CHARACTERIZATION OF NUCLEAR PSEUDO-MULTIPLIERS ASSOCIATED TO THE HARMONIC OSCILLATOR

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In this paper we study pseudo-multipliers associated to the harmonic oscillator (also called Hermite pseudo-multipliers) belonging to the ideal of r-nuclear operators on Lebesgue spaces. Our main result is Theorem 1.1 where we classify the r-nuclearity of pseudo-multipliers. We also investigate the nuclear trace of these operators.

Keywords: Harmonic oscillator, Fourier multiplier, Hermite multiplier, nuclear operator, traces.
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1. Introduction

1.1. Outline of the paper

In this paper, we are interested in the r-nuclearity of pseudo-multipliers associated to the harmonic oscillator (also called Hermite pseudo-multipliers) on $L^p(\mathbb{R}^n)$-spaces. This paper is the continuation of the work [2] where the authors have given necessary conditions for the r-nuclearity of Hermite multipliers. Our main result is Theorem 1.1 where we classify the r-nuclearity of pseudo-multipliers. In order to present our result we recall some notions. Let us consider the sequence of Hermite functions on $\mathbb{R}^n$,

$$\phi_\nu = \Pi_{j=1}^n \phi_{\nu_j}, \;\; \phi_{\nu_j}(x_j) = (2^{\nu_j}\nu_j!\sqrt{\pi})^{-\frac{1}{2}} H_{\nu_j}(x_j) e^{-\frac{1}{2}x_j^2} \qquad (1)$$

where $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, $\nu = (\nu_1, \cdots, \nu_n) \in \mathbb{N}_0^n$, and $H_{\nu_j}(x_j)$ denotes the Hermite polynomial of order ν_j. It is well known that the Hermite functions provide a complete and orthonormal system in $L^2(\mathbb{R}^n)$. If we consider the operator $L = -\Delta + |x|^2$ acting on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, where Δ is the standard Laplace operator on $\mathbb{R}^n$, then we have the relation $L\phi_\nu = \lambda_\nu \phi_\nu$, $\nu \in \mathbb{N}_0^n$. The operator L is symmetric and positive in $L^2(\mathbb{R}^n)$ and admits a self-adjoint extension H whose

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domain is given by

$$\mathrm{Dom}(H)=\left\{\sum_{\nu\in\mathbb{N}_0^n}\langle f,\phi_\nu\rangle_{L^2}\phi_\nu:\sum_{\nu\in\mathbb{N}_0^n}|\lambda_\nu\langle f,\phi_\nu\rangle_{L^2}|^2<\infty\right\}. \tag{2}$$

So, for $f\in\mathrm{Dom}(H)$, we have

$$(Hf)(x)=\sum_{\nu\in\mathbb{N}_0}\lambda_\nu\widehat{f}(\phi_\nu)\phi_\nu(x),\quad \widehat{f}(\phi_\nu)=\langle f,\phi_\nu\rangle_{L^2}. \tag{3}$$

The operator H is precisely the quantum harmonic oscillator on $\mathbb{R}^n$ (see [24]). The sequence $\{\widehat{f}(\phi_v)\}$ determines the Fourier-Hermite transform of f, with corresponding inversion formula

$$f(x)=\sum_{\nu\in\mathbb{N}_0^n}\widehat{f}(\phi_v)\phi_\nu(x). \tag{4}$$

On the other hand, pseudo-multipliers are defined by the quantization process that associates to a function m on $\mathbb{R}^n\times\mathbb{N}_0^n$ a linear operator T_m of the form:

$$T_mf(x)=\sum_{\nu\in\mathbb{N}_0^n}m(x,\nu)\widehat{f}(\phi_\nu)\phi_\nu(x),\quad f\in\mathrm{Dom}(T_m). \tag{5}$$

The function m on $\mathbb{R}^n\times\mathbb{N}_0^n$ is called the symbol of the pseudo-multiplier T_m. If in (5), $m(x,\nu)=m(\nu)$ for all x, the operator T_m is called a multiplier. Multipliers and pseudo-multipliers have been studied, for example, in the works [1, 30, 31, 32, 33, 34] (and references therein) principally by its mapping properties on L^p spaces. In order that the operator $T_m:L^{p_1}(\mathbb{R}^n)\to L^{p_2}(\mathbb{R}^n)$ extends to a r-nuclear operator, in this paper we provide necessary and sufficient conditions on the symbol m.

