ACCELERATED RHSS ITERATION METHODS FOR SADDLE-POINT LINEAR SYSTEMS

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Bai and Benzi (2016) recently studied a class of regularized Hermitian and skew-Hermitian splitting (RHSS) methods for the solution of large, sparse linear systems in saddle-point form. In this paper, we establish a class of accelerated regularized Hermitian and skew-Hermitian splitting (ARHSS) iteration methods for large sparse saddle point linear systems by making use of the regularized Hermitian and skew-Hermitian splitting (RHSS) iteration technique. These methods involve two iteration parameters whose special choices can recover the known preconditioned HSS iteration methods, as well as yield new ones. Theoretical analysis shows that the new methods converge unconditionally to the unique solution of the saddle point problem. In addition, theoretical properties of the preconditioned Krylov subspace methods such as GMRES are investigated when the ARHSS iterations are employed as their preconditioners. Numerical experiments confirm the correctness of the theory and the effectiveness of the methods.

Keywords: Saddle-point linear system, Hermitian and skew-Hermitian splitting, Iterative methods, Preconditioning, Convergence

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1. Introduction

We consider the iterative solution of large sparse saddle point linear systems of the form

$$Ax \equiv \begin{pmatrix} B & E \\ -E^* & O \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \equiv b \tag{1}$$

where $B \in \mathbb{C}^{p\times p}$ is Hermitian positive definite, $O \in \mathbb{C}^{q\times q}$ is zero, $E \in \mathbb{C}^{p\times q}$ has full column rank, $p \geq q$, $f \in \mathbb{C}^p$ and $g \in \mathbb{C}^q$. These assumptions guarantee the existence and uniqueness of the solution of the system of linear equations (1). Here and in the sequel, we indicate by $(.)^*$ the conjugate transpose of either a vector or a matrix of suitable dimension, and we let $n = p + q$.

Such systems typically result from mixed or hybrid finite-element approximations of second-order elliptic problems, elasticity problems or the Stokes equations [8] and from Lagrange multiplier methods [9]. There exists an extensive literature concerning structured preconditionners [2, 10, 11, 12] and iterative methods [14, 15, 16, 17, 18] for these problems. Since the above problem is large and sparse, iterative methods for solving equation (1) are effective because of storage requirements and preservation of sparsity. The best known and the oldest methods is Uzawa algorithms [21]. The well-known SOR method, which is a simple iterative method that is popular in engineering applications, cannot be applied directly to system (1) because of the singularity of the block diagonal part of the coefficient matrix. Recently, several versions for generalizing the SOR method to system (1) have been proposed [22, 23]. Also, Bai et al. [4] presented the Hermitian/skew-Hermitian splitting (HSS) method to solve non-Hermitian positive definite system of linear equations. After

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that, this method gains peoples attention and proposed different variants of the method. We refer to Benzi et al. [13] for a comprehensive survey.

Recently, Bai and Benzi in [1] proposed a class of regularized Hermitian and skew-Hermitian splitting (RHSS) methods for the solution of large, sparse linear systems in saddle-point form. These methods can be used as stationary iterative solvers or as preconditioners for Krylov subspace methods. They establish unconditional convergence of the stationary iterations.

Bai and Benzi split the coefficient matrix $A \in \mathbb{C}^{n\times n}$ in (1) into

$$\begin{aligned} A &= \begin{pmatrix} B & O \\ O & Q \end{pmatrix} + \begin{pmatrix} O & E \\ -E^* & -Q \end{pmatrix} = H_+ + S_- \\ &= \begin{pmatrix} O & E \\ -E^* & Q \end{pmatrix} + \begin{pmatrix} B & O \\ O & -Q \end{pmatrix} = S_+ + H_-. \end{aligned} \tag{2}$$

wherein $Q \in \mathbb{C}^{q\times q}$ is Hermitian positive semidefinite matrix. They call the collection of these two splittings a regularized Hermitian and skew-Hermitian (RHS) splitting. Also, they call Q the regularization matrix as the matrix Q plays a regularizing role in the HS splitting. So according to this splitting, they proposed RHSS iteration method. In order to further improve the convergence behavior of the RHSS iteration method, in this paper we propose accelerated RHSS (ARHSS) method. These methods are two-parameter generalizations of the RHSS iteration methods and they can recover the RHSS methods as well as yielding new ones by suitable choices of the two arbitrary parameters.

The organization of the paper is as follows. In Section 2 we present the algorithmic description of the ARHSS iteration method. In Section 3, we prove the unconditional convergence of the ARHSS iteration method and we analyze clustering properties of the eigenvalues of the ARHSS-preconditioned matrix in Section 4. Finally, a numerical experiment on saddle point problems arising from the discretization of the Stokes equations are presented to illustrate the feasibility and effectiveness of this method and preconditioners.

