

## UNITARY DUAL OF THE STANDARD SHEARLET GROUP, IN ARBITRARY SPACE DIMENSIONS

Masoumeh Zare<sup>1</sup>, Rajab Ali Kamyabi-Gol<sup>2</sup>

*This paper is devoted to definition standard higher dimension shearlet group  $\mathbb{S} = \mathbb{R}^+ \times \mathbb{R}^{n-1} \times \mathbb{R}^n$  in arbitrary space dimensions and concerned with the application of the Mackey Machine in order to determine the unitary dual of  $\mathbb{S}$ , using the theory of induced unitary representations for locally compact groups, via the action of  $\mathbb{R}^+ \times \mathbb{R}^{n-1}$  on  $\widehat{\mathbb{R}^n}$ . Also we give a characterisation of irreducible sub-representations of the quasi-regular representation of  $\mathbb{S}$ .*

**Keywords:** Standard shearlet group, unitary dual, irreducible representation, Mackey theory

**MSC2010:** 22D10, 22D30.

### 1. Introduction

Locally compact groups arise in many diverse areas of mathematics, the physical sciences, and engineering. The presence of the group is usually felt through unitary representations of the group. This observation underlines the importance of understanding such representations and how they may be constructed, combined, or decomposed. Of particular importance are the irreducible unitary representations. Irreducible unitary representations of a locally compact group are the basic building blocks of the harmonic analysis associated with a locally compact groups. In the middle of the last century, G. W. Mackey initiated a program to develop a systematic method for identifying all the irreducible unitary representations of a given locally compact group  $G$  in the series of papers [10, 11, 12, 13]. The set of all unitarily equivalence class of irreducible unitary representations of  $G$  is denoted by  $\widehat{G}$ . The program Mackey initiated, received contributions from many researchers with some of the most substantial advances made by Blattner [2] and Fell [5]. Fell's work is particularly important in studying  $\widehat{G}$  as a topological space. At the core of Mackey's analysis is the inducing construction, which is a method of building a unitary representation of a group from a representation of a subgroup. It is worthwhile to know that the induced representations for finite groups were introduced in 1898 by Frobenius [7], where the idea is by no means limited to the case of finite groups, but the theory in that case is particularly well-behaved. For general locally compact groups, the notion of induced unitary representations was to a large extent developed by Mackey in the 1950s [11, 12, 13]. Mackey confined himself to second countable groups  $G$  and separable Hilbert spaces  $\mathcal{H}$  and the Imprimitivity Theorem is the foundation of his method. The objects appearing in this parametrization are easiest to deal with when  $G$  splits as a semi-direct product of the Abelian subgroup  $N$  and another locally compact group  $H$  and carefully study the orbit space formed by  $G = N \times H$  acting on  $\widehat{N}$ . In this paper we describe a Mackey procedure for constructing the unitary dual of the standard shearlet group in arbitrary space dimensions. The semi-direct product structure

<sup>1</sup> Department of Pure Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran, e-mail: zare.masume@gmail.com

<sup>2</sup> Department of Pure Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran

of the shearlet group in arbitrary space dimensions, is the useful feature to determine the unitary dual (the class of irreducible representations) of this group. Note that we are interested in investigating the standard shearlet group  $\mathbb{S}$  in arbitrary space dimensions, which is introduced in two dimension space by Kutyniok et .al in [8], and the class of the unitary dual of  $\mathbb{S}$ , because of the importance of the irreducible sub-representations of the quasi-regular representation of  $\mathbb{S}$  and specially the shearlet transform.

More precisely this article is organized as follows. We collect the necessary background and tools containing inducing a representation from a subgroup to the full group, the class of semi-direct product of locally compact group, and also the unitary dual of semi-direct product group, in section 2. Section 3, is devoted to determine the standard shearlet group and its semi-direct product structure in arbitrary space dimensions. Finally, in section 4, we describe the unitary dual of the shearlet group in arbitrary space dimensions, using the Mackey theory for the semi-direct product groups.

## 2. Preliminaries and notation

The single most important method for producing representations of a locally compact group is the procedure of inducing representations from subgroups. If  $H$  is a closed subgroup of a locally compact group  $G$  and  $\pi$  is a unitary representation of  $H$  on the Hilbert space  $\mathcal{H}_\pi$ , then  $\text{ind}_H^G \pi$  is a unitary representation of  $G$  that is constructed by combining the action of  $\pi$  with the algebraic and measure-theoretic inter-relation of  $G$ ,  $H$  and  $G/H$ , where  $G/H$  admits a quasi-invariant measure  $\mu$ . The Hilbert space on which  $\text{ind}_H^G \pi$  acts, can be precisely identified as the space  $\mathcal{F}$ , of all  $\mathcal{H}_\pi$ -valued functions on  $G$  that are measurable and square-integrable and that satisfy the appropriate covariance equation with respect to  $H$  i.e.,

$$\mathcal{F} = \{f : G \rightarrow \mathcal{H}_\pi; f(gh) = \pi(h^{-1})f(g), \text{ for } g \in G, h \in H \text{ and } \int_{G/H} |f(g)|^2 d\mu(gH) < \infty\}.$$

To summarize, the Hilbert space associated with the representation  $\text{ind}_H^G \pi$  is the completion of  $\mathcal{F}$  and

$$(\text{ind}_H^G \pi(x))f(g) = f(x^{-1}g),$$

for  $x, g \in G$  and  $f \in \mathcal{F}$ . General theoretical results on induced representation can be found in [6, 10].

We continue this section with a review of the basic definitions and notations of the case that  $G$  is a semi-direct product group. We shall use the following conventions of semi-direct product groups throughout the paper.

