U.P.B. Sci. Bull., Series A, Vol. 83, Iss. 1, 2021 ISSN 1223-7027

φ-BIPROJECTIVE AND φ-APPROXIMATE HELEMSKII BIFLAT BANACH ALGEBRAS

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This study aim to introduce the concept of φ-approximate Helemskii biflat Banach algebra A, where φ is a continuous Banach algebra homomorphism. Then a relation between φ-biprojectivity and φ-Helemskii biflatness with φ-amenability is proved. At the end, it is show that $l^1(\mathbb{N}_\wedge)$ is a φ-approximate Helemskii biflat Banach algebra.

Keywords: φ-biprojective, φ-approximate Helemskii biflat, φ-amenability.

1. Introduction

Amenable Banach algebras were introduced by Johnson in [8]. He showed that A is an amenable Banach algebra if and only if A has a virtual diagonal, that is, for some M in $(A\widehat{\otimes}A)^{**}$ we have $M\cdot a=a\cdot M$ and $\pi^{**}(M)\cdot a=a$ for all $a\in A$. The notions of biflat and biprojective Banach algebra were introduced by Helemskii [6, 7]. In fact, a Banach algebra A is called biprojective if there exists a bounded A-bimodule map $\theta:A\longrightarrow A\widehat{\otimes}A$ such that $\pi\circ\theta=id_A$, where π is the product morphism from $A\widehat{\otimes}A$ into A defined by $\pi(a\otimes b)=ab$ for all $a,b\in A$ and id_A denotes the identity map on A. He proved that a Banach algebra A is amenable if A is biflat and it has a bounded approximate identity [5, 7]. In fact, A is called biflat if there is a bounded A-bimodule map $\theta:(A\widehat{\otimes}A)^*\longrightarrow A^*$ such that $\theta\circ\pi^*=id_{A^*}$.

Recently, some authors have added a kind of twist to the amenability definition. Given a continuous homomorphism φ from A into A. The authors in [10, 11, 4] defined and studied φ-derivations and φ-amenability.

Motivated by this consideration the authors in [13] introduced a generalization of Helemskiis concept like φ-biprojectivity, where φ is a continuous Banach algebra homomorphism. They stated a Banach algebra A is φ-biprojective if there exists a bounded φ-A-bimodule homomorphism $\theta:A\longrightarrow(A\widehat{\otimes}A)$ such that $\pi\circ\theta=\varphi$.

In this paper, we introduce a new concept of φ-approximate biflatness and then a relation between φ-biflatness and φ-amenability is proved. We show that $l^1(\mathbb{N}_\wedge)$ is φ-approximate Helemskii biflat if we choose $\varphi\in Hom(l^1(\mathbb{N}_\wedge))$ such

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that φ has dense range, then $l^1(\mathbb{N}_\wedge)$ is not φ-Helemskii biflat and so $l^1(\mathbb{N}_\wedge)$ is not biflat Banach algebra. Finally, we give an examples of φ-biprojective Banach algebra which is not biprojective.

2. φ-biprojective Banach algebras

Let A be a Banach algebra and $\varphi \in Hom(A)$, where $Hom(A)$ contains all continuous homomorphisms from A into itself. Let X and Y be Banach A-bimodules, a φ-A-bimodule morphism from X into Y is a morphism $T: X \longrightarrow Y$ such that for every $a \in A,\ x \in X$ and $\varphi \in Hom(A)$, we have

$$T(a \cdot x) = \varphi(a) \cdot T(x), \qquad T(x \cdot a) = T(x) \cdot \varphi(a).$$

In the following result, $\varphi \in Hom(A)$ and I is a closed ideal of A which is φ-invariant, that is, $\varphi(I) \subset I$, and also we consider the map $\tilde{\varphi}: A/I \longrightarrow A/I$ defined by $\tilde{\varphi}(a + I) = \varphi(a) + I$.

Theorem 2.1. Suppose that A is a φ-biprojective Banach algebra. If I is a closed ideal of A, then A/I is $\tilde{\varphi}$-biprojective.