2. **The ARHSS Iterative Method**

In this section, we derive the ARHSS iteration method for the saddle-point linear system (1) for the saddle-point matrix $A \in \mathbb{C}^{n\times n}$. By applying the regularized Hermitian and skew-Hermitian (RHS) splitting to (2), we then obtain the iteration scheme

$$\begin{cases} (\Lambda + H_+)x^{(k+\frac{1}{2})} &= (\Lambda - S_-)x^{(k)} + b, \\ (\Lambda + S_+)x^{(k+1)} &= (\Lambda - H_-)x^{(k+\frac{1}{2})} + b, \end{cases} \tag{3}$$

where

$$\Lambda = \begin{pmatrix} \alpha I & O \\ O & \beta I \end{pmatrix},$$

with α and β positive constants, or in their blockwise forms and after straightforward computations, we can rewrite

$$\begin{pmatrix} \alpha I + B & \frac{1}{\alpha}(\alpha I + B)E \\ -E^* & \beta I + Q \end{pmatrix} \begin{pmatrix} y^{(k+1)} \\ z^{(k+1)} \end{pmatrix} = \begin{pmatrix} \alpha I - B & -\frac{1}{\alpha}(\alpha I - B)E \\ E^* & \beta I + Q \end{pmatrix} \begin{pmatrix} y^{(k+\frac{1}{2})} \\ z^{(k+\frac{1}{2})} \end{pmatrix} + 2\begin{pmatrix} f \\ g \end{pmatrix}. \tag{4}$$

Remark 2.1. *Note that when $Q = (\beta - \alpha)I + \tilde{Q}$, the RHSS method becomes the ARHSS method. Also, note that when $\alpha = \beta$, the ARHSS methods automatically reduces to the RHSS method.*

We obtain the accelerated regularized HSS (or in short, ARHSS) iteration method for solving the saddle-point linear system (4) as follows.

The ARHSS Iteration Method:

Given an initial guess $x^{(0)} = (y^{(0)*}, z^{(0)*})^{(*)} \in \mathbb{C}^{n\times n}$. For $k = 0, 1, 2, ...,$ until $\{x^{(k)}\} =$

$\{(y^{(k)*}, z^{(k)*})^{(*)}\} \subset \mathbb{C}^{n\times n}$ converges, compute the next iterate $x^{(k+1)} = (y^{(k+1)*}, z^{(k+1)*})^{(*)}$ by solving the linear system (4), where α and β are given positive constants.

From (4), we easily know that the ARHSS iteration method can be equivalently rewritten as

$$\begin{pmatrix} y^{(k+1)} \\ z^{(k+1)} \end{pmatrix} = L(\alpha,\beta)\begin{pmatrix} y^{(k)} \\ z^{(k)} \end{pmatrix} + C(\alpha,\beta)\begin{pmatrix} f \\ g \end{pmatrix} \tag{5}$$

where

$$L(\alpha,\beta) = \begin{pmatrix} \alpha I + B & \frac{1}{\alpha}(\alpha I + B)E \\ -E^* & \beta I + Q \end{pmatrix}^{-1} \begin{pmatrix} \alpha I - B & -\frac{1}{\alpha}(\alpha I - B)E \\ E^* & \beta I + Q \end{pmatrix} \tag{6}$$

and

$$C(\alpha,\beta) = 2\begin{pmatrix} \alpha I + B & \frac{1}{\alpha}(\alpha I + B)E \\ -E^* & \beta I + Q \end{pmatrix}^{-1}.$$

Here, $L(\alpha,\beta)$ is the iteration matrix of the ARHSS iteration. In fact, (5) may also result from the splitting

$$A = M(\alpha,\beta) - N(\alpha,\beta)$$

of the coefficient matrix A, with

$$M(\alpha,\beta) = \frac{1}{2}\begin{pmatrix} \alpha I + B & \frac{1}{\alpha}(\alpha I + B)E \\ -E^* & \beta I + Q \end{pmatrix}$$

$$N(\alpha,\beta) = \frac{1}{2}\begin{pmatrix} \alpha I - B & -\frac{1}{\alpha}(\alpha I - B)E \\ E^* & \beta I + Q \end{pmatrix}$$

In actual computations, at each iterate of the ARHSS iterations, we need to solve a linear system with the coefficient matrix $M(\alpha,\beta)$.

By making use of block-triangular factorization of the matrix $M(\alpha,\beta)$, we can straightforwardly obtain the following algorithmic version of the ARHSS iteration method.