1.2. Nuclearity of pseudo-multipliers

We recall the notion of r-nuclearity as follows. By following A. Grothendieck [22], we can recall that a linear operator $T:E\to F$ (E and F Banach spaces) is r-nuclear, if there exist sequences $(e_n')_{n\in\mathbb{N}_0}$ in E' (the dual space of E) and $(y_n)_{n\in\mathbb{N}_0}$ in F such that

$$Tf=\sum_{n\in\mathbb{N}_0}e_n'(f)y_n,\quad\text{and}\quad\sum_{n\in\mathbb{N}_0}\|e_n'\|_{E'}^r\|y_n\|_F^r<\infty. \tag{6}$$

The class of $r-$nuclear operators is usually endowed with the quasi-norm

$$n_r(T):=\inf\left\{\left\{\sum_n\|e_n'\|_{E'}^r\|y_n\|_F^r\right\}^{\frac{1}{r}}:T=\sum_n e_n'\otimes y_n\right\} \tag{7}$$

and, if $r=1$, $n_1(\cdot)$ is a norm and we obtain the ideal of nuclear operators. In addition, when $E=F$ is a Hilbert space and $r=1$ the definition above agrees with the concept of trace class operators. For the case of Hilbert spaces H, the set of r-nuclear operators agrees with the Schatten-von Neumann class of order r (see Pietsch [25, 26]).

In order to study the r-nuclearity and the spectral trace of Hermite pseudo-multipliers, we will use results from J. Delgado [8], on the characterization of nuclear integral operators on $L^p(X,\mu)$ spaces, which in this case can be applied to L^p spaces on $\mathbb{R}^n$. Indeed, we will prove that under certain conditions, a r-nuclear operator $T_m : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ has a nuclear trace given by

$$\mathrm{Tr}(T_m) = \int_{\mathbb{R}^n} \sum_{\nu\in\mathbb{N}_0^n} m(x,\nu)\phi_\nu(x)^2 dx. \tag{8}$$

It was proved in [2] that a multiplier T_m with symbol satisfying one of the following conditions

- $1 \le p_2 < 4$, $\frac{4}{3} < p_1 < \infty$ and

$$\varkappa(m,p_1,p_2) := \sum_{s=0}^{n} \sum_{\nu\in I_s} k^{\frac{sr}{2}(\frac{1}{p_2}-\frac{1}{p_1})} (\prod_{\nu_j>k} \nu_j)^{\frac{r}{2}(\frac{1}{p_2}-\frac{1}{p_1})} |m(\nu)|^r < \infty, \tag{9}$$

- $1 \le p_2 < 4$, $p_1 = \frac{4}{3}$ and

$$\varkappa(m,p_1,p_2) := \sum_{s=0}^{n} \sum_{\nu\in I_s} k^{\frac{sr}{2}(\frac{1}{p_2}-\frac{3}{4})} (\ln k)^{sr} \cdot \prod_{\nu_j>k} [\nu_j^{\frac{r}{2}(\frac{1}{p_2}-\frac{3}{4})} (\ln(\nu_j))^r] |m(\nu)|^r < \infty, \tag{10}$$

- $1 \le p_2 < 4$, $1 < p_1 < \frac{4}{3}$ and

$$\varkappa(m,p_1,p_2) := \sum_{s=0}^{n} \sum_{\nu\in I_s} k^{\frac{sr}{2}(\frac{1}{p_2}+\frac{1}{3p_1}-1)} \cdot (\prod_{\nu_j>k} \nu_j)^{\frac{r}{2}(\frac{1}{p_2}+\frac{1}{3p_1}-1)} |m(\nu)|^r < \infty, \tag{11}$$

- $p_2 = 4$, $\frac{4}{3} < p_1 < \infty$ and

$$\varkappa(m,p_1,p_2) := \sum_{s=0}^{n} \sum_{\nu\in I_s} k^{\frac{sr}{2}(\frac{1}{4}-\frac{1}{p_1})} (\ln(k))^{sr} \prod_{\nu_j>k} [(\ln(\nu_j))^r \nu_j^{\frac{r}{2}(\frac{1}{4}-\frac{1}{p_1})}] |m(\nu)|^r < \infty, \tag{12}$$

