_{k=1}^{n} |\langle x_n, e_k\rangle|^2 \le \|x_n\|^2$.
Step 1. Since $\|\cdot\|_\alpha^s; s = -, +;$ are crisp norms on X, we have

$$\sum_{k=1}^{n} |\langle x_n, e_k\rangle|^2 \le \|x_n\|_\alpha^{-2} \quad \textit{and} \quad \sum_{k=1}^{n} |\langle x_n, e_k\rangle|^2 \le \|x_n\|_\alpha^{+2}.$$

Since $\langle\cdot,\cdot\rangle$ is continuous, passing the limit above yields,

$$\sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 \le \|x\|_\alpha^{-2} \quad \textit{and} \quad \sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 \le \|x\|_\alpha^{+2}.$$

Step 2. In this case, the norm is Felbin-type induced by the fuzzy inner product on X and let $[\langle x_n, e_k\rangle]_\alpha = [\langle x_n, e_k\rangle_\alpha^-, \langle x_n, e_k\rangle_\alpha^+]$ and $[|\langle x_n, e_k\rangle|]_\alpha = [m_\alpha, m'_\alpha]$, where $m_\alpha = \max(0, \langle x_n, e_k\rangle_\alpha^-, -\langle x_n, e_k\rangle_\alpha^+), m'_\alpha = \max(|\langle x_n, e_k\rangle_\alpha^-|, |\langle x_n, e_k\rangle_\alpha^+|)$ for $\alpha \in (0, 1]$. Invoking the Theorem 2.4, we have $\langle x_n, e_k\rangle \to \langle x, e_k\rangle$. Hence $\langle x_n, e_k\rangle_\alpha^\pm \to \langle x, e_k\rangle_\alpha^\pm$, with respect to n. We have $||\langle x_n, e_k\rangle_\alpha^\pm| - |\langle x, e_k\rangle_\alpha^\pm|| \le |\langle x_n, e_k\rangle_\alpha^\pm - \langle x, e_k\rangle_\alpha^\pm|$, hence

$$|\langle x_n, e_k\rangle_\alpha^\pm|^2 \to |\langle x, e_k\rangle_\alpha^\pm|^2. \tag{8}$$

Also from Theorem 2.4 again, $\langle x_n, x_n\rangle \to \langle x, x\rangle$, then we have $\langle x_n, x_n\rangle_\alpha^\pm \to \langle x, x\rangle_\alpha^\pm$. Therefore

$$\|x_n\|_\alpha^\pm \to \|x\|_\alpha^\pm. \tag{9}$$

By Schwarz inequality, we have $|\langle x_n, e_k\rangle| \preceq \|x_n\|\|e_k\|$ and since $\|e_k\| = 1$, we have $|\langle x_n, e_k\rangle|^2 \preceq \|x_n\|^2$ then $|\langle x_n, e_k\rangle_\alpha^\pm|^2 \le \|x_n\|_\alpha^{\pm 2}$. From (8) and (9) with taking lim as $n \to \infty$, $|\langle x, e_k\rangle_\alpha^\pm|^2 \le \|x\|_\alpha^{\pm 2}$, and since $\langle x, e_k\rangle = \begin{cases} 1, & x = e_k \\ 0, & x \ne e_k \end{cases}$, we have

$$\sum_{k=1}^{\infty} |\langle x, e_k\rangle_\alpha^-|^2 \le \|x\|_\alpha^{-2} \tag{10}$$

and

$$\sum_{k=1}^{\infty} |\langle x, e_k\rangle_\alpha^+|^2 \le \|x\|_\alpha^{+2}, \tag{11}$$

from (10) and (11) we have

$$\sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 \preceq \|x\|^2.$$

This completes the proof. □

Theorem 5.2. *(Bessel's inequality) Let H be a fuzzy Hilbert space. If $\{e_k\}$ is a fuzzy orthonormal sequence in H, then*

$$\sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 \preceq \|x\|^2, \quad x \in H,$$

which is a special case of Theorem 5.1.

6. **Conclusion**

As the idea of fuzzy Hilbert space is relatively new and the classic form of theorems plays the role of a prototype in our discussion of this paper, it is natural to ask: do ordinary theorems in this discussion hold in Felbin's fuzzy Hilbert space? In this paper, we consider the fuzzy norm in the sense of Felbin. Some concepts have been introduced. Some theorems have been established in fuzzy setting. Since these theorems have many applications in functional analysis, it is expected that the results of this paper will be helpful for the researchers to develop fuzzy functional analysis.

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