For two locally compact groups  $H$  and  $N$ , let  $h \mapsto \tau_h$  be a homomorphism of  $H$  into the group of automorphisms of  $N$  denoted by  $\text{Aut}(N)$ . Also assume that the mapping  $(h, n) \mapsto \tau_h n$ , from  $H \times N$  (endowed with the product topology) onto  $N$  is continuous. Then the set  $H \times N$  with the operations:

$$(h, n)(h', n') := (hh', n\tau_h(n')),$$

and

$$(h, n)^{-1} = (h^{-1}, \tau_{h^{-1}}(n^{-1})),$$

is a locally compact group. This group is denoted by  $H \rtimes_\tau N$  and called the semi-direct product of  $H$  and  $N$ . The left Haar measure of  $G = H \rtimes_\tau N$  is  $d\mu_G(h, n) = \delta(h)d\mu_H(h)d\mu_N(n)$ , where  $d\mu_H$  and  $d\mu_N$  are the left Haar measures on  $H$  and  $N$ , respectively and  $\delta$  is a positive continuous homomorphism on  $H$  which is given by

$$d\mu_N(n) = \delta(h)d\mu_N(\tau_h(n)).$$

Also the right Haar measure on  $G = H \rtimes_\tau N$  is

$$d\nu_G(h, n) = d\nu_H(h)d\nu_N(n),$$

where  $d\nu_H$  and  $d\nu_N$  are the right Haar measure on  $H$  and  $N$ , respectively. For more details on semi-direct product groups one can see [1, 9]. Without further comment, we identify  $N$  and  $H$  with the obvious closed subgroups of  $G$ . Then  $N$  is a normal subgroup of  $G$ . Let us return to our situation in which  $N$  is Abelian. For each  $\chi \in \widehat{N}$ , denoted by  $G_\chi$ , the stabilizer group associated with  $\chi$  and defined by  $G_\chi = \{g \in G, g \cdot \chi = \chi\}$ , which is closed subgroup of  $G$ . In this case, it is reasonable to formulate all of the  $G$ -orbit information in terms of  $H$ . Consider the stabilizer subgroups  $H_\chi = G_\chi \cap H$ , for each  $\chi \in \widehat{N}$ . Since  $N$  is Abelian subgroup of  $G$ , so  $N \subset G_\chi$  and then  $G_\chi = H_\chi \rtimes_\tau N$  and  $O_\chi = \{h \cdot \chi; h \in H\}$  also  $H_\chi = G_\chi/N$ . Moreover, for any  $\chi \in \widehat{N}$ ,  $G/G_\chi$  can be identified with  $H/H_\chi$  via the mapping  $(n, h)G_\chi \rightarrow hH_\chi$  for  $h \in H$  and  $n \in N$ . Note that if  $G$  is second countable, then  $G$  acts regularly on  $\widehat{N}$ , if there is a Borel set  $X$  in  $\widehat{N}$ , such that intersects each orbit in exactly one point (this assertion was proved by Mackey [10]). Actually the existence of the Borel set implies that the orbit space is countably separated and structure of the orbits is that of homogeneous space. The Borel set  $X$  with this property is called a cross-section of the  $G$ -orbits in  $\widehat{N}$ .

By using the following proposition, we introduce a procedure by which the unitary dual of a semi-direct product group with an appropriate Abelian subgroup, can be completely constructed. This procedure is often called Mackey theory or more informally, the Mackey Machine for Abelian subgroups [6].

**Proposition 2.1.** *Suppose  $G = H \rtimes_\tau N$ , where  $N$  is Abelian and  $G$  acts regularly on  $\widehat{N}$ . Also  $X$  be a cross-section of the  $G$ -orbits in  $\widehat{N}$ , then*

$$\widehat{G} = \{\text{ind}_{G_\chi}^G(\pi \times \chi); \pi \in \widehat{H_\chi}, \chi \in X\}.$$

In this proposition,  $(\pi \times \chi)$  is a representation of  $G_\chi = H_\chi \rtimes N$  into the unitary operators on  $\mathcal{H}_\pi$ , defined by  $(\pi \times \chi)(h, n) = \pi(h)\chi(n)$  for  $h \in H_\chi$  and  $n \in N$ , also  $\pi$  is an irreducible representation of  $H_\chi$  and  $\chi \in \widehat{N}$ . This shows that how one can classify the irreducible representation of a semi-direct product  $G = H \rtimes_\tau N$  in terms of the irreducible representation of  $N$  (i.e., the characters  $\chi \in \widehat{N}$ ) and the irreducible representation of the stabilizer subgroup  $H_\chi$ .

Note that we are far from knowing the complete unitary dual of  $\widehat{G}$ , more precisely  $\text{ind}_{G_\chi}^G(\pi \times \chi)$  and  $\text{ind}_{G_\chi}^G(\pi \times \chi')$  are equivalent if and only if  $\chi$  and  $\chi'$  belong to the same orbit i.e., we can give a formula for a parametrized set of irreducible representations that are mutually inequivalent and such that any other irreducible representation is equivalent to one in this set. So the unitary dual of the groups are the explicit description of the class of irreducible unitary representations of the groups. The reader is directed to [6] for general overview of these preliminaries.

### 3. Standard shearlet group in arbitrary space dimension

In this section, we introduce the definition and basic properties of standard shearlet group in arbitrary space dimensions, which is introduced in full version by Dahlke et al. [3]. For analysing data in  $\mathbb{R}^n, n \geq 3$ , Dahlke et al. [3], generalized the two dimensional full shearlet transform which is introduced in [4], to the arbitrary space dimensions, with the same technique, as follows.

Let  $I_n$  denote the  $n \times n$  identity matrix, also  $0_n$  be the vector with  $n$  zero value entries. For  $a \in \mathbb{R}^* := \mathbb{R} \setminus \{0\}$  and  $s \in \mathbb{R}^{n-1}$

$$A_a = \begin{pmatrix} a & 0_{n-1} \\ 0_{n-1}^T & \text{sgn}(a)|a|^{\frac{1}{n}}I_{n-1} \end{pmatrix} \quad \text{and} \quad S_s = \begin{pmatrix} 1 & s \\ 0_{n-1}^T & I_{n-1} \end{pmatrix},$$

denote the parabolic scaling matrix and the shear matrix, respectively, where  $\text{sgn}(a)$  denotes the sign of  $a$ . The choice of  $S_s$  leads shearlet transform to be a square-integrable group

representation and in order to have directional selectivity, the dilation factors at the diagonal of  $A_a$  is chosen in an anisotropic way. The set  $\mathbb{S} = \mathbb{R}^* \times \mathbb{R}^{n-1} \times \mathbb{R}^n$  for  $n \geq 2$ , endowed with the operation

$$(a, s, t) \circ (a', s', t') = (aa', s + |a|^{1-\frac{1}{n}}s', t + S_s A_a t')$$

is a locally compact group, which is called full shearlet group. The left and right Haar measures on  $\mathbb{S}$  are given by

$$d\nu_l(a, s, t) = \frac{1}{|a|^{n+1}} dadsdt \quad \text{and} \quad d\nu_r(a, s, t) = \frac{1}{|a|} dadsdt,$$

respectively. For  $f \in L^2(\mathbb{R}^n)$  the map  $\varrho : \mathbb{S} \rightarrow \mathcal{U}(L^2(\mathbb{R}^n))$ , defined by

$$\varrho(a, s, t)f(x) = |a|^{\frac{1-2n}{2n}} f(A_a^{-1} S_s^{-1}(x - t)) \quad (1)$$

is a unitary representation of locally compact group  $\mathbb{S}$  on the Hilbert space  $L^2(\mathbb{R}^n)$ , with respect to the Haar measure  $d\nu_l$ .