- $p_2 = 4$, $p_1 = \frac{4}{3}$ and

$$\varkappa(m,p_1,p_2) := \sum_{s=0}^{n} \sum_{\nu\in I_s} k^{-\frac{sr}{4}} (\ln k)^{2sr} \prod_{\nu_j>k} [\nu_j^{-\frac{r}{4}} (\ln \nu_j)^{2r}] \cdot |m(\nu)|^r < \infty, \tag{13}$$

- $p_2 = 4$, $1 < p_1 < \frac{4}{3}$ and

$$\varkappa(m,p_1,p_2) := \sum_{s=0}^{n} \sum_{\nu\in I_s} k^{\frac{sr}{6}(\frac{1}{p_1}-\frac{9}{4})} (\ln(k))^{sr} \prod_{\nu_j>k} [\nu_j^{\frac{r}{6}(\frac{1}{p_1}-\frac{9}{4})} \ln(\nu_j)^r] \cdot |m(\nu)|^r < \infty, \tag{14}$$

- $4 < p_2 \le \infty$, $\frac{4}{3} < p_1 < \infty$ and

$$\varkappa(m,p_1,p_2) := \sum_{s=0}^{n} \sum_{\nu\in I_s} k^{\frac{sr}{2}(\frac{1}{3p_2'}-\frac{1}{p_1})} (\prod_{\nu_j>k} \nu_j)^{\frac{r}{2}(\frac{1}{3p_2'}-\frac{1}{p_1})} |m(\nu)|^r < \infty, \tag{15}$$

- $4 < p_2 \leq \infty$, $p_1 = \frac{4}{3}$ and

$$\varkappa(m,p_1,p_2) := \sum_{s=0}^{n}\sum_{\nu\in I_s} k^{-\frac{sr}{6}(\frac{1}{p_2}+\frac{5}{4})}(\ln(k))^{sr}\prod_{\nu_j>k}[\nu_j^{-\frac{r}{6}(\frac{1}{p_2}+\frac{5}{4})}(\ln(\nu_j))^r]|m(\nu)|^r < \infty, \tag{16}$$

- $4 < p_2 \leq \infty$, $1 < p_1 < \frac{4}{3}$ and

$$\varkappa(m,p_1,p_2) := \sum_{s=0}^{n}\sum_{\nu\in I_s} k^{\frac{sr}{6}(\frac{1}{p_1}-\frac{1}{p_2}-2)}\cdot(\prod_{\nu_j>k}\nu_j)^{\frac{r}{6}(\frac{1}{p_1}-\frac{1}{p_2}-2)}|m(\nu)|^r < \infty, \tag{17}$$

where $\{I_s\}_{s=0}^n$ is a suitable partition of $\mathbb{N}_0^n$, can be extended to a r-nuclear operator from $L^{p_1}(\mathbb{R}^n)$ into $L^{p_2}(\mathbb{R}^n)$. Although is easy to see that similar necessary conditions apply for pseudo-multipliers, and that such conditions can be useful for applications because they can verified, for example, numerically for m given, in this paper we want to characterize the r-nuclearity of pseudo-multipliers by using abstract conditions depending on the existence of certain measurable functions. In fact, the main result of this paper is the following.

Theorem 1.1. *$T_m : L^{p_1}(\mathbb{R}^n) \to L^{p_2}(\mathbb{R}^n)$ is r-nuclear, if and only if, for every $\nu \in \mathbb{N}_0^n$, the function $m(\cdot,\nu)\phi_\nu$ admits a decomposition of the form*

$$m(x,\nu) = \phi_\nu(x)^{-1}\sum_{k=1}^{\infty} h_k(x)\widehat{g}(\phi_\nu),\ \ x\in\{\varkappa\in\mathbb{R}^n : \phi_\nu(\varkappa)\neq 0\}, \tag{18}$$

where $\{g_k\}_{k\in\mathbb{N}}$ and $\{h_k\}_{k\in\mathbb{N}}$ are sequences of functions satisfying

$$\sum_{k=0}^{\infty}\|g_k\|^r_{L^{p_1'}}\|h_k\|^r_{L^{p_2}} < \infty. \tag{19}$$

Some remarks about our main theorem are the following.