Proof. Let $\theta: A \longrightarrow (A \widehat{\otimes} A)$ be a φ-A bimodule morphism such that $\pi \circ \theta = \varphi$ and $q: A \longrightarrow A/I$ be the quotient map. We define the map $\tilde{\theta}: A/I \longrightarrow (A/I \widehat{\otimes} A/I)$ by $a + I \mapsto (q \widehat{\otimes} q) \circ \theta(a)$ $(a \in A)$ and prove that $\tilde{\theta}$ is an $\tilde{\varphi}$-A/I- bimodule map. To do this, take $a, b, c \in A$, so we get

$$\begin{aligned}
&\tilde{\theta}((a + I)(b + I)(c + I)) = \tilde{\theta}(abc + I) \\
&= (q \widehat{\otimes} q) \circ \theta(abc) \\
&= (q \widehat{\otimes} q)(\varphi(a) \cdot \theta(b) \cdot \varphi(c)) \\
&= (\varphi(a) + I) \cdot \tilde{\theta}(b + I) \cdot (\varphi(c) + I) \\
&= \tilde{\varphi}(a + I) \cdot \tilde{\theta}(b + I) \cdot \tilde{\varphi}(c + I),
\end{aligned}$$

and also we have

$$\begin{aligned}
&\pi_{A/I} \circ \tilde{\theta} = \pi_{A/I} \circ (q \widehat{\otimes} q) \circ \theta(a) \\
&= q \circ \pi_A \circ \theta(a) \\
&= q(\varphi(a)) = \varphi(a) + I = \tilde{\varphi}(a + I).
\end{aligned}$$

That is, A/I is $\tilde{\varphi}$-biprojective.

Theorem 2.2. Suppose that A is a φ-biprojective Banach algebra. If I is a closed ideal of A with one sided bounded approximate identity and $\varphi(I) \subset I$, then I is $\varphi|_I$-biprojective.

Proof. Assume that $\theta: A \longrightarrow (A \widehat{\otimes} A)$ is a φ-A bimodule morphism such that $\pi \circ \theta = \varphi$. Let $\iota: I \hookrightarrow A$ be the inclusion map. Then $\theta|_I = \theta \circ \iota: I \longrightarrow (A \widehat{\otimes} A)$ is a $\varphi|_I$-I-bimodule homomorphism. We put $I^3 = \overline{span\{abc:\ a,b,c \in I\}}$. Since I has a one sided bounded approximate identity, it follows that $I^3 = I$. Moreover, we have

$$\begin{aligned}\theta|_I &= \theta(I)\\ &= \theta(I^3)\\ &\subseteq span\{a \cdot \theta(b) \cdot c\}^-\\ &\subseteq span\{a \cdot m \cdot c: a, c \in I, m \in A \widehat{\otimes} A\}^- \subseteq I \widehat{\otimes} I.\end{aligned}$$

Therefore, for every $a \in I$, we get

$$\begin{aligned}\pi \circ \theta|_I(a) &= \pi(\theta(a))\\ &= \varphi(a).\end{aligned}$$

Recall that a character on A is a non-zero homomorphism from A into the scalar field. The set of all characters on A is called the character space of A and is denoted by Φ_A.

Proposition 2.1. Let A be a unital Banach algebra, B be a Banach algebra containing a non-zero idempotent b_0, $\varphi \in Hom(A)$, and $\psi \in Hom(B)$. If $A \widehat{\otimes} B$ is $\varphi \otimes \psi$-biprojective, then A is φ-biprojective.

Proof. There is a $\varphi \otimes \psi - A \widehat{\otimes} B$-bimodule $\theta: A \widehat{\otimes} B \longrightarrow (A \widehat{\otimes} B) \widehat{\otimes} (A \widehat{\otimes} B)$ with $\pi_{A \widehat{\otimes} B} \circ \theta = \varphi \otimes \psi$. We consider $A \widehat{\otimes} B$ as an A-bimodule with the actions defined by

$$a_1 \cdot (a_2 \otimes b) = a_1 a_2 \otimes b, \quad and \quad (a_2 \otimes b) \cdot a_1 = a_2 a_1 \otimes b \quad (a_1, a_2 \in A,\ b \in B).$$

Thus for every $a_1, a_2 \in A$, we have

$$\begin{aligned}\theta(a_1 a_2 \otimes b_0) &= \theta((a_1 \otimes b_0)(a_2 \otimes b_0))\\ &= (\varphi(a_1) \otimes \psi(b_0)) \cdot \theta(a_2 \otimes b_0)\\ &= \varphi(a_1) \cdot (e_A \otimes b_0) \cdot \theta(a_2 \otimes b_0)\\ &= \varphi(a_1) \cdot \theta(a_2 \otimes b_0).\end{aligned}$$

Similarly, we can show a right-module version of this equation. So, we have

$$\theta(a_1 a_2 \otimes b_0) = \varphi(a_1) \cdot \theta(a_2 \otimes b_0) = \theta(a_1 \otimes b_0) \cdot \varphi(a_2) \quad (a_1, a_2 \in A).$$

We take $f \in \Phi_A$ with $f(b_0) = 1$ and define

$$\rho: (A \widehat{\otimes} B) \widehat{\otimes} A \widehat{\otimes} B \longrightarrow (A \widehat{\otimes} A), \quad (a_1 \otimes b_1) \otimes (a_2 \otimes b_2) \mapsto f(b_1 b_2) a_1 \otimes a_2$$

Then ρ is a φ-A-bimodule morphism.