Given an initial guess $x^{(0)} = (y^{(0)*}, z^{(0)*})^{(*)} \in \mathbb{C}^{n\times n}$ and two positive constants α and β. For $k = 0, 1, 2, ...,$ until $\{x^{(k)}\} = \{(y^{(k)*}, z^{(k)*})^{(*)}\} \subset \mathbb{C}^{n\times n}$ converges, compute the next iterate $x^{(k+1)} = (y^{(k+1)*}, z^{(k+1)*})^{(*)}$ according to the following procedure:

1− solve $y^{(k+\frac{1}{2})}$ from the linear sub-system

$$(\alpha I + B)y^{(k+\frac{1}{2})} = \alpha y^{(k)} - Ez^{(k)} + f;$$

2− compute

$$f^{(k+\frac{1}{2})} = (\alpha I - B)y^{(k+\frac{1}{2})} + f \quad \text{and} \quad g^{(k+\frac{1}{2})} = (\alpha I + Q)z^{(k)} + E^* y^{(k)} + 2g;$$

3− solve $z^{(k+1)} \in \mathbb{C}^q$ from the linear sub-system

$$\left(\beta I + Q + \frac{1}{\alpha}E^*E\right) z^{(k+1)} = \frac{1}{2}E^* f^{(k+\frac{1}{2})} + f^{(k+\frac{1}{2})};$$

4− compute

$$y^{(k+1)} = \frac{1}{\alpha}\left(-Ez^{(k+1)} + f^{(k+\frac{1}{2})}\right).$$

So, at each of the ARHSS iteration steps, we have to solve two subsystems of linear equations with the coefficient matrix $(\alpha I + B)$ and one subsystem of linear equations with the coefficient matrix $(\beta I + Q + \frac{1}{\alpha}E^*E)$. The latter matrix can be expected to be better conditioned than $(\alpha I + Q + \frac{1}{\alpha}E^*E)$, arising in the RHSS iteration method, consider the following Lemma.

Lemma 2.1. *Let $Q \in \mathbb{C}^{q\times q}$ be Hermitian positive semi-definite matrix, $E \in \mathbb{C}^{p\times q}$ is a rectangular matrix of full column rank and $\beta \geqslant \alpha > 0$. Then*

$$\mathrm{Cond}(\beta I + Q + \frac{1}{\alpha}E^*E) \leqslant \mathrm{Cond}(\alpha I + Q + \frac{1}{\alpha}E^*E).$$

Proof. Let $A = \alpha I + Q + \frac{1}{\alpha}E^*E$, so $\beta I + Q + \frac{1}{\alpha}E^*E = (\beta-\alpha)I + A$. Since A is Hermitian, suppose $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$ are the eigenvalues of A. Then we get

$$\begin{aligned}
(\beta-\alpha)\lambda_1(A) &\geq (\beta-\alpha)\lambda_n(A)\\
(\beta-\alpha)\lambda_1(A) + \lambda_1(A)\lambda_n(A) &\geq (\beta-\alpha)\lambda_n(A) + \lambda_1(A)\lambda_n(A)\\
\lambda_1(A)\left((\beta-\alpha) + \lambda_n(A)\right) &\geq \lambda_n(A)((\beta-\alpha) + \lambda_1(A))\\
\frac{\lambda_1(A)}{\lambda_n(A)} &\geq \frac{(\beta-\alpha) + \lambda_1(A)}{(\beta-\alpha) + \lambda_n(A)}
\end{aligned}$$

So,

$$\mathrm{Cond}(A) = \frac{\lambda_1(A)}{\lambda_n(A)} \quad \geq \quad \frac{(\beta-\alpha) + \lambda_1(A)}{(\beta-\alpha) + \lambda_n(A)} = \mathrm{Cond}((\beta-\alpha)I + A).$$

□

Remark 2.2. *Alternatively, the ARHSS iteration method can be regarded as a special case of the PHSS iteration method developed in [6] with the particular preconditioning matrix*

$$P = \begin{pmatrix} I & O \\ O & \frac{\beta}{\alpha}I + \frac{1}{\alpha}Q \end{pmatrix}.$$

3. Convergence analysis and the ARHSS Preconditioner

In this section, we prove the unconditional convergence of the ARHSS iteration method. also we consider two preconditioner for Krylov subspace methods applied to the nonsingular saddle point problem (1).

Theorem 3.1. *([1])Let $B \in \mathbb{C}^{p\times p}$ be Hermitian positive definite, $E \in \mathbb{C}^{p\times q}$ be of full column rank, and $\alpha > 0$ be a given constant. Assume that $Q \in \mathbb{C}^{q\times q}$ is a Hermitian positive semidefinite matrix. Then the RHSS iteration method converges unconditionally to the exact solution of the saddle point linear system (1).*

In the following theorem, we prove the unconditional convergence of the ARHSS iteration method.