- A consequence of the above theorem is that symbols associated to nuclear multipliers admit decompositions of the form

$$m(\nu) = \sum_{k=0}^{\infty}\widehat{h}_k(\phi_\nu)\widehat{g}_k(\phi_\nu). \tag{20}$$

  This can be obtained multiplying both sides of (18) by ϕ_ν and later integrating both sides over $\mathbb{R}^n$.
- Our approach is an adaptation to the non-compact case of $\mathbb{R}^n$ of techniques used in the work [19] by M. B. Ghaemi, M. Jamalpour Birgani, and M. W. Wong.
- For every ν, the function ϕ_ν has only finitely many zeros. So, the set $M = \{x : \phi_\nu(x) = 0 \text{ for some } \nu\}$ is a countable subset of $\mathbb{R}^n$. According to (18), outside of the set M we have

$$m(x,\nu) = \phi_\nu(x)^{-1}\sum_{k=1}^{\infty} h_k(x)\widehat{g}(\phi_\nu). \tag{21}$$

So, our main result can be formulated as follows: a pseudo-multiplier T_m can be extended to a r-nuclear operator from L^{p_1} into L^{p_2} if and only if (21) holds true almost everywhere where the functions h_k and g_k satisfy the condition (19).

- Let us recall that $L^2(\mathbb{R}^n)$ is a Hilbert space and consequently, the ideal of r-nuclear operators on $L^2(\mathbb{R}^n)$ coincides with the class of Schatten-von Neumann of order r, $S_r(L^2)$ for all $0 < r \leq 1$. Although our main theorem classify those pseudo-multipliers in the ideal $S_r(L^2)$, explicit conditions in order that the operators $T_m \in S_r(L^2)$ have been proved in Cardona [3].

1.3. Related works

Now, we include some references on the subject. Sufficient conditions for the r-nuclearity of spectral multipliers associated to the harmonic oscillator, but, in modulation spaces and Wiener amalgam spaces have been considered by J. Delgado, M. Ruzhansky and B. Wang in [10, 11]. The Properties of these multipliers in L^p-spaces have been investigated in the references S. Bagchi, S. Thangavelu [1], J. Epperson [18], K. Stempak and J.L. Torrea [30, 31, 32], S. Thangavelu [33, 34] and references therein. Hermite expansions for distributions can be found in B. Simon [29]. The r-nuclearity and Grothendieck-Lidskii formulae for multipliers and other types of integral operators can be found in [9, 11]. Sufficient conditions for the nuclearity of pseudo-differential operators on the torus can be found in [7, 19]. The references [12, 13, 14, 15] and [17] include a complete study on the r-nuclearity, $0 < r \leq 1$, of multipliers (and pseudo-differential operators) on compact Lie groups, and more generally on compact manifolds, with explicit conditions on symbols of operators providing an useful tool for applications (see [5]). For compact and Hausdorff groups, the work [20] by M. B. Ghaemi, M. Jamalpour Birgani, and M. W. Wong characterize in terms of the existence of certain measurable functions the nuclearity of pseudo-differential operators. On Hilbert spaces the class of r-nuclear operators agrees with the Schatten-von Neumann class $S_r(H)$; in this context operators with integral kernel on Lebesgue spaces and, in particular, operators with kernel acting of a special way with anharmonic oscillators of the form $E_a = -\Delta_x + |x|^a$, $a > 0$, have been considered on Schatten classes on $L^2(\mathbb{R}^n)$ in J. Delgado and M. Ruzhansky [16].

The proof of our results will be presented in the next section.

2. Nuclear pseudo-multipliers associated to the Harmonic oscillator

2.1. Characterization of nuclear pseudo-multipliers

In this section we prove our main result for pseudo-multipliers T_m. Our criteria will be formulated in terms of the symbols m. First, let us observe that every multiplier T_m is an operator with kernel $K_m(x, y)$. In fact, straightforward computation show that

$$T_m f(x) = \int_{\mathbb{R}^n} K_m(x,y) f(y)dy, \; K_m(x,y) := \sum_{\nu \in \mathbb{N}_0^n} m(x,\nu)\phi_\nu(x)\phi_\nu(y) \tag{22}$$

for every $f \in \mathcal{D}(\mathbb{R}^n)$. In order to analyze the r-nuclearity of T_m we study its kernel K_m by using the following theorem (see J. Delgado [6, 8]).