We now define $\tilde{\theta}: A \longrightarrow (A \widehat{\otimes} A)$ by

$$\tilde{\theta}(a) = \rho \circ \theta(a \otimes b_0), \quad (a \in A).$$

Thus $\tilde{\theta}$ is a φ-A-bimodule morphism and also

$$\pi_A \circ \rho = (id_A \otimes f) \circ \pi_{A\widehat{\otimes}B}.$$

Therefore,

$$\begin{aligned} \pi_A \circ \tilde{\theta}(a) &= \pi_A \circ \rho \circ \theta(a \otimes b_0) \\ &= (id_A \otimes f) \circ \pi_{A\widehat{\otimes}B} \circ \theta(a \otimes b_0) \\ &= \varphi(a). \end{aligned}$$

That is, A is φ-biprojective.

Let A be a φ-biprojective Banach algebra and B be a ψ-biprojective Banach algebra, where $\varphi \in Hom(A)$ and $\psi \in Hom(B)$. Then $A \widehat{\otimes} B (A \oplus B)$ is a $\varphi \otimes \psi (\varphi \oplus \psi)$- biprojective Banach algebra.

Here, we now give an examples of φ-biprojective Banach algebra which is not biprojective.

Example 2.1. Let $\mathcal{V}$ be a Banach space, and let $f \in \mathcal{V}^*$ be a non-zero element such that $\| f \| \leq 1$. Then $\mathcal{V}$ equipped with the product defined by $ab := f(a)b$ for $a, b \in \nu$ is a Banach algebra which is denoted by $\mathcal{V}_f$. In general, $\mathcal{V}_f$ is a non- commutative and non-unital Banach algebra without right approximate identity, but it is not amenable. Hence, $(\mathcal{V}_f)^\sharp$ (unitization of $\mathcal{V}_f$) is not biprojective. If we define $\varphi: (\mathcal{V}_f)^\sharp \longrightarrow (\mathcal{V}_f)^\sharp$ by $\varphi(a + \lambda e) = \lambda$ for $a \in \mathcal{V}_f$ and $\lambda \in \mathbb{C}$, then Example 3.2 [10] shows that $(\mathcal{V}_f)^\sharp$ is a φ-contactible Banach algebra. Since φ is an idempotent homomorphism, thus by Theorem 4.3 [13] $(\mathcal{V}_f)^\sharp$ is φ-biprojective.

Example 2.2. The Banach algebra l^1 with respect to pointwise product is non-amenable and biprojective Banach algebra [3, Example 4.1.42]. Hence, $(l^1)^\sharp$ (unitization of l^1) is not biprojective. If we define $\varphi: (l^1)^\sharp \longrightarrow (l^1)^\sharp$ by $\varphi(a + \lambda e) = \lambda$ for $a \in l^1$ and $\lambda \in \mathbb{C}$, then Example 3.2 [10] shows that $(l^1)^\sharp$ is a φ-contactible Banach algebra, since φ is an idempotent homomorphism, hence by Theorem 4.3 [13], $(l^1)^\sharp$ is φ-biprojective.

Definition 2.1. Let A be a Banach algebra and φ be a continuous homomorphism from A into A. Then A is called φ-Helemskii biflat if there exists a φ-A-bimodule homomorphism $\theta: A \longrightarrow (A \widehat{\otimes} A)^{**}$ such that $\pi^{**} \circ \theta = \kappa_A \circ \varphi$, where κ_A is the canonical embedding map of A into A^{**}.

The notion of φ-biflat in [4] gave a slight generalization of biflat. The above definition is a new concept of φ-biflatness as a generalization of the notion of biflatness, because for every $\varphi \in Hom(A)$, any biflat Banach algebra is φ-Helemskii biflat.

Proposition 2.2. Let A be a unital Banach algebra and B be a Banach algebra containing a non-zero idempotent b_0. If $A \widehat{\otimes} B$ is $\varphi \otimes \psi$-Helemskii biflat for $\varphi \in Hom(A)$ and $\psi \in Hom(B)$, then A is φ-Helemskii biflat.