Theorem 3.2. *Let $B \in \mathbb{C}^{p\times p}$ be Hermitian positive definite, $E \in \mathbb{C}^{p\times q}$ be of full column rank, and $\alpha, \beta > 0$ be given constants. Assume that $Q \in \mathbb{C}^{q\times q}$ is a Hermitian positive semidefinite matrix. Then the ARHSS iteration method converges unconditionally to the exact solution of the saddle-point linear system (1), i.e., $\rho(L(\alpha,\beta)) < 1$, where $\rho(L(\alpha,\beta))$ is denote the spectral radius of the iteration matrix of the ARHSS iteration.*

Proof. If $\beta \geqslant \alpha$, according to ARHS splitting, we obtain

$$\begin{aligned}
A &= (\Lambda + H_+) - (\Lambda - S_-)\\
&= \left(\alpha I + \begin{pmatrix} B & O \\ O & \tilde{Q} \end{pmatrix}\right) - \left(\alpha I + \begin{pmatrix} O & E \\ -E^* & -\tilde{Q} \end{pmatrix}\right)
\end{aligned} \tag{7}$$

and

$$\begin{aligned}
A &= (\Lambda + S_+) - (\Lambda - H_-)\\
&= \left(\alpha I + \begin{pmatrix} O & E \\ -E^* & \tilde{Q} \end{pmatrix}\right) - \left(\alpha I + \begin{pmatrix} B & O \\ O & -\tilde{Q} \end{pmatrix}\right)
\end{aligned} \tag{8}$$

where $\tilde{Q} = (\beta-\alpha)I + Q$. Relations (7) and (8) are exactly RHS splitting, and $\tilde{Q}$ is Hermitian positive semidefinite matrix, so by using Theorem 3.1 for $\alpha > 0$, this split is converges to the exact solution of the saddle point linear system (1).
If $\beta \leqslant \alpha$, according to ARHS splitting, we get

$$\begin{aligned}
A &= (\Lambda + H_+) - (\Lambda - S_-)\\
&= \left(\beta I + \begin{pmatrix} B & O \\ O & \tilde{Q} \end{pmatrix}\right) - \left(\beta I + \begin{pmatrix} O & E \\ -E^* & -\tilde{Q} \end{pmatrix}\right)
\end{aligned} \tag{9}$$

and

$$\begin{aligned} A &= (\Lambda + S_+) - (\Lambda - H_-) \\ &= \left(\beta I + \begin{pmatrix} O & E \\ -E^* & \tilde{Q} \end{pmatrix}\right) - \left(\beta I + \begin{pmatrix} B & O \\ O & -\tilde{Q} \end{pmatrix}\right) \end{aligned} \tag{10}$$

where $\tilde{Q} = (\alpha - \beta)I + Q$. Relations (9) and (10) are exactly RHS splitting, and $\tilde{Q}$ is Hermitian positive semidefinite matrix, so by using to Theorem 3.1 for $\beta > 0$, this split is converges to the exact solution of the saddle point linear system (1), Then we obtain $\rho(L(\alpha, \beta)) < 1$. □

As is well known, for some cases, Krylov subspace methods, like GMRES [19], are all likely to suffer from slow convergence for large scale linear systems that arise from some typical applications. Preconditioning is a key ingredient for the useful of Krylov subspace methods in some applications. The efficiency of Krylov subspace methods is effected by the spectral distribution of the coefficient matrix and the degree of its polynomial. Preconditioning attempts to improve the spectral properties of the coefficient matrix A by transforming the linear system $Ax = b$ into another system $M^{-1}Ax = M^{-1}b$ with more favourable properties for iterative solution. By preconditioning, we expect that the preconditioned matrix will have a smaller spectral condition number or the preconditioned matrix has a minimum polynomial of small degree. Now, we analyze spectral properties of the preconditioned matrix $M(\alpha, \beta)^{-1}A$.

Theorem 3.3. *Let $B \in \mathbb{C}^{p\times p}$ be Hermitian positive definite, $B \in \mathbb{C}^{p\times q}$ be of full column rank, and $\alpha, \beta > 0$ be given constants. Assume that $B \in \mathbb{C}^{q\times q}$ is a Hermitian positive semidefinite matrix. If $Q = I$, α is fixed and β is close to ∞, then the eigenvalues of the preconditioned matrix $A(\alpha, \beta) = M(\alpha, \beta)^{-1}A$ are clustered 0_+ and 2_-.*

Proof. Let λ be an eigenvalue of the matrix $M(\alpha, \beta)^{-1}A$. So there exists a nonzero vector x such that

$$Ax = \lambda M(\alpha, \beta)x. \tag{11}$$

Let $x = \begin{pmatrix} y \\ z \end{pmatrix}$. Then the equation (11) can be equivalently written as

$$\begin{cases} 2(By + Ez) & = \lambda(\alpha I + B)(y + \frac{1}{\alpha}Ez), \\ -2E^*y & = \lambda(-E^*y + (\beta I + Q)z) \end{cases} \tag{12}$$