In this paper, we introduce the standard shearlet group, again denoted by  $\mathbb{S}$ , as follows. For  $a \in \mathbb{R}^+$  and  $s \in \mathbb{R}^{n-1}$ , let

$$A_a = \begin{pmatrix} a & 0_{n-1} \\ 0_{n-1}^T & a^{\frac{1}{n}} I_{n-1} \end{pmatrix}, \quad S_s = \begin{pmatrix} 1 & s \\ 0_{n-1}^T & I_{n-1} \end{pmatrix},$$

denote the parabolic scaling matrix and the shear matrix, respectively. The standard shearlet group  $\mathbb{S}$  is then defined to be the set  $\mathbb{R}^+ \times \mathbb{R}^{n-1} \times \mathbb{R}^n$  endowed with the group operation

$$(a, s, t) \circ (a', s', t') = (aa', s + a^{1-\frac{1}{n}}s', t + S_s A_a t')$$

and

$$(a, s, t)^{-1} = (a^{-1}, -a^{\frac{1}{n}-1}s, -A_a^{-1} S_s^{-1}t).$$

It is easy to prove that  $\mathbb{S}$  is a locally compact group. Actually we can consider  $\mathbb{S}$ , as the 3-fold semi-direct product group  $(\mathbb{R}^+ \rtimes_{\tau} \mathbb{R}^{n-1}) \rtimes_{\lambda} \mathbb{R}^n$ , where the homomorphism  $\lambda : \mathbb{R}^+ \times \mathbb{R}^{n-1} \rightarrow \text{Aut}(\mathbb{R}^n)$  is defined by

$$\lambda_{S_s A_a}(t) = S_s A_a t,$$

and the homomorphism  $\tau : \mathbb{R}^+ \rightarrow \text{Aut}(\mathbb{R}^{n-1})$  is defined by

$$\tau_a(s) = a^{\frac{n-1}{n}} s.$$

In fact the group laws of  $\mathbb{R}^+ \rtimes_{\tau} \mathbb{R}^{n-1}$  are given by

$$(a, s)(a', s') = (aa', s + a^{1-\frac{1}{n}}s') \quad \text{and} \quad (a, s)^{-1} = (a^{-1}, -a^{\frac{1}{n}-1}s).$$

**Lemma 3.1.** *The left and right Haar measures of  $\mathbb{S}$  are given by*

$$d\mu_l(a, s, t) = \frac{1}{a^{n+1}} dadsdt,$$

and

$$d\mu_r(a, s, t) = \frac{1}{a} dadsdt,$$

respectively.

*Proof.* For  $a \in \mathbb{R}^+$ ,  $s \in \mathbb{R}^{n-1}$  and  $t \in \mathbb{R}^n$  we have

$$\begin{aligned} d\mu(t) &= \delta_{\lambda}(a, s) d\mu(\lambda_{(a,s)}(t)) \\ &= \delta_{\lambda}(a, s) a^{\frac{2n-1}{n}} d\mu(t). \end{aligned}$$

This implies  $\delta_{\lambda}(a, s) = a^{\frac{1-2n}{n}}$ . On the other hand, we obtain

$$\begin{aligned} d\mu(s) &= \delta_{\tau}(a) d\mu(\tau_a(s)) \\ &= \delta_{\tau}(a) a^{(1-\frac{1}{n})(n-1)} d\mu(s). \end{aligned}$$

Thus  $\delta_\tau(a) = a^{(1-\frac{1}{n})(1-n)}$  and the left Haar measure of  $\mathbb{R}^+ \rtimes_\tau \mathbb{R}^{n-1}$  is  $d\mu_l(a, s) = \frac{dads}{a^{\frac{n^2-n+1}{n}}}$ .

Therefore the left Haar measure of  $\mathbb{S}$  is  $d\mu_l(a, s, t) = \frac{1}{a^{n+1}}dadsdt$ . Similarly, one can show that  $d\mu_r(a, s, t) = \frac{1}{a}dadsdt$  is the right Haar measure of  $\mathbb{S}$ .  $\square$

The quasi-regular representation of  $\mathbb{S}$  on the Hilbert space  $L^2(\mathbb{R}^n)$  is defined by

$$\rho(a, s, t)f(x) = a^{\frac{1-2n}{2n}} f(A_a^{-1}S_s^{-1}(x-t)), \quad (2)$$

which is the natural combination of translation on  $\mathbb{R}^n$  with dilation by members of  $\mathbb{R}^+ \times \mathbb{R}^{n-1}$ , that is not irreducible representation. This is not the case for the Full shearlet group, in particular the representation (1) is irreducible [3]. In the sequel we will detect the irreducible sub-representations of the representation which is defined in (2).

#### 4. Unitary dual of the shearlet group

The main goal of this section is to describe the unitary dual of the standard shearlet group  $\mathbb{S}$  which is defined in Section 3. As we mentioned that the standard shearlet group  $\mathbb{S}$  is a 3-fold semi-direct product group, so the unitary dual of the group  $\mathbb{R}^+ \rtimes_\tau \mathbb{R}^{n-1}$  has an effective influence on the construction of the unitary dual of  $\mathbb{S}$ . Therefore we need to clarify the unitary dual of the group  $\mathbb{R}^+ \rtimes_\tau \mathbb{R}^{n-1}$ .