Proof. There is a $\varphi \otimes \psi - A \widehat{\otimes} B$-bimodule $\theta: A \widehat{\otimes} B \longrightarrow (A \widehat{\otimes} B) \widehat{\otimes} (A \widehat{\otimes} B)^{**}$ with $\pi^{**}_{A\widehat{\otimes}B} \circ \theta = \kappa_{A\widehat{\otimes}B} \circ (\varphi \otimes \psi)$.

The following proof is similar to the proof of Proposition 2.1. We now define $\tilde{\theta}: A \longrightarrow (A \widehat{\otimes} A)^{**}$ by

$$\tilde{\theta}(a) = \rho^{**} \circ \theta(a \otimes b_0) \qquad (a \in A).$$

Then $\tilde{\theta}$ is a φ-A-bimodule morphism and

$$\pi^{**}_A \circ \tilde{\theta} = \kappa_A \circ \varphi.$$

That is, A is φ-Helemskii biflat.

Let A be a φ-Helemskii biflat Banach algebra and B be a ψ-Helemskii biflat Banach algebra, where $\varphi \in Hom(A)$ and $\psi \in Hom(B)$. Then $A \widehat{\otimes} B (A \oplus B)$ is a $\varphi \otimes \psi(\varphi \oplus \psi)$-Helemskii biflat Banach algebra.

Proposition 2.3. Let A be a Banach algebra. If A is φ-Helemskii biflat, then A is $\varphi \circ \psi$-Helemskii biflat for any $\psi \in Hom(A)$. In particular, A is φ^n-Helemskii biflat.

Proof. There exists a φ-A-bimodule homomorphism $\theta: A \longrightarrow (A \widehat{\otimes} A)^{**}$ such that $\pi^{**} \circ \theta = \kappa_A \circ \varphi$. We define $\tilde{\theta} = \theta \circ \psi$. For every $a, b \in A$, we have

$$\begin{aligned} \tilde{\theta}(ab) &= \theta \circ \psi(ab) \\ &= \theta(\psi(a)\psi(b)) \\ &= \varphi \circ \psi(a)\theta(\psi(b)) = \varphi \circ \psi(a)\tilde{\theta}(a). \end{aligned}$$

and also

$$\begin{aligned} \pi_A \circ \tilde{\theta}(a) &= \pi_A \circ \theta \circ \psi(a) \\ &= \varphi \circ \psi(a). \end{aligned}$$

That is, A is $\varphi \circ \psi$-Helemskii biflat.

Let A be a Banach algebra. If A is a φ-biprojective, then A is a φ-Helemskii biflat Banach algebra. The next result can be found in [10].

Lemma 2.1. Let A be a Banach algebra. Then there is an A-bimodule homomorphism $\gamma: (A \widehat{\otimes} A)^* \longrightarrow (A^{**} \otimes A^{**})^*$ such that for any functional $f \in (A \widehat{\otimes} A)^*$, elements $\varphi, \psi \in A^{**}$ and nets $(a_\alpha), (b_\beta)$ in A with $w^* - lim_\alpha a_\alpha = \varphi$ and $w^* - lim_\beta b_\beta = \psi$, we have

$$\gamma(f)(\varphi \otimes \psi) = lim_\alpha lim_\beta f(a_\alpha \otimes b_\beta).$$

Theorem 2.3. Suppose that A is a Banach algebra and $\varphi \in Hom(A)$. If A^{**} is φ^{**}-biprojective, then A is φ-Helemskii biflat.

Proof. Let $\kappa: A \longrightarrow A^{**}$, $\kappa_1: A^* \longrightarrow A^{***}$ and $\kappa_*: A^{**} \longrightarrow A^{****}$ denotes the natural inclusion π ($^{**}\pi$, respectively) product map on A (A^{**}, respectively) and also γ be defined as in Lemma 2.1 Then for any $a^* \in A^*$, elements $a_1^{**}, a_2^{**} \in A^{**}$ and nets $(a_\alpha), (b_\beta) \subset A$ with $w^* - \lim_\alpha a_\alpha = a_1^{**}, w^* - \lim_\beta b_\beta = a_2^{**}$, we obtain

$$
\begin{aligned}
&(\gamma(\pi^*(a^*)))(a_1^{**} \otimes a_2^{**}) = \lim_\alpha \lim_\beta \pi^*(a^*)(a_\alpha \otimes b_\beta) \\
&= \lim_\alpha \lim_\beta a^*(a_\alpha b_\beta) \\
&= w^* - \lim_\alpha w^* - \lim_\beta \kappa(a_\alpha b_\beta)(a^*) \\
&= \kappa_1(a^*)(a_1^{**} a_2^{**}) \\
&= \kappa_1(a^*)(^{**}\pi(a_1^{**} \otimes a_2^{**})) = (^{**}\pi^*(\kappa_1(a^*)))(a_1^{**} \otimes a_2^{**}).
\end{aligned}
$$