Theorem 2.1. *Let us consider* $1 \leq p_1, p_2 < \infty$, $0 < r \leq 1$ *and let* p_i' *be such that* $\frac{1}{p_i} + \frac{1}{p_i'} = 1$. *Let* (X_1, μ_1) *and* (X_2, μ_2) *be* σ*-finite measure spaces. An operator* $T : L^{p_1}(X_1, \mu_1) \to L^{p_2}(X_2, \mu_2)$ *is* r*-nuclear if and only if there exist sequences* $(h_k)_k$ *in* $L^{p_2}(\mu_2)$, *and* (g_k) *in* $L^{p_1'}(\mu_1)$, *such that*

$$\sum_k \|h_k\|^r_{L^{p_2}} \|g_k\|^r_{L^{p_1'}} < \infty, \text{ and } Tf(x) = \int_{X_1} (\sum_k h_k(x) g_k(y)) f(y) d\mu_1(y), \text{ a.e.w. } x, \tag{23}$$

for every $f \in L^{p_1}(\mu_1)$. *In this case, if* $p_1 = p_2$, *and* $\mu_1 = \mu_2$, (see Section 3 of [6]) *the nuclear trace of* T *is given by*

$$\mathrm{Tr}(T) := \int_{X_1} \sum_k g_k(x) h_k(x) d\mu_1(x). \tag{24}$$

Now, we prove our main theorem.

Theorem 2.2. *Let* $0 < r \leq 1$. *The operator* $T_m : L^{p_1}(\mathbb{R}^n) \to L^{p_2}(\mathbb{R}^n)$ *extends to a* r*-nuclear operator, if and only if, for every* $\nu \in \mathbb{N}_0^n$, *the function* $m(\cdot, \nu)\phi_\nu$ *admits a decomposition of the form*

$$m(x,\nu) = \phi_\nu(x)^{-1} \sum_{k=1}^{\infty} h_k(x) \widehat{g}(\phi_\nu), \;\; x \in \{\varkappa \in \mathbb{R}^n : \phi_\nu(\varkappa) \neq 0\}, \tag{25}$$

where $\{g_k\}_{k\in\mathbb{N}}$ *and* $\{h_k\}_{k\in\mathbb{N}}$ *are sequences of functions satisfying*

$$\sum_{k=0}^{\infty} \|g_k\|^r_{L^{p_1'}} \|h_k\|^r_{L^{p_2}} < \infty. \tag{26}$$

Proof. Let us assume that $T_m : L^{p_1}(\mathbb{R}^n) \to L^{p_2}(\mathbb{R}^n)$ is a r-nuclear operator. Then there exist sequences h_k in L^{p_2} and g_k in $L^{p_1'}$ satisfying

$$Tf(x) = \int_{\mathbb{R}^n} \left(\sum_{k=1}^{\infty} h_k(x) g_k(y) \right) f(y) dy, \;\; f \in L^{p_1}, \tag{27}$$

with

$$\sum_{k=0}^{\infty} \|g_k\|^r_{L^{p_1'}} \|h_k\|^r_{L^{p_2}} < \infty. \tag{28}$$

Since every Hermite function ϕ_ν belongs to the Schwartz class which is contain in L^p-spaces, for $f = \phi_\nu \in L^{p_1}$ we have

$$T_m(\phi_\nu) = \int_{\mathbb{R}^n} \left(\sum_{k=1}^{\infty} h_k(x) g_k(y) \right) \phi_\nu(y) dy = \sum_{k=1}^{\infty} h_k(x) \widehat{g}_k(\nu).$$

Now, if we compute $T_m(\phi_\nu)$ from the definition of pseudo-multipliers (5), we obtain

$$T_m(\phi_\nu)(x) = m(x,\nu)\phi_\nu(x) \tag{29}$$

where we have used the L^2-orthogonality of Hermite functions. Consequently, we deduce the identity

$$m(x,\nu) = \phi_\nu(x)^{-1}\sum_{k=1}^{\infty} h_k(x)\widehat{g}(\phi_\nu), \;\; x \in \{\varkappa \in \mathbb{R}^n : \phi_\nu(\varkappa) \neq 0\}. \tag{30}$$