We know that $y \neq 0$. the second equality in (12) gives

$$z = (\frac{\lambda - 2}{\lambda})(\beta I + Q)^{-1}E^*y. \tag{13}$$

Let $E(\beta) = E(\beta I + Q)^{-1}E^*$. Substituting (13) into the first equality in (12), after suitable manipulations we get

$$\begin{aligned} \lambda^2(\alpha I + B)(\alpha I + E(\beta))y &- 2\lambda((\alpha I + B)(\alpha I + E(\beta)) - \alpha(\alpha I - E(\beta))y \\ &+ 4\alpha E(\beta)y = 0 \end{aligned} \tag{14}$$

With the change of variable

$$\tilde{y} = (\alpha I + E(\beta))y,$$

the equality (14) can be rewritten as

$$\begin{aligned} \lambda^2(\alpha I + B)\tilde{y} &- 2\lambda\left((\alpha I + B) - \alpha(\alpha I - E(\beta))(\alpha I + E(\beta))^{-1}\right)\tilde{y} \\ &+ 4\alpha E(\beta)(\alpha I + E(\beta))^{-1}\tilde{y} = 0. \end{aligned} \tag{15}$$

By multiplying both sides of (15) from left with $\tilde{y}^*$ and we adopt the notation,

$$\nu = \tilde{y}^*B\tilde{y} \quad \text{and} \quad \delta(\alpha, \beta) = \tilde{y}^*(\alpha I - E(\beta))(\alpha I + E(\beta))^{-1}\tilde{y},$$

we obtain

$$(\alpha + \nu)\lambda^2 - 2\left(\alpha + \nu - \alpha\delta(\alpha, \beta)\right)\lambda + 2\alpha(1 - \delta(\alpha, \beta)) = 0.$$

The two roots of this quadratic equation are

$$\lambda_{1,2} = \frac{\alpha(1-\delta(\alpha,\beta)) + \nu \pm \sqrt{\nu^2 - \alpha^2(1-\delta^2)}}{\alpha+\nu}. \tag{16}$$

ν is bounded and for $\alpha, \beta > 0$, also $\delta(\alpha,\beta)$ is bounded ($|\delta(\alpha,\beta)| \leq 1$), and so $\lim_{\beta\to\infty}\delta(\alpha,\beta) = 1$. By taking limits in the formula (16), we see that $\lim_{\beta\to\infty}\lambda_1 = 0$ and $\lim_{\beta\to\infty}\lambda_2 = 2$. $\square$

In order to get a better approximation of the coefficient matrix A, we propose an improved RHSS preconditioner (IRHSS preconditioner) as the following:

$$\tilde{M}(\alpha,\beta) = \begin{pmatrix} B & E + \frac{1}{\alpha}BE \\ -E^* & \beta I + Q \end{pmatrix} \tag{17}$$

Acturally, in order to analyze conveniently, we have eliminated the factor $\frac{1}{2}$ for $\tilde{M}(\alpha,\beta)$, we also know that this factor has no effect on the preconditioned linear system, but it has impact on the eigenvalue distribution of the preconditioned matrix. Therefore, in order to compare with the spectrum distributions of other preconditioners, we add the factor $\frac{1}{2}$ to $\tilde{M}(\alpha,\beta)$ in our following numerical experiments, we still denote it by $\tilde{M}(\alpha,\beta)$.

We find that the difference between the preconditioner $\tilde{M}(\alpha,\beta)$ and the coefficient matrix A is given by

$$\tilde{N}(\alpha,\beta) = \tilde{M}(\alpha,\beta) - A = \begin{pmatrix} O & \frac{1}{\alpha}BE \\ O & \beta I + Q \end{pmatrix} \tag{18}$$

which is better approximation than the preconditioner $M(\alpha,\beta)$. Now, we analyze the spectrum properties of the preconditioned matrix $\tilde{M}(\alpha,\beta)^{-1}A$.

Theorem 3.4. *Let the preconditioner $\tilde{M}(\alpha,\beta)$ be defined as in (14). Then the preconditioned matrix $\tilde{M}(\alpha,\beta)^{-1}A$ has the following properties:*

i. *$\tilde{M}(\alpha,\beta)^{-1}A$ has an eigenvalue 1 with multiplicity at least n, and the corresponding eigenvectors are of the form $(e_i, 0)$, where e_i, $i = 1,2,...,p$, are the linearly independent vectors in $\mathbb{C}^p$.*
ii. *The remaining eigenvalues of $\tilde{M}(\alpha,\beta)^{-1}A$ are given by μ_i, where μ_i, $i = 1,2,...,q$, are the eigenvalues of matrix $(\beta I+Q)^{-1}E^*\tilde{B}^{-1}E$ with $\tilde{B} = B + \frac{1}{\alpha}(\alpha I + B)E(\beta I+Q)^{-1}E^*$, and the corresponding eigenvalue vectors are of the form $(\frac{1}{\lambda-1}\tilde{B}^{-1}E\tilde{z}; \tilde{z})$, where $\tilde{z}$ is an eigenvector of $(\beta I+Q)^{-1}E^*\tilde{B}^{-1}E$.*