Thus $\gamma \circ \pi^* =^{**} \pi^* \circ \kappa_1$ and then $\pi^{**} \circ \gamma^* = \kappa_1^* \circ^{**} \pi^{**}$. Since A^{**} is $\varphi^{**} -$ biprojective, so there is a φ^{**}-A-bimodule map $\theta_0: A^{**} \longrightarrow (A^{**} \widehat{\otimes} A^{**})$ such that $\pi \circ \theta_0 = \varphi^{**}$. By putting $\theta: = \gamma^* \circ \theta_0 \circ \kappa$, then for every $a \in A$, we have

$$
\begin{aligned}
&\pi^{**} \circ \theta(a) = \pi^{**} \circ \gamma^* \circ \theta_0 \circ \kappa(a) \\
&= \kappa_1^* \circ \pi^{**} \circ \theta_0 \circ \kappa \circ \varphi(a) \\
&= \kappa_1^*(\varphi^{**}(a) = \varphi(a).
\end{aligned}
$$

That is, A is φ-Helemskii biflat.

Theorem 2.4. Suppose that A is a φ-Helemskii biflat Banach algebra with bounded approximate identity. If φ is a dense range map, then A is an amenable Banach algebra.

Proof. Let $\theta: A \longrightarrow (A \widehat{\otimes} A)^{**}$ be a bounded φ-A-bimodule map such that $\pi^{**} \circ \theta = \kappa_A \circ \varphi$. Let (e_α) be a bounded approximate identity for A and we define $M = w^* - \lim_\alpha \theta(e_\alpha)$. Then we get

$$
\begin{aligned}
&\varphi(a) \cdot M = w^* - \lim_\alpha \varphi(a) \cdot \theta(e_\alpha) \\
&= w^* - \lim_\alpha \theta(a e_\alpha) \\
&= w^* - \lim_\alpha \theta(e_\alpha a) \\
&= w^* - \lim_\alpha \theta(e_\alpha) \cdot \varphi(a) \\
&= M \cdot \varphi(a).
\end{aligned}
$$

and so

$$
\begin{aligned}
&\pi^{**}(M) \cdot \varphi(a) = w^* - \lim_\alpha \pi^{**} \circ \theta(e_\alpha) \cdot \varphi(a) \\
&= w^* - \lim_\alpha \kappa_A \circ \varphi(e_\alpha) \cdot \varphi(a) = \varphi(a).
\end{aligned}
$$

Let $a \in A$ and take a sequence $(a_n)_n \subseteq A$ such that $\lim_n \varphi(a_n) = a$. Hence

$$
a \cdot M = \lim_n \varphi(a_n) \cdot M
$$

$$= \lim_n M \cdot \varphi(a_n)$$
$$= M \cdot a.$$

and so $\pi^{**}(M) \cdot a = a$.

Thus M is a virtual diagonal for A. Hence by Theorem 2.9.65 [3] A is amenable.

Corollary 2.1. Suppose that A is a φ-Helemskii biflat Banach algebra with bounded approximate identity. If φ is a dense range map, then A is a φ-amenable Banach algebra.

Let A be a Banach algebra. An element M in $(A \widehat{\otimes} A)^{**}$ is said to be a φ-virtual diagonal for A if $M \cdot \varphi(a) = \varphi(a) \cdot M$ and $\pi^{**}(M) \cdot \varphi(a) = \varphi(a)$ for every $a \in A$.

Theorem 2.5. Suppose that A is a φ-amenable Banach algebra with bounded approximate identity. Then A is a φ-Helemskii biflat Banach algebra

Proof. Let A be a φ-amenable Banach algebra with bounded approximate identity. By Theorem 2.2 [4] A has a φ -virtual diagonal M. We define $\theta: A \longrightarrow (A \widehat{\otimes} A)^{**}$ by $a \mapsto \varphi(a) \cdot M$. For every $a \in A$, we have

$$\pi^{**} \circ \theta(a) = \pi^{**} \circ (\varphi(a) \cdot M)$$
$$= \pi^{**} \circ (M \cdot \varphi(a))$$
$$= \varphi(a) = \kappa_A \circ \varphi(a).$$

That is, A is a φ-Helemskii biflat Banach algebra.