Let  $\mathbb{G} = \mathbb{R}^+ \rtimes_\tau \mathbb{R}^{n-1}$ . We identify the Abelian subgroup  $N = \{1\} \times \mathbb{R}^{n-1}$  of  $\mathbb{G}$  with  $\mathbb{R}^{n-1}$  and then  $\widehat{N}$  with  $\mathbb{R}^{n-1}$  via the map  $y \rightarrow \gamma_y$  defined by  $\gamma_y((t_1, \dots, t_{n-1})) = e^{2\pi i(y_1 t_1 + \dots + y_{n-1} t_{n-1})}$ , for  $(t_1, \dots, t_{n-1}) \in N$  and  $(y_1, \dots, y_{n-1}) \in \mathbb{R}^{n-1}$ . With this identification, the action of  $\mathbb{G}$  on  $\widehat{N}$  is

$$(a, s) \cdot \gamma_y = \gamma_y \circ \tau_{(a,s)}^{-1} = \gamma_{ya^{\frac{1}{n}-1}}. \quad (3)$$

In the next lemma we show that  $\mathbb{G}$  acts regularly on  $\widehat{N}$ , via the action (3).

**Lemma 4.1.**  $\mathbb{G}$  acts regularly on  $\widehat{N}$  via the action (3).

*Proof.* Since  $\mathbb{R}^+, \mathbb{R}^{n-1}$  are second countable, so is  $\mathbb{G}$ . It is enough to show the existence of a cross section for the  $\mathbb{G}$ -orbits of the action (3). For each  $y = (y_1, \dots, y_{n-1}) \in \mathbb{R}^{n-1}$ , the orbit of each point  $\gamma_y \in \widehat{N}$  is the set

$$O_{\gamma_y} = \{(a, s)\gamma_y; (a, s) \in \mathbb{G}\} = \{\gamma_{ya^{\frac{1}{n}-1}}; a \in \mathbb{R}^+\}.$$

Easy calculation show that there are  $3^{n-1}$  disjoint orbits in  $\widehat{N}$ , such that the choice of them depends on the sign of  $y_i$  to be positive, negative or zero, for  $i = 1, \dots, n-1$ . Consider the set

$$\Omega = \{(x_1, \dots, x_{n-1}); x_i \in \{0, 1, -1\}, i = 1, \dots, n-1\}. \quad (4)$$

Therefore  $\gamma_x$  meets each orbit once, for any  $x \in \Omega$ . So  $\overline{\Omega} = \{\gamma_x, x \in \Omega\}$  provides a cross-section of the  $\mathbb{G}$ -orbits.  $\square$

Now we can give an explicit description of the unitary dual of  $\mathbb{G}$  by investigating each orbit in  $\widehat{N}$ . Consider  $0_{n-1}$ , the zero vector in  $\Omega$ , defined in (4). Then the stabilizer group  $\mathbb{G}_{\gamma_{0_{n-1}}}$  is equal to  $\mathbb{G}$ , therefore the irreducible representations of  $\mathbb{G}$  is therefore the representations  $\omega_\alpha(a, b) = a^{i\alpha}$ ,  $\alpha \in \mathbb{R}$ , with respect to the orbit  $O_{\gamma_{0_{n-1}}}$ . By analysing other orbits  $O_{\gamma_x}$  associate with every  $x \in \Omega \setminus 0_{n-1}$ , easy calculation shows that the stabilizer group  $\mathbb{G}_{\gamma_x}$  is equal to  $\mathbb{R}^{n-1}$ . Consider  $\gamma_x$  as a base point of each orbit for  $x \in \Omega \setminus 0_{n-1}$ . Then the irreducible representation of  $\mathbb{G}$  obtained by inducing each  $\gamma_x$  from  $\mathbb{R}^{n-1}$  (identified with the obvious Abelian subgroup) can be realized on  $L^2(\mathbb{R}^{n-1})$  via

$$U^x = \text{ind}_{\mathbb{R}^{n-1}}^{\mathbb{G}} \gamma_x \quad (5)$$

for  $x \in \Omega \setminus x^0$ .

However we can conclude that the unitary dual of the semi-direct product group  $\mathbb{G}$  is

$$\widehat{\mathbb{G}} = \{\omega_\alpha; \alpha \in \mathbb{R}\} \cup \{U^x; x \in \Omega \setminus x^0\}. \quad (6)$$

Let us turn next to the standard shearlet group  $\mathbb{S}$ , where the goal of this paper is to find the unitary dual of it, by applying the construction of unitary dual for semi-direct product groups.

As we mentioned in section 3, the standard shearlet group  $\mathbb{S}$  is the semi-direct product  $\mathbb{G}$  with  $\mathbb{R}^n$ ,  $n \geq 2$ . Let  $\tilde{N} = \{(1, 0_{n-1}, t), t \in \mathbb{R}^n\}$ , which is isomorphic with  $\mathbb{R}^n$ , be the Abelian subgroup of  $\mathbb{S}$ . Then  $\widehat{\tilde{N}}$  is isomorphic to  $\mathbb{R}^n$  via  $y \rightarrow \chi_y$ , where  $\chi_y(t) = e^{2\pi i \langle y, t \rangle}$ , for  $t \in \tilde{N}$  and  $y \in \mathbb{R}^n$ . The action of  $(a, s, x) \in \mathbb{S}$  on  $\mathbb{R}^n$  is denoted by  $t \rightarrow S_s A_a t$  and the corresponding action of  $(a, s, x) \in \mathbb{S}$  on  $\chi \in \widehat{\tilde{N}}$  and  $(a, s, x) \cdot \chi \in \widehat{\tilde{N}}$  is given by

$$(a, s, x) \cdot \chi := \chi \circ \lambda_{(a,s)}^{-1}. \quad (7)$$

Since  $\mathbb{R}^n$  has not any influence in this action, so it is possible to say that  $\mathbb{G}$  acts on  $\widehat{\tilde{N}}$ , via this action. Indeed

$$\begin{aligned} (a, s) \cdot \chi(t) &= \chi \circ \lambda_{(a,s)}^{-1}(t) = \chi(A_a^{-1} S_s^{-1} t) = \chi(S_{\frac{-s}{\sqrt[n]{a^{n-1}}}} A_{\frac{1}{a}} t) \\ &= e^{2\pi i \chi \cdot S_{\frac{-s}{\sqrt[n]{a^{n-1}}}} A_{\frac{1}{a}} t} = e^{2\pi i A_{\frac{1}{a}} S_{\frac{-s}{\sqrt[n]{a^{n-1}}}}^T \chi \cdot t} = A_{\frac{1}{a}} S_{\frac{-s}{\sqrt[n]{a^{n-1}}}}^T \chi(t), \end{aligned}$$

where  $a \in \mathbb{R}^+$  and  $s = (s_1, \dots, s_{n-1}) \in \mathbb{R}^{n-1}$ . Therefore  $(a, s) \cdot \chi = A_{\frac{1}{a}} S_{\frac{-s}{\sqrt[n]{a^{n-1}}}}^T \chi$ .