So, we have proved the first part of the theorem. Now, if we assume that the symbol m of a multiplier T_m and every Hermite function ϕ_ν satisfies the decomposition formula 30 for fixed sequences h_k in L^{p_2} and g_k in $L^{p_1'}$ satisfying (28), then from (5) we can write

$$\begin{aligned}
T_m f(x) &= \sum_{\nu\in\mathbb{N}_0^n} m(x,\nu)\phi_\nu(x)\widehat{f}(\phi_\nu) = \sum_{\nu\in\mathbb{N}_0^n}\sum_{k=1}^{\infty} h_k(x)\widehat{g}_k(\nu)\widehat{f}(\phi_\nu) \\
&= \sum_{\nu\in\mathbb{N}_0^n}\sum_{k=1}^{\infty} h_k(x)\int_{\mathbb{R}^n} g_k(y)\phi_\nu(y)dy\widehat{f}(\phi_\nu) \\
&= \int_{\mathbb{R}^n}\left(\sum_{k=1}^{\infty} h_k(x)g_k(y)\right)\left(\sum_{\nu\in\mathbb{N}_0^n}\widehat{f}(\phi_\nu)\phi_\nu(y)\right)dy \\
&= \int_{\mathbb{R}^n}\left(\sum_{k=1}^{\infty} h_k(x)g_k(y)\right)f(y)dy,
\end{aligned}$$

where in the last line we have used the inversion formula (4). So, by Delgado Theorem (Theorem 2.1) we end the proof. □

2.2. Traces of nuclear pseudo-multipliers of the harmonic oscillator

If $T : E \to E$ is r-nuclear, with the Banach space E satisfying the Grothendieck approximation property (see Grothendieck[22]), then there exist sequences $(e_n')_{n\in\mathbb{N}_0}$ in E' (the dual space of E) and $(y_n)_{n\in\mathbb{N}_0}$ in E such that

$$Tf = \sum_{n\in\mathbb{N}_0} e_n'(f)y_n, \quad \text{and} \quad \sum_{n\in\mathbb{N}_0}\|e_n'\|_{E'}^r\|y_n\|_F^r < \infty. \tag{31}$$

In this case the nuclear trace of T is given by $\mathrm{Tr}(T) = \sum_{n\in\mathbb{N}_0^n} e_n'(f_n)$. L^p-spaces have the Grothendieck approximation property and as consequence we can compute the nuclear trace of every r-nuclear pseudo-multipliers. For to do so, let us consider a $r-$nuclear pseudo-multiplier $T_m : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$. Since the function, $\varkappa(x,y) := \sum_k h_k(x)g_k(y)$ is defined a.e.w., let us choose $z \in \mathbb{R}^n$ such that $\varkappa(x,z)$ is finite $a.e.w.$ Let us consider $B(z,r)$, the ball centered at z with radius $r > 0$. Let us denote by $|B(z,r)|$ the Lebesgue measure of $B(z,r)$. If $f = |B(z,r)|^{-1}\cdot 1_{B(z,r)}$, where $1_{B(z,r)}$

is the characteristic function of $B(z,r)$, we obtain

$$T_m(|B(z,r)|^{-1}\cdot 1_{B(z,r)}) = \frac{1}{|B(z,r)|}\int_{B(z,r)}\left(\sum_{k=1}^{\infty} h_k(x)g_k(y)\right)dy \tag{32}$$

but, we also have

$$T_m(|B(z,r)|^{-1}\cdot 1_{B(z,r)}) = \frac{1}{|B(z,r)|}\int_{B(z,r)} K_m(x,y)dy, \tag{33}$$

where K_m is defined as in (22). So, we have the identity

$$\frac{1}{|B(z,r)|}\int_{B(z,r)} K_m(x,y)dy = \frac{1}{|B(z,r)|}\int_{B(z,r)}\left(\sum_{k=1}^{\infty} h_k(x)g_k(y)\right)dy \tag{34}$$

for every $r>0$. Taking limit as $r\to 0^+$ and by applying Lebesgue differentiation Theorem, we obtain

$$K_m(x,z) = \sum_{k=1}^{\infty} h_k(x)g_k(z), \ \text{a.e.w.} \tag{35}$$

Finally, the nuclear trace of T_m can be computed from (24). So, we have

$$\mathrm{Tr}(T_m) = \int_{\mathbb{R}^n} K_m(x,x)dx = \int_{\mathbb{R}^n}\sum_{\nu\in\mathbb{N}_0^n} m(x,\nu)\phi_\nu(x)^2dx. \tag{36}$$