We now give an example of φ-Helemskii biflat Banach algebra which is not biflat.

Example 2.3. Note that the Banach algebra l^1 is a biflat Banach algebra because it is biprojective [3, see Example 4.1.42]. So $(l^1)^\sharp$ is a non-amenable [12]. By [5], $(l^1)^\sharp$ is not biflat and if we define $\varphi: (l^1)^\sharp \longrightarrow (l^1)^\sharp$ by $\varphi(a + \lambda e) = \lambda$ for $a \in l^1 and\ \lambda \in \mathbb{C}$, then Example 3.2 [10] shows that $(l^1)^\sharp$ is a φ-amenable Banach algebra. By Theorem 2.5, $(l^1)^\sharp$ is a φ-Helemskii biflat Banach algebra.

The next example shows that a φ-Helemskii biflat Banach algebra need not to be a φ-amenable.

Example 2.4. The Banach algebra l^1 is non-amenable and a biflat Banach algebra [3, see Example 4.1.42]. Then for any $\varphi \in Hom(l^1)$, l^1 is φ-Helemskii biflat. If we choose $\varphi \in Hom(l^1)$ such that φ be an epimorphism, then l^1 is not φ-amenable, by Proposition 2.3 [11].

3. φ-approximate Helemskii biflat Banach algebras

We start this section by introducing the following:

Denition 3.1. Let A be a Banach algebra. Then A is called φ-approximate Helemskii biflat if there is a net $\theta_\alpha: A \longrightarrow (A \widehat{\otimes} A)^{**}$, $(\alpha \in I)$ of φ-A-bimodule morphisms such that $\pi^{**} \circ \theta_\alpha(a) \to \varphi(a)$.

Theorem 3.1. Suppose that A is a φ-approximate Helemskii biflat Banach algebra with one sided bounded approximate identity. If I is a closed ideal of A, then A/I is $\tilde{\varphi}$-approximate Helemskii biflat.

Proof. Let $\theta_\alpha: A \longrightarrow (\mathrm{A} \widehat{\otimes} A)^{**}$ be a bounded φ-A-bimodule map such that $\lim_\alpha \pi^{**} \circ \theta_\alpha(a) = \varphi(a)$ and $q: A \longrightarrow A/I$ be the quotient map. We define the map $\tilde{\theta}_\alpha: A/I \longrightarrow (\mathrm{A}/I \widehat{\otimes} A/I)^{**}$ by $a + I \mapsto (\mathrm{q} \widehat{\otimes} q)^{**} \circ \theta_\alpha(a)$ $(a \in A)$. Hence $\tilde{\theta}_\alpha$ is well-defined. If (e_β) is a bounded left approximate identity for A (the right case is similar), then

$$\| (\mathrm{q} \widehat{\otimes} q)^{**}(\theta_\alpha(a)) \| = \lim_\beta \| (\mathrm{q} \widehat{\otimes} q)^{**}(\theta_\alpha(e_\beta a)) \|$$
$$= \lim_\beta \| q(a)(q \widehat{\otimes} q)^{**}(\theta_\alpha(e_\beta)) \|$$
$$\leq \| q \|^2 \| \theta_\alpha \| \sup_\beta \| e_\beta \| \| q(a) \|.$$

We also obtain

$$\lim_\alpha \pi_{A/I}^{**} \circ \tilde{\theta}_\alpha(a + I) = \lim_\alpha \pi_{A/I}^{**} \circ (q \widehat{\otimes} q)^{**} \circ \theta_\alpha(a)$$
$$= \lim_\alpha q^{**} \circ \pi_A^{**} \circ \theta_\alpha(a) \longrightarrow q(\varphi(a)) = \tilde{\varphi}(a + I).$$

That is, A/I is $\tilde{\varphi}$-approximate Helemskii biflat.

Since the proof of the next result is similar to Theorem 2.3, so we omit it.

Theorem 3.2. Suppose that A is a Banach algebra. If A^{**} is φ^{**}-approximate Helemskii biflat, then A is φ-approximate Helemskii biflat.

Definition 3.2. Let A be a Banach algebra and $\varphi \in Hom(A)$. We say that A is $\varphi -$ pseudo amenable if A admits a $\varphi -$ approximate virtual diagonal, i.e., there is a net $(m_\alpha) \subset \mathrm{A} \widehat{\otimes} A$ (not necessary bounded) such that $m_\alpha \cdot \varphi(a) - \varphi(a) \cdot m_\alpha \longrightarrow 0$ and $\pi(m_\alpha) \cdot \varphi(a) \longrightarrow \varphi(a)$ $(a \in A)$.