**Lemma 4.2.** *The standard shearlet group  $\mathbb{S}$  acts regularly on  $\widehat{\mathbb{R}^n}$ .*

*Proof.* With simplify the action, we have

$$\begin{aligned} (a, s) \cdot \chi &= A_{\frac{1}{a}} S_{\frac{-s}{\sqrt[n]{a^{n-1}}}}^T \chi = \begin{pmatrix} \frac{1}{a} & 0_{n-1} \\ 0_{n-1}^T & a^{-\frac{1}{n}} I_{n-1} \end{pmatrix} \begin{pmatrix} 1 & 0_{n-1} \\ \frac{-s^T}{\sqrt[n]{a^{n-1}}} & I_{n-1} \end{pmatrix} \begin{pmatrix} \chi_1 \\ \vdots \\ \chi_n \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{a} & 0_{n-1} \\ \frac{-s^T}{a} & a^{-\frac{1}{n}} I_{n-1} \end{pmatrix} \begin{pmatrix} \chi_1 \\ \vdots \\ \chi_n \end{pmatrix}; \end{aligned}$$

i.e.,

$$\begin{pmatrix} \frac{1}{a} & 0 & 0 & \cdots & 0 \\ \frac{-s_1}{a} & \frac{1}{\sqrt[n]{a}} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{-s_{n-2}}{a} & 0 & \cdots & \frac{1}{\sqrt[n]{a}} & 0 \\ \frac{-s_{n-1}}{a} & 0 & \cdots & 0 & \frac{1}{\sqrt[n]{a}} \end{pmatrix} \begin{pmatrix} \chi_1 \\ \vdots \\ \chi_n \end{pmatrix} = \begin{pmatrix} \frac{\chi_1}{a} \\ \frac{-s_1 \chi_1}{a} + \frac{\chi_2}{\sqrt[n]{a}} \\ \vdots \\ \frac{-s_{n-1} \chi_1}{a} + \frac{\chi_n}{\sqrt[n]{a}} \end{pmatrix}.$$

Furthermore, this action has  $2 + 3^{n-1}$  disjoint orbit such that the structure of them depends on the sign of  $\chi_i$ ,  $i = 1, \dots, n$  to be 1, -1 or zero. On the other hand, for  $\chi = (\chi_1, \dots, \chi_n) \in \widehat{\mathbb{R}^n}$ , if  $\text{sgn}(\chi_1) = 1$ , then the orbit associated with  $\chi$  is the set  $O_\chi = \{(x_1, \dots, x_n); x_1 > 0\}$ , if  $\text{sgn}(\chi_1) = -1$ , then the orbit associated with  $\chi$  is the set  $O_\chi = \{(x_1, \dots, x_n); x_1 < 0\}$ , else if  $\chi_1 = 0$  then the structure of the orbits depend on the  $\text{sgn}(\chi_i)$ ,  $i = 2, \dots, n$ . Actually since  $\mathbb{S}$  is second countable and also because of the existence of the set

$$X = \{(\pm 1, 0_{n-1})\} \cup \{(0, \eta_1, \dots, \eta_{n-1}); \eta_i \in \{0, -1, 1\}, i = 1, \dots, n-1\}, \quad (8)$$

which intersects each orbit exactly once, hence  $\mathbb{S}$  acts regularly on  $\widehat{\mathbb{R}^n}$ .  $\square$

Now by considering the sets  $\Omega$  and  $X$ , which are defined in (4) and (8) respectively and also the unitary dual of  $\mathbb{G}$ , defined in (6), we can construct the unitary dual of the standard shearlet group  $\mathbb{S}$  as follows.

**Theorem 4.1.** *Let  $\mathbb{S} = \mathbb{R}^+ \times \mathbb{R}^{n-1} \times \mathbb{R}^n$ . Then the unitary dual of  $\mathbb{S}$  is the set*

$$\begin{aligned} \widehat{\mathbb{S}} &= \{ind_{\mathbb{R}^{n-1}}^{\mathbb{S}}(\pm 1, 0_{n-1})\} \cup \{\pi^\alpha; \alpha \in \mathbb{R}\} \cup \{\pi^x; x \in \Omega \setminus 0_{n-1}\}, \\ &\cup \{ind_{(\mathbb{R}^{n-1} \times \mathbb{R}^n)}^{\mathbb{S}}(\xi \times \eta); \eta \in X \setminus \{0_n, (\pm 1, 0_{n-1})\}, \xi \in \widehat{\mathbb{R}^{n-1}}\}, \end{aligned}$$

such that  $\pi^\alpha(a, s, t) = a^{i\alpha}$  and  $\pi^x(a, s, t) = U^x(a, s)$ .

*Proof.* Let every element of  $X$  be the representative of each  $2 + 3^{n-1}$  orbit. Then for  $0_{n-1} \in X$ , the stabilizer subgroup associated with the orbit  $O_{0_n}$  is equal with  $\mathbb{G}$ . So the irreducible representations of  $\mathbb{S}$  associated to the orbit  $O_{0_n}$ , are the irreducible representations of  $\mathbb{G}$ , lifted to  $\mathbb{S}$ , defined by

$$\pi_0^\alpha(a, s, t) = \{a^{i\alpha}; \alpha \in \mathbb{R}\}, \quad \pi_0^x(a, s, t) = U^x(a, s),$$

for  $x \in \Omega = \{(x_1, \dots, x_{n-1}); x_i \in \{0, 1, -1\}, i = 1, \dots, n-1\} \setminus 0_{n-1}$ . The action (7) follows that the stabilizer group  $\mathbb{S}_{\chi_x}$  is equal with  $\mathbb{R}^n$ , whenever  $x = (1, 0_{n-1})$  or  $(-1, 0_{n-1})$ , whereas if  $x \in X \setminus \{0_n, (\pm 1, 0_{n-1})\}$ , then  $\mathbb{G}_{\chi_x} = \mathbb{R}^{n-1}$  and thus  $\mathbb{S}_{\chi_x} = \mathbb{R}^{n-1} \times \mathbb{R}^n$ .