Proof. From (14), it has

$$\tilde{M}(\alpha,\beta) = \begin{pmatrix} I & E + \frac{1}{\alpha}BE \\ O & \beta I + Q \end{pmatrix} \begin{pmatrix} \tilde{B} & O \\ -(\beta I+Q)^{-1}E^* & I \end{pmatrix}$$

where $\tilde{B} = B + \frac{1}{\alpha}(\alpha I + B)E(\beta I+Q)^{-1}E^*$, and $\tilde{B}$ is nonsingular. Then, we can obtain

$$\tilde{M}(\alpha,\beta)^{-1} = \begin{pmatrix} \tilde{B}^{-1} & O \\ (\beta I+Q)^{-1}E^*\tilde{B}^{-1} & I \end{pmatrix} \begin{pmatrix} I & -(E + \frac{1}{\alpha}BE)(\beta I+Q)^{-1} \\ O & (\beta I+Q)^{-1} \end{pmatrix}$$

or

$$\tilde{M}(\alpha,\beta)^{-1} = \begin{pmatrix} \tilde{B}^{-1} & -\tilde{B}^{-1}(E + \frac{1}{\alpha}BE)(\beta I+Q)^{-1} \\ (\beta I+Q)^{-1}E^*\tilde{B}^{-1} & (\beta I+Q)^{-1} - (\beta I+Q)^{-1}E^*\tilde{B}^{-1}(E + \frac{1}{\alpha}BE)(\beta I+Q)^{-1} \end{pmatrix}$$

From (18) and the above equation, it has

$$\begin{aligned}\tilde{M}(\alpha,\beta)^{-1}A &= I-\tilde{M}(\alpha,\beta)^{-1}\tilde{N}(\alpha,\beta)\\ &= \begin{pmatrix} I & O\\ O & I\end{pmatrix}-\begin{pmatrix} O & -\tilde{B}^{-1}E\\ O & I-(\beta I+Q)^{-1}E^*\tilde{B}^{-1}E\end{pmatrix}\\ &= \begin{pmatrix} I & \tilde{B}^{-1}E\\ O & (\beta I+Q)^{-1}E^*\tilde{B}^{-1}E\end{pmatrix}\end{aligned}$$

Thus, we can assert that the preconditioned matrix $\tilde{M}(\alpha,\beta)^{-1}A$ has at least n eigenvalues equal to 1, and the other eigenvalues are equal to the ones of $(\beta I+Q)^{-1}E^*\tilde{B}^{-1}E$.

Now, we analyze the corresponding eigenvectors of them. Let $\tilde{\lambda}$ be an eigenvalue of $\tilde{M}(\alpha,\beta)^{-1}A$, $(\tilde{y}^*,\tilde{z}^*)^*$ be the corresponding eigenvector. Then we obtain

$$\begin{pmatrix} I & \tilde{B}^{-1}E\\ O & (\beta I+Q)^{-1}E^*\tilde{B}^{-1}E\end{pmatrix}\begin{pmatrix}\tilde{y}\\ \tilde{z}\end{pmatrix}=\tilde{\lambda}\begin{pmatrix}\tilde{y}\\ \tilde{z}\end{pmatrix},$$

and so

$$\begin{cases} \tilde{y}+\tilde{B}^{-1}E\tilde{z}=\tilde{\lambda}\tilde{y}\\ (\beta I+Q)^{-1}E^*\tilde{B}^{-1}E\tilde{z}=\tilde{\lambda}\tilde{z}.\end{cases} \tag{19}$$

If $\tilde{z}=0$, the first equation in (19) implies that $\tilde{\lambda}=1$, otherwise, $\tilde{y}=0$, this eliminate with the fact that $(\tilde{y}^*,\tilde{z}^*)^*$ is an eigenvector. Conversely, if $\tilde{\lambda}=1$, from the first equation of (19), it holds $\tilde{B}^{-1}E\tilde{z}=0$, substituting it into the second equation of (19), we obtain $\tilde{z}=0$. Thus, eigenvectors corresponding to eigenvalue 1 are of the form $(\tilde{y};0)$, moreover, we can replace $\tilde{y}$ by e_i, $i=1,2,...,p$, the coordinate vectors of $\mathbb{C}^p$. In the case that $\tilde{\lambda}\neq 1$, from the first equation of (19), we get

$$\tilde{y}=\frac{1}{\tilde{\lambda}-1}\tilde{B}^{-1}E\tilde{z},$$

and by using the second equation of (19), it follows that the eigenvectors corresponding to nonunit eigenvalues are of the form $(\frac{1}{\tilde{\lambda}-1}\tilde{B}^{-1}E\tilde{z};\tilde{z})$, where $\tilde{z}$ is the eigenvector of matrix $(\beta I+Q)^{-1}E^*\tilde{B}^{-1}E$. □