Theorem 3.3. Let A be a Banach algebra with an approximate identity and φ be an idempotent homomorphism on A. Then A is $\varphi -$ pseudo amenable $(\varphi \in Hom(A))$ if and only if A is $\varphi -$ approximate Helemskii biflat.

Proof. Let $(e_\beta)_{\beta \in I}$ be an approximate identity for A and $\theta_\alpha: A \longrightarrow (A \widehat{\otimes} A)^{**}$ $(\alpha \in \triangle)$ satisfies $\pi^{**} \circ \theta_\alpha \circ (a) \to \varphi(a)$ $(a \in A)$. Then for every $a \in A$ and $f \in (A \widehat{\otimes} A)^*$, we obtain

$$\lim_\beta\lim_\alpha\langle f,\theta_\alpha(\varphi(e_\beta))\cdot\varphi(a)-\varphi(a)\cdot\theta_\alpha(\varphi(e_\beta))\rangle =$$
$$\lim_\beta\lim_\alpha\langle f,\theta_\alpha(\varphi(e_\beta)\varphi(a)$$
$$-\varphi(a)\varphi(e_\beta)\ \rangle$$
$$=\lim_\beta\lim_\alpha\langle f,\theta_\alpha\big(\varphi(e_\beta a-ae_\beta)\ \big)\rangle$$
$$=0.$$

Also, for $a\in A$ and $\psi\in A^*$, we get

$$\lim_\beta\lim_\alpha\langle\psi,\varphi(a)\cdot\pi^{**}\circ\theta_\alpha(\varphi(e_\beta))\rangle=\lim_\beta\langle\psi,\varphi(a)\varphi^2(e_\beta)\rangle$$
$$=\lim_\beta\langle\psi,\varphi(ae_\beta)\rangle$$
$$=\langle\psi,\varphi(a)\rangle.$$

Let $E=I\times\triangle^I$ be directed by the product ordering and for any $\lambda=(\beta,\alpha)\in E$, define $m_\lambda=\theta_\alpha(\varphi(e_\beta))$. Using the iterated limit theorem [9, Theorem 2.4], the above calculation gives

$$w^*-\lim_\lambda(m_\lambda\cdot\varphi(a)-\varphi(a)\cdot m_\lambda\ )=0\quad(a\in A),$$

and

$$w^*-\lim_\lambda\varphi(a)\cdot\pi^{**}(m_\lambda)=\varphi(a)\quad(a\in A).$$

By Goldestin's theorem we can assume that $(m_\lambda)\subset(A\widehat{\otimes}A)$ and we can replace weak * convergence in equations by weak convergence. Applying Mazur's theorem, we obtain a net $(m'_\lambda)\subset(A\widehat{\otimes}A)$ of convex combinations of (m_λ) such that

$$m'_\lambda\cdot\varphi(a)-\varphi(a)\cdot m'_\lambda\to 0,$$

and also

$$\varphi(a)\cdot\pi^{**}(m'_\lambda)\to\varphi(a)\quad(a\in A).$$

That is, A is $\varphi-$ pseudo amenable. Conversely, let (m_β) be a $\varphi-$approximate virtual diagonal for A and define $\theta_\beta: A\longrightarrow(A\widehat{\otimes}A)^{**}$ by $a\mapsto\varphi(a)\cdot m_\beta$. Clearly, θ_β is linear and a $\varphi-A-$bimodule morphism. Also, for every $a\in A$, we have

$$\pi^{**}\circ\theta_\beta\circ(a)=\pi^{**}\circ(\varphi(a)\cdot m_\beta)$$
$$=\varphi(a)\pi^{**}(m_\beta)$$
$$\to\varphi(a).$$

Proposition 3.1. Let A be a $\varphi-$amenable Banach algebra and φ be an idempotent homomorphism on A. Then A is φ-approximate Helemskii biflat.