The above mentioning with the Mackey theory for sem-direct product group, which is explain in section 2, describe a procedure by which  $\widehat{\mathbb{S}}$ , the unitary dual of the standard shearlet group  $\mathbb{S}$ , completely constructed.  $\square$

Let us take a close look at the  $ind_{\mathbb{R}^n}^{\mathbb{S}}(\pm 1, 0_{n-1})$ . Realize the irreducible representation  $\Phi^{(1, 0_{n-1})} = ind_{\mathbb{R}^n}^{\mathbb{S}}(1, 0_{n-1})$  on the Hilbert space  $L^2(\mathbb{R}^+ \times \mathbb{R}^{n-1})$ . For  $(a, s, t) \in \mathbb{S}$  and  $f \in L^2(\mathbb{R}^+ \times \mathbb{R}^{n-1})$ ,  $\Phi^{(1, 0_{n-1})}(a, s, t)f$  is defined by

$$\Phi^{(1, 0_{n-1})}(a, s, t)f(a', s') = e^{2\pi i(\frac{1}{a'} - \frac{s'}{a'}) \cdot t} f\left(\frac{a'}{a}, \frac{s' - s}{\sqrt{a^{n-1}}}\right), \quad (9)$$

**Theorem 4.2.** *With the above notations,  $\Phi^{(1, 0_{n-1})}$  is unitary equivalent with an irreducible sub-representation of the representation  $\rho$ , defined in (2).*

*Proof.* First we move  $\Phi^{(1, 0_{n-1})}$  over to  $L^2(O_{(1, 0_{n-1})})$ .

Consider  $L^2(O_{(1, 0_{n-1})}, \mu_{\widehat{\mathbb{R}^n}})$  ( $\mu_{\widehat{\mathbb{R}^n}}$  is the Lebesgue measure on  $\widehat{\mathbb{R}^n}$ ) simply as  $L^2(O_{(1, 0_{n-1})})$  and identify it with the closed subspace of  $L^2(\widehat{\mathbb{R}^n})$ , consisting of all functions in  $L^2(\widehat{\mathbb{R}^n})$ , which are supported on  $O_{(1, 0_{n-1})}$ . For  $\xi \in L^2(O_{(1, 0_{n-1})})$ , we consider  $W\xi$  on  $\mathbb{R}^+ \times \mathbb{R}^{n-1}$  defined by

$$W\xi((a, s)) = a^{\frac{1-2n}{2n}} \xi((a, s) \cdot (1, 0_{n-1})).$$

It is straightforward to show that  $W$  is a unitary map of  $L^2(O_{(1, 0_{n-1})})$  onto  $L^2(\mathbb{R}^+ \times \mathbb{R}^{n-1})$ . We define  $\pi_{(1, 0_{n-1})}$  to be the representation  $\Phi^{(1, 0_{n-1})}$  transferred by  $W$ , which is, for  $(a, s, t) \in \mathbb{S}$ ,

$$\pi_{(1, 0_{n-1})}(a, s, t) = W^{-1}\Phi^{(1, 0_{n-1})}(a, s, t)W.$$

That  $\pi_{(1, 0_{n-1})}$  is a unitary representation can be verified by straightforward computations. By the fact that for a fixed  $\chi \in O_{(1, 0_{n-1})}$ , there exist  $(a', s') \in \mathbb{R}^+ \times \mathbb{R}^{n-1}$  such that

$\chi = (a', s') \cdot (1, 0_{n-1}) = (a'^{-1}, -s'a'^{-1})$  and also for  $(a, s, t) \in \mathbb{S}$ ,  $\xi \in L^2(O_{(1,0_{n-1})})$ , we get

$$\begin{aligned} & \pi_{(1,0_{n-1})}(a, s, t)\xi(\chi) \\ &= W^{-1}\Phi^{(1,0_{n-1})}(a, s, t)W\xi(a'^{-1}, -s'a'^{-1}) \\ &= a'^{\frac{2n-1}{2n}}\Phi^{(1,0_{n-1})}(a, s, t)W\xi(a', s') \\ &= a'^{\frac{2n-1}{2n}}(a'^{-1}, -s'a'^{-1})(t)W\xi((a, s)^{-1}(a', s')) \\ &= a'^{\frac{2n-1}{2n}}(a'^{-1}, -s'a'^{-1})(t)a^{\frac{2n-1}{2n}}a'^{\frac{1-2n}{2n}}\xi(((a, s)^{-1}(a', s')) \cdot (1, 0_{n-1})) \\ &= a^{\frac{2n-1}{2n}}\chi(t)\xi((a, s)^{-1} \cdot \chi), \end{aligned}$$

i.e.,  $\pi_{(1,0_{n-1})}(a, s, t)\xi(\chi) = a^{\frac{2n-1}{2n}}\chi(t)\xi(A_a S_s^T \chi)$ . If we define a representation  $\pi$  of  $\mathbb{S}$  on  $L^2(\widehat{\mathbb{R}^n})$  by

$$\pi(a, s, t)\xi(\chi) = a^{\frac{2n-1}{2n}}\chi(t)\xi(A_a S_s^T \chi)$$

for  $\chi \in \widehat{\mathbb{R}^n}$ ,  $\xi \in L^2(\widehat{\mathbb{R}^n})$ , and  $(a, s, t) \in \mathbb{S}$ , then obviously  $L^2(O_{(1,0_{n-1})})$  is a  $\pi$ -invariant subspace of  $L^2(\widehat{\mathbb{R}^n})$ . Let  $\mathcal{F} : L^2(\mathbb{R}^n) \rightarrow L^2(\widehat{\mathbb{R}^n})$  denote the Plancherel transform. Using this unitary operator, we can move  $\pi$  over to a representation acting on  $L^2(\mathbb{R}^n)$ . Let  $\rho(a, s, t) = \mathcal{F}^{-1}\pi(a, s, t)\mathcal{F}$ , for  $(a, s, t) \in \mathbb{S}$ . Then

$$\begin{aligned} \mathcal{F}^{-1}\pi(a, s, t)\mathcal{F}f(x) &= \int_{\widehat{\mathbb{R}^n}} \pi(a, s, t)\widehat{f}(\xi)\xi(x)d\xi \\ &= \int_{\widehat{\mathbb{R}^n}} a^{\frac{2n-1}{2n}}\xi(t)\widehat{f}(A_a S_s^T \xi)\xi(-x)d(\xi) \\ &= \int_{\widehat{\mathbb{R}^n}} a^{\frac{2n-1}{2n}}\widehat{f}(\xi)\xi(A_{a^{-1}}S_{-s}(x-t))a^{\frac{1-2n}{2n}}d(\xi) = a^{\frac{1-2n}{2n}}f(A_{a^{-1}}S_{-s}(x-t)), \end{aligned}$$