For Krylov subspace methods, such as GMRES, from the proposition 2 in [7], we know that the solution $x^{(k)}$ produced by GMRES at step k is exact if and only if the degree of the minimal polynomial of the initial residual vector $r_0=\tilde{M}(\alpha,\beta)^{-1}b-\tilde{M}(\alpha,\beta)^{-1}Ax^{(0)}$ is equal to k. This means the GMRES method will terminate when the degree of the minimal polynomial is attained. Moreover, based on the proposition 6.2 in [19], the degree of the minimal polynomial of the initial residual vector is equal to the dimension of the corresponding Krylov subspace $\mathcal{K}(\tilde{M}(\alpha,\beta)^{-1}A,r^{(0)})$. Therefore, we analyze the degree of the minimal polynomial of the initial residual vector $r^{(0)}$.

Theorem 3.5. *The degree of the minimal polynomial of the preconditioned matrix $\tilde{M}(\alpha,\beta)^{-1}A$ is at most $q+1$. Thus, the dimension of the Krylov subspace $\mathcal{K}(\tilde{M}(\alpha,\beta)^{-1}A,r^{(0)})$ is at most $q+1$.*

Proof. Based on the form of the preconditioned matrix $\tilde{M}(\alpha,\beta)^{-1}A$ and the eigenvalue distribution described in Theorem 3.4, we know that the characteristic polynomial of $\tilde{M}(\alpha,\beta)^{-1}A$ is

$$(\tilde{M}(\alpha,\beta)^{-1}A-I)^n\prod_{i=1}^{m}(\tilde{M}(\alpha,\beta)^{-1}A-\theta_i I).$$

By expanding the above polynomial $(\tilde{M}(\alpha,\beta)^{-1}A-I)\prod_{i=1}^{m}(\tilde{M}(\alpha,\beta)^{-1}A-\theta_i I)$, we get

$$(\tilde{M}(\alpha,\beta)^{-1}A-I)\prod_{i=1}^{m}(\tilde{M}(\alpha,\beta)^{-1}A-\theta_i I)=\begin{pmatrix} O & \tilde{B}^{-1}E\prod_{i=1}^{m}(\Delta-\theta_i I) \\ O & (\Delta-I)\prod_{i=1}^{m}(\Delta-\theta_i I)\end{pmatrix} \tag{20}$$

wherein $\Delta=(\beta I+Q)^{-1}E^*\tilde{B}^{-1}E$. Since the $\theta_i (i=1,2,...,q)$ are eigenvalues of Δ, it follows that

$$\prod_{i=1}^{m}(\Delta-\theta_i I)=0.$$

Thus, the degree of the minimal polynomial of $\tilde{M}(\alpha,\beta)^{-1}A$ is at most $q+1$, which means that the dimension of the corresponding Krylov subspace $\mathcal{K}(\tilde{M}(\alpha,\beta)^{-1}A,r^{(0)})$ is at most $q+1$.

If the matrix Δ has k distinct eigenvalues θ_i of multiplicity δ_i, we can write the characteristic polynomial of $\tilde{M}(\alpha,\beta)^{-1}A$ as

$$(\tilde{M}(\alpha,\beta)^{-1}A-I)^{n-1}\left[\prod_{i=1}^{m}(\tilde{M}(\alpha,\beta)^{-1}A-\theta_i I)^{(\delta_i-1)}\right]$$
$$\times(\tilde{M}(\alpha,\beta)^{-1}A-I)\prod_{i=1}^{m}(\tilde{M}(\alpha,\beta)^{-1}A-\theta_i I).$$

By expanding $(\tilde{M}(\alpha,\beta)^{-1}A-I)\prod_{i=1}^{m}(\tilde{M}(\alpha,\beta)^{-1}A-\theta_i I)$, we can get the same form as (20). □

Theorem 3.5 indicates that, if the preconditioned Krylov subspace iteration methods with an optimal or Galerkin property [19] are used to solve the preconditioned saddle point linear system with coefficient matrix $\tilde{M}(\alpha,\beta)^{-1}A$, they will converge to the exact solution of the original linear system within at most $q+1$ iterations or less.

4. Numerical results

In this section, we present a numerical example arising from the Stokes problem to examine the convergence behavior of the ARHSS iteration method for solving saddle point problems. That is compared with the HSS preconditioner and RHSS preconditioner. All computations are carried out in MATLAB (version 8.3.0.532 (R2014a)) on a personal computer with 2.60 GHz central processing unit (Intel(R) Core(TM) i5-4210) and 4.00G memory. Starting with zero initial guesses, we compare the tested methods from the point of view of the total iteration steps (denoted by 'IT') and elapsed CPU times in seconds (denoted by 'CPU'). All iteration processes are terminated when the current residuals satisfy $\| b-Ax^{(k)}\|_2\leq 10^{-5}\times\| b\|_2$, with $x^{(k)}$ the current approximation solutions.