Now, in order to determinate a relation with the eigenvalues of T_m we recall that, the nuclear trace of an r-nuclear operator on a Banach space coincides with the spectral trace, provided that $0<r\leq\frac{2}{3}$. For $\frac{2}{3}\leq r\leq 1$ we recall the following result (see [27]).

Theorem 2.3. *Let $T: L^p(X,\mu)\to L^p(X,\mu)$ be a r-nuclear operator as in* (31)*. If $\frac{1}{r}=1+|\frac{1}{p}-\frac{1}{2}|$, then,*

$$\mathrm{Tr}(T) := \sum_{n\in\mathbb{N}_0^n} e_n'(f_n) = \sum_n \lambda_n(T), \tag{37}$$

where $\lambda_n(T)$, $n\in\mathbb{N}$ is the sequence of eigenvalues of T with multiplicities taken into account.

As an immediate consequence of the preceding theorem, if $T_m: L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)$ is a r-nuclear pseudo-multiplier and $\frac{1}{r}=1+|\frac{1}{p}-\frac{1}{2}|$ then,

$$\mathrm{Tr}(T_m) = \int_{\mathbb{R}^n}\sum_{\nu\in\mathbb{N}_0^n} m(x,\nu)\phi_\nu(x)^2dx = \sum_n \lambda_n(T), \tag{38}$$

where $\lambda_n(T)$, $n\in\mathbb{N}$ is the sequence of eigenvalues of T_m with multiplicities taken into account.

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REFERENCES

[1] *S. Bagchi and S. Thangavelu*, On Hermite pseudo-multipliers. J. Funct. Anal. **268**(2015), No. 1, 140–170.

[2] *E. S. Barraza and D. Cardona*, On nuclear L^p-multipliers associated to the Harmonic oscillator, in: Analysis in Developing Countries, Springer Proceedings in Mathematics & Statistics, Springer, 2018, M. Ruzhansky and J. Delgado (Eds), to appear.

[3] *D. Cardona*, A brief description of operators associated to the quantum harmonic oscillator on Schatten-von Neumann classes, Rev. Integr. Temas Mat. 36(2018), No. 1, 49-57.

[4] *D. Cardona*, Nuclear pseudo-differential operators in Besov spaces on compact Lie groups. J. Fourier Anal. Appl. 23(2017), No. 5, 1238-1262.

[5] *D. Cardona and C. Del Corral*, The Dixmier trace and the non-commutative residue for multipliers on compact manifolds, submitted. arXiv:1703.07453

[6] *J. Delgado*, A trace formula for nuclear operators on L^p, in: Schulze, B.W., Wong, M.W. (eds.) Pseudo-Differential Operators: Complex Analysis and Partial Differential Equations, Operator Theory: Advances and Applications, 205, 181–193. Birkhäuser, Basel (2010)

[7] *J. Delgado and M. W. Wong*, L^p-nuclear pseudo-differential operators on $\mathbb{Z}$ and $\mathbb{S}^1$. Proc. Amer. Math. Soc., 141(2013), No. 11, 3935–394.

[8] *J. Delgado*, The trace of nuclear operators on $L^p(\mu)$ for σ-finite Borel measures on second countable spaces. Integral Equations Operator Theory, 68(2010), No. 1, 61–74.

[9] *J. Delgado*, On the r-nuclearity of some integral operators on Lebesgue spaces. Tohoku Math. J. 67(2015), No. 2 125–135.

[10] *J. Delgado, M. Ruzhansky and B. Wang*, Approximation property and nuclearity on mixed-norm L^p, modulation and Wiener amalgam spaces. J. Lond. Math. Soc. 94(2016), 391–408,

[11] *J. Delgado, M. Ruzhansky, and B. Wang*, Grothendieck-Lidskii trace formula for mixed-norm L^p and variable Lebesgue spaces. to appear in J. Spectr. Theory.