for  $x \in \mathbb{R}^n$ ,  $f \in L^2(\mathbb{R}^n)$  and  $(a, s, t) \in \mathbb{S}$ . So

$$\rho(a, s, t)f(x) = a^{\frac{1-2n}{2n}}f(A_{a^{-1}}S_{-s}(x-t)).$$

Let  $\mathcal{H}_{O_{(1,0_{n-1})}}^2 = \{f \in L^2(\mathbb{R}^2); \widehat{f} \in L^2(O_{(1,0_{n-1})})\}$ . Then  $\mathcal{H}_{O_{(1,0_{n-1})}}^2$  is a  $\rho$ -invariant subspace and  $\rho_{O_{(1,0_{n-1})}}$ , formed by restricting  $\rho$  to  $\mathcal{H}_{O_{(1,0_{n-1})}}^2$ , is another representation in the equivalent class of irreducible representation  $ind_{\mathbb{R}^n}^{\mathbb{S}}\omega$ , for  $\omega \in \widehat{\mathbb{R}^n}$ .  $\square$

Also we can conclude that the representation

$$\Phi^{(-1,0_{n-1})} = ind_{\mathbb{R}^n}^{\mathbb{S}}(-1, 0_{n-1}),$$

is equivalent with another irreducible sub-representation of  $\rho$ , denoted by  $\rho_{O_{(-1,0_{n-1})}}$ , defined by

$$\rho_{O_{(-1,0_{n-1})}}(a, s, t)f(x) = a^{\frac{1-2n}{2n}}f(A_{a^{-1}}S_{-s}(x-t)),$$

on the Hilbert space  $\mathcal{H}_{O_{(-1,0_{n-1})}}^2$ . By restricting  $\mu_{\widehat{\mathbb{R}^n}}$ , the Lebesgue measure on  $\widehat{\mathbb{R}^n}$ , we have two non-zero open free  $(\mathbb{R}^+ \times \mathbb{R})$ -orbits  $O_{(1,0_{n-1})}$  and  $O_{(-1,0_{n-1})}$ . So  $\rho = \rho_{O_{(1,0_{n-1})}} \oplus \rho_{O_{(-1,0_{n-1})}}$ , where  $\rho_{O_{(1,0_{n-1})}}$  and  $\rho_{O_{(-1,0_{n-1})}}$  are irreducible and inequivalent infinite-dimensional representations.

In the following example we determine the unitary dual of standard 3 dimensional shearlet group.

**Example 4.1.** Let  $\mathbb{S} = \mathbb{R}^+ \times \mathbb{R}^2 \times \mathbb{R}^3$ . Then the action of  $\mathbb{S}$  on  $\widehat{\mathbb{R}^3}$  is given by  $(a, s, t) \cdot \gamma := \gamma \circ \lambda_{(a,s)}^{-1}$  for  $(a, (s_1, s_2), t) \in \mathbb{S}$  and  $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \widehat{\mathbb{R}^3}$ . Indeed for

$$A_a = \begin{pmatrix} a & 0 & 0 \\ 0 & \sqrt[3]{a} & 0 \\ 0 & 0 & \sqrt[3]{a} \end{pmatrix}, \quad S_s = \begin{pmatrix} 1 & s_1 & s_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$



we have  $(a, s, t) \cdot \gamma = A_{\frac{1}{a}} S_{\frac{-s}{\sqrt[3]{a^2}}}^T \gamma = \begin{pmatrix} \frac{1}{a} \gamma_1 \\ \frac{-s_1}{a} \gamma_1 + \frac{1}{\sqrt[3]{a}} \gamma_2 \\ \frac{-s_2}{a} \gamma_1 + \frac{1}{\sqrt[3]{a}} \gamma_3 \end{pmatrix}$ .

Therefore  $\begin{pmatrix} \frac{1}{a} \gamma_1 \\ \frac{-s_1}{a} \gamma_1 + \frac{1}{\sqrt[3]{a}} \gamma_2 \\ \frac{-s_2}{a} \gamma_1 + \frac{1}{\sqrt[3]{a}} \gamma_3 \end{pmatrix} = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix}$ , yields  $(a, (s_1, s_2)) = (1, (0, 0))$ .

The above mentioned action has 11 orbits as follows:

$$\begin{aligned} O_{(0,0,0)} &= \{(0, 0, 0)\}, \\ O_{(\gamma_1, \gamma_2, \gamma_3)} &= \{(x, y, z) \in \mathbb{R}^3 : x > 0\}, \gamma_1 > 0, \\ O_{(\gamma_1, \gamma_2, \gamma_3)} &= \{(x, y, z) \in \mathbb{R}^3 : x < 0\}, \gamma_1 < 0, \\ O_{(0, \gamma_2, \gamma_3)} &= \{(0, y, z) \in \mathbb{R}^3 : y > 0\}, \gamma_2 > 0, \\ O_{(0, \gamma_2, \gamma_3)} &= \{(0, y, z) \in \mathbb{R}^3 : y < 0\}, \gamma_2 < 0, \\ O_{(0, \gamma_2, \gamma_3)} &= \{(0, y, z) \in \mathbb{R}^3 : z > 0\}, \gamma_3 > 0, \\ O_{(0, \gamma_2, \gamma_3)} &= \{(0, y, z) \in \mathbb{R}^3 : z < 0\}, \gamma_3 < 0, \\ O_{(0, \gamma_2, \gamma_3)} &= \{(0, y, z) \in \mathbb{R}^3 : y > 0, z > 0\}, \gamma_2 > 0, \gamma_3 > 0, \\ O_{(0, \gamma_2, \gamma_3)} &= \{(0, y, z) \in \mathbb{R}^3 : y > 0, z < 0\}, \gamma_2 > 0, \gamma_3 < 0, \\ O_{(0, \gamma_2, \gamma_3)} &= \{(0, y, z) \in \mathbb{R}^3 : y < 0, z > 0\}, \gamma_2 < 0, \gamma_3 > 0, \\ O_{(0, \gamma_2, \gamma_3)} &= \{(0, y, z) \in \mathbb{R}^3 : y < 0, z < 0\}, \gamma_2 < 0, \gamma_3 < 0. \end{aligned}$$