Example 4.1. [20] *Consider the Stokes problem: Find u and p such that*

$$\begin{cases} -\Delta u+\nabla p=\tilde{f}, \\ \nabla . u=0, \end{cases} \tag{21}$$

under the boundary and the normalization conditions $u=0$ on $\partial\Omega$ and $\int_\Omega=0$, where $\Omega=(0,1)\times(0,1)\subset\mathbb{R}^2$, $\partial\Omega$ is the boundary of Ω, Δ is the componentwise Laplacian operator, ∇ and $\nabla.$ denote the gradient and the divergence operators, u is a vector-valued function representing the velocity, $\tilde{f}$ is a vector-valued function that represent the external forces applied to the fluid and p is a scalar function representing the pressure. By discretizing this problem with the upwind finite-difference scheme, we obtain the saddle-point linear system (1), in which

$$B=\begin{pmatrix} I\otimes\Upsilon+\Upsilon\otimes I & O \\ O & I\otimes\Upsilon+\Upsilon\otimes I\end{pmatrix} \quad\text{and}\quad E=\begin{pmatrix} I\otimes\Psi \\ \Psi\otimes I\end{pmatrix},$$

with

$$\Upsilon = \frac{1}{h^2}\text{tridiag}(-1,2,-1) \in \mathbb{R}^{m\times m} \quad \text{and} \quad \Psi = \frac{1}{h}\text{tridiag}(-1,1,0) \in \mathbb{R}^{m\times m},$$

and $f = (1,\ldots,1)^T \in \mathbb{R}^{2m^2}$ *and* $g = (0,\ldots,0)^T \in \mathbb{R}^{m^2}$ *being constant vectors. Here* $h = \frac{1}{m+1}$ *represents the discretization stepsize and* $\otimes$ *denotes the Kronecker product symbol.*

In order to give better performance, before solving the Example 1, we first scale the coefficient matrix A by the matrix $D = diag(diag(B), I)$, which results its nonzero diagonal entries are all equal to 1. In our implementations, we use the Cholesky factorization to solve the sub-systems with the coefficient matrices $\alpha I + B$, $\alpha I + \frac{1}{\alpha}E^*E$, $\beta I + Q$ and $\beta I + Q + \frac{1}{\alpha}E^*E$ in both HSS, RHSS methods and ARHSS method. And we choose the regularization matrix $Q = \gamma E^*E$ to implement the ARHSS iteration methods, where γ is a regularization parameter. In addition, we choose $Q = I$ for GMRES method while it use preconditioners $M(\alpha,\beta)$ and $\tilde{M}(\alpha,\beta)$.

In Table 1, we list performances of the HSS, RHSS methods and ARHSS method for Example 1 with different size. The results indicate that the ARHSS method is better than the RHSS method and HSS method.

Table 1. IT, CPU and RES for Example 1

Method	Index	$m = 16$	$m = 32$	$m = 64$
HSS	α_{opt}	0.38	0.27	0.21
	IT	91	149	245
	CPU	0.041	0.125	0.622
RHSS	α_{opt}	0.18	0.14	0.10
	γ_{opt}	4	3	3
	IT	59	100	152
	CPU	0.032	0.099	0.322
ARHSS	α_{opt}	0.18	0.14	0.1
	β_{opt}	0.10	0.13	0.15
	γ_{opt}	4	3	3
	IT	55	95	131
	CPU	0.013	0.074	0.097

Table 2. Results for Example 1 as preconditiners with m=16.

Method	α	β	IT	CPU	RES
GMRES(10)	–	–	204	3.12	7.28e-06
HSS-GMRES(10)	0.01	–	10	2.41	6.23e-06
RHSS-GMRES(10)	0.01	–	8	1.23	6.23e-06
ARHSS-GMRES(10)	0.01	100	7	1.00	1.23e-06
IARHSS-GMRES(10)	0.01	150	3	0.12	8.53e-07

In Table 2,3, we use the preconditioned GMRES method with different preconditioners to solve Example 5.1. From these results, we find that ARHSS-GMRES and IARHSS-GMRES are better than RHSS-GMRES method, and the iteration counts and CPU time decrease with the decreasing of m for both three preconditioned GMRES methods.

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TABLE 3. Results for Example 1 as preconditiners with m=32.

Method	α	β	IT	CPU	RES
GMRES	–	–	352	21.08	9.256e-6
HSS-GMRES	0.01	–	120	15.10	5.574e-6
RHSS-GMRES	0.01	–	43	10.50	9.256e-6
ARHSS-GMRES	0.01	100	15	8.14	9.256e-6
IARHSS-GMRES(10)	0.01	200	4	7.21	9.256e-7

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