[12] *J. Delgado and M. Ruzhansky*, L^p-nuclearity, traces, and Grothendieck-Lidskii formula on compact Lie groups., J. Math. Pures Appl. **(102)**(2014), No. 1, 153-172.

[13] *J. Delgado and M. Ruzhansky*, Schatten classes on compact manifolds: Kernel conditions. J. Funct. Anal., 267(2014) No. 3, 772–798.

[14] *J. Delgado and M. Ruzhansky*, Kernel and symbol criteria for Schatten classes and r-nuclearity on compact manifolds., C. R. Acad. Sci. Paris. Ser. I. 352(2014), No. 10, 779–784.

[15] *J. Delgado and M. Ruzhansky*, Fourier multipliers, symbols and nuclearity on compact manifolds.arXiv:1404.6479

[16] *J. Delgado and M. Ruzhansky*, Schatten-von Neumann classes of integral operators. arXiv:1709.06446.

[17] *J. Delgado, M. Ruzhansky, and N. Tokmagambetov*, Schatten classes, nuclearity and nonharmonic analysis on compact manifolds with boundary. arXiv:1505.02261

[18] *J. Epperson*, Hermite multipliers and pseudo-multipliers, Proc. Amer. Math. Soc. 124(1996), No. 7, 2061–2068.

[19] *M. B. Ghaemi, M. JamalpourBirgani, and M. W. Wong*, Characterizations of nuclear pseudo-differential operators on $\mathbb{S}^1$ with applications to adjoints and products. J. Pseudo-Differ. Oper. Appl. 8(2017), No. 2, 191–201.

[20] *M. B. Ghaemi, M. JamalpourBirgani, and M. W. Wong*, Characterization, adjoints and products of nuclear pseudo-differential operators on compact and Hausdorff groups. U.P.B. Sci. Bull., Series A, Vol. 79(2017), No. 4, 207–220.

[21] *A. Grothendieck*, La theorie de Fredholm, Bull. Soc. Math. France 84(1956), 319–384.

[22] *A. Grothendieck*, Produits tensoriels topologiques et espaces nucléaires, Memoirs Amer. Math. Soc. 16, Providence, 1955 (Thesis, Nancy, 1953).

[23] *H. Koch and D. Tataru*, L^p-eigenfunction bounds for the Hermite operator. Duke Math. J. 128(2005), 369–392.

[24] *E. Prugovečki*, Quantum mechanics in Hilbert space. Second edition. Pure and Applied Mathematics, 92. Academic Press, Inc, New York-London, 1981.

[25] *A. Pietsch*, Operator ideals. Mathematische Monographien, 16. VEB Deutscher Verlag der Wissenschaften, Berlin, 1978.

[26] *A. Pietsch*, History of Banach spaces and linear operators. Birkhäuser Boston, Inc., Boston, MA, 2007.

[27] *O.I. Reinov and Q. Latif*, Grothendieck-Lidskii theorem for subspaces of Lp-spaces. Math. Nachr. 286(2013), No. 2-3, 279–282.

[28] *F. Nicola and L. Rodino*, Global pseudo-differential calculus on Euclidean spaces. Pseudo-Differential Operators. Theory and Applications, 4. Birkhäuser Verlag, Basel, 2010.

[29] *B. Simon*, Distributions and their Hermite expansions. J. Math. Phys. 12(1971), 140–148.

[30] *K. Stempak*, Multipliers for eigenfunction expansions of some Schrödinger operators, Proc. Amer. Math. Soc(1985), 93, 477–482.

[31] *K. Stempak and J. L. Torre*, On g-functions for Hermite function expansions, Acta Math. Hung. 109(2005), 99–125.

[32] *K. Stempak ad J. L. Torre*, BMO results for operators associated to Hermite expansions, Illinois J. Math. 49, (2005), 1111–1132,

[33] *S. Thangavelu*, Lectures on Hermite and Laguerre Expansions, Math. Notes, vol. 42, Princeton University Press, Princeton, 1993.

[34] *S. Thangavelu*, Hermite and special Hermite expansions revisited Duke Mathematical Journal, 94(1998), No. 2, 257–278.