Clearly we can realize that the set

$$\begin{aligned} \Lambda &= \{(0, 0, 0), (1, 0, 0), (-1, 0, 0), (0, 1, 0), (0, -1, 0), (0, 0, 1), \\ &\quad (0, 0, -1), (0, 1, 1), (0, 1, -1), (0, -1, 1), (0, -1, -1)\}, \end{aligned}$$

meet each orbit in exactly one point. By choosing every element of  $\Lambda$  as a representative of each  $\mathbb{S}$ -orbit, we have the stabilizer groups

$$\mathbb{S}_{(0,0,0)} = \mathbb{S}, \quad \mathbb{S}_{(\pm 1, 0, 0)} = \mathbb{R}^3,$$

and other stability groups associated with the  $\mathbb{S}$ -orbits are equal with  $\mathbb{R}^2 \times_{\lambda} \mathbb{R}^3$ . Thus Theorem 4.1 tells us that

$$\begin{aligned} \widehat{\mathbb{S}} &= \{ \text{ind}_{\mathbb{R}^3}^{\mathbb{S}}(\pm 1, 0, 0) \\ &\quad \cup \{ \text{ind}_{\mathbb{R}^2 \times \mathbb{R}^3}^{\mathbb{S}}(\xi \times \eta); \xi \in \widehat{\mathbb{R}^2}, \eta \in \Lambda \setminus \{(0, 0, 0), (\pm 1, 0, 0)\} \} \\ &\quad \cup \{ \pi_0^\alpha; \alpha \in \mathbb{R} \} \\ &\quad \cup \{ \text{ind}_{\mathbb{R}^2}^{\mathbb{R}^+ \times \mathbb{R}^2} \gamma'_x; x \in \{(1, 0), (-1, 0), (0, 1), (0, -1), (1, 1), (1, -1), \\ &\quad (-1, 1), (-1, -1)\} \}, \end{aligned}$$

where  $\pi_0^\alpha(a, s, t) = a^{i\alpha}$  and  $\gamma'_x \in \widehat{\mathbb{R}^2}$ .

Note that, the non-zero measure orbits are

$$O_{(1,0,0)} = \{(x, y, z) \in \mathbb{R}^3 : x > 0\}, \quad O_{(-1,0,0)} = \{(x, y, z) \in \mathbb{R}^3 : x < 0\}.$$

Define  $A_+ := O_{(1,0,0)}$  and  $A_- := O_{(-1,0,0)}$ . Then  $\sigma_+ : \mathbb{S} \rightarrow \mathcal{U}(\mathcal{H}_{A_+}^2)$  and  $\sigma_- : \mathbb{S} \rightarrow \mathcal{U}(\mathcal{H}_{A_-}^2)$  defined by

$$\sigma_+(a, s, t)\psi(x) = a^{\frac{1}{2n}-1} \psi(A_a^{-1} S_s^{-1}(x-t)) = a^{-\frac{5}{6}} \psi\left(\begin{pmatrix} \frac{1}{a} & \frac{-s}{\sqrt[3]{a}} \\ 0_2 & \frac{1}{\sqrt[3]{a}} I_2 \end{pmatrix} (x-t)\right),$$

and

$$\sigma_-(a, s, t)\phi(x) = a^{\frac{1}{2n}-1}\phi(A_a^{-1}S_s^{-1}(x-t)) = a^{\frac{-s}{6}}\phi\left(\begin{pmatrix} \frac{1}{a} & \frac{-s}{a} \\ 0 & \frac{1}{\sqrt{a}}I_2 \end{pmatrix}(x-t)\right),$$

are the irreducible sub-representations of  $\sigma$ , the quasi-regular representation of 3-D standard shearlet group, for  $\psi \in \mathcal{H}_{A_+}^2$  and  $\varphi \in \mathcal{H}_{A_-}^2$ . Therefore  $\sigma = \sigma_+ \oplus \sigma_-$ . Actually  $\sigma_+$  and  $\sigma_-$  are unitary equivalent with the irreducible representations  $\text{ind}_{\mathbb{R}^3}^{\mathbb{S}}(1, 0, 0)$  and  $\text{ind}_{\mathbb{R}^3}^{\mathbb{S}}(-1, 0, 0)$ , respectively.

## REFERENCES

- [1] S. T. Ali, J. P. Antoine and J. P. Gazeau, Coherent States, Wavelets and Their Generalizations. New York: Springer-Verlag, 2000.
- [2] R. J. Blattner, On induced representations. Amer. J. Math., **83**(1961), 79-98.
- [3] S. Dahlke and G. Teschke, The continuous shearlet transform in arbitrary space dimensions, J. Fourier Anal. Appl., **16**(2010), 340-364.
- [4] S. Dahlke, G. Kutyniok, G. Steidl, and G. Teschke, Shearlet coorbit spaces and associated Banach frames. Appl. Comput. Harm. Anal., **27**(2009), No. 2, 195-214.
- [5] J. M. G. Fell, A new look at Mackeys imprimitivity theorem. In: Conference on Harmonic Analysis, Lecture Notes in Mathematics **266**. Berlin: Springer-Verlag (1972), 43-58.
- [6] G. B. Folland, A Course in Abstract Harmonic Analysis. Boca Katon: CRC Press, 1995.
- [7] F. G. Frobenius, U ber Relationen zwischen den Charakteren einer Gruppe und denen ihrer Untergruppen. Sitz. Preuss. Akad. Wiss. (1898), 50115; also pp. 104- 18 in Frobeniuss Gesammelte Abhandlungen, Vol. III. Berlin: Springer, 1968.
- [8] K. Guo, G. Kutyniok and D. Labate, Sparse multidimensional representations using anisotropic dilation and shear operators , Wavelets and Splines (Athens, GA, 2005), Nashboro Press, Nashville, TN, (2006), 189201.
- [9] E. Hewitt and K. A. Ross, Abstract Harmonic Analysis, Vol. 1, Springer-Verlag, Berlin, 1963.
- [10] G. W. Mackey, Borel structures in groups and their duals, Trans. Amer. Math. Soc. **85**(1957), 134-165.
- [11] G. W. Mackey, Induced representations of locally compact groups. I. Ann. of Math., **55**(1952), 101-39.
- [12] G. W. Mackey, Induced representations of locally compact groups. II: The Frobenius theorem. Ann. of Math., **58**(1953), 193-220.
- [13] G. W. Mackey, On induced representations of groups. Amer. J. Math., **73**(1951), 576-592.