Proof. By Proposition 4.1 [11] A has a bounded approximate identity $(e_\alpha)_{\alpha\in I}$. Let E be a w^* −cluster point of $(\varphi(e_\alpha)\otimes\varphi(e_\alpha))_\alpha$ in $(A\widehat{\otimes}A)^{**}$. We define a $\varphi-$derivation $D: A\longrightarrow (A\widehat{\otimes}A)^{**}$ by $D(a)=\varphi(a)\cdot E-E\cdot\varphi(a)\ (a\in A)$. Then for every $a\in A$

$$\pi^{**}(D(a))=\lim_\alpha \pi[(\varphi(a)(\varphi(e_\alpha)\otimes\varphi(e_\alpha))-(\varphi(e_\alpha)\otimes\varphi(e_\alpha))\varphi(a)]$$
$$=\lim_\alpha \varphi(ae_\alpha)\varphi(e_\alpha)-\varphi(e_\alpha)\varphi(e_\alpha a)$$
$$=\lim_\alpha \varphi(ae_\alpha^2)-\varphi(e_\alpha^2 a)$$
$$=\lim_\alpha \varphi(ae_\alpha^2-e_\alpha^2 a)=0.$$

Therefore, $D(A)\subseteq ker(\pi^{**})=(ker\pi)^{**}$. Thus there exists $N\in(ker\pi)^{**}$ such that $D=ad_{\varphi;N}$. Put $M=E-N$. Then for every $a\in A$, we have

$$\pi^{**}(M)\cdot\varphi(a)=\pi^{**}(E-N)\cdot\varphi(a)=\pi^{**}(E)\cdot\varphi(a)$$
$$=w^*-\lim_\alpha \pi(\varphi(e_\alpha)\otimes\varphi(e_\alpha))\cdot\varphi(a)$$
$$=w^*-\lim_\alpha \varphi(e_\alpha^2 a)=\varphi(a).$$

Let $(m_\alpha)_\alpha$ be a net in $(A\widehat{\otimes}A)$ such that $M=w^*-\lim_\alpha m_\alpha$. Then $w-\lim_\alpha(m_\alpha\cdot\varphi(a)-\varphi(a)\cdot m_\alpha)=0$ and $w-\lim_\alpha(\pi(m_\alpha)\cdot\varphi(a)-\varphi(a))=0$ $(a\in A)$. Following the argument given in the proof of Lemma 2.9.64 [3] we can show that there exists a net $(m_\beta)_\beta$ in $(A\widehat{\otimes}A)$ such that each m_β is a convex combination of m_α's with $m_\beta\cdot\varphi(a)-\varphi(a)\cdot m_\beta\longrightarrow 0$ and $\pi(m_\beta)\cdot\varphi(a)\longrightarrow\varphi(a)$ $(a\in A)$. Thus A is $\varphi-$ pseudo amenable and so by Theorem 3.3 A is φ-approximate Helemskii biflat.

Using the proof of the above proposition we obtain the following results.

Corollary 3.1. Let $L^1(G)$ be a φ −amenable Banach algebra $(\varphi\in Hom(L^1(G)))$ and φ be an idempotent homomorphism on $L^1(G)$. Then $L^1(G)$ is φ-approximate Helemskii biflat.

Denition 3.3. Let A be a Banach algebra with the norm $\|.\|_A$. Then a Banach algebra B with the norm $\|.\|_B$ is said to be an abstract Segal algebra with respect to A if

(i) B is a dense left ideal in A;

(ii) there exists $M>0$ such that $\|b\|_A\leq M\|b\|_B$ for all $b\in B$;

(iii) there exists $C>0$ such that $\|ab\|_B\leq C\|a\|_A\|b\|_B$ for all $a,\ b\in B$.

Theorem 3.4. Let A be a Banach algebra and B be an abstract Segal algebra with respect to A. Suppose that $\varphi\in Hom(B)$ and B contains a net $(e_\alpha)_\alpha$ such that $(\varphi(e_\alpha^2))$ is an approximate identity for B and $a\varphi(e_\alpha)=\varphi(e_\alpha)a$ for all $a\in A$. If A is $\overline{\varphi}$-approximate Helemskii biflat, then B is φ-approximate Helemskii biflat.

Proof. Let $T_\alpha: A\widehat{\otimes} A \longrightarrow B\widehat{\otimes} B$ defined by $a\otimes b \mapsto a\varphi(e_\alpha)\otimes b\varphi(e_\alpha)$. Since A is $\overline{\varphi}$-approximate Helemskii biflat, so there is a net $(\theta_\beta)_\beta$ with $\theta_\beta: A\longrightarrow (A\widehat{\otimes} A)^{**}$ such that $\pi_A^{**}\circ\theta_\beta\circ(a)\rightarrow\overline{\varphi}(a) \quad (a\in A)$. For $\lambda=(\alpha,\gamma)$ we define $\theta_\lambda: B\longrightarrow (B\widehat{\otimes} B)^{**}$ by $\theta_\lambda := T_\alpha^{**}\circ\theta_\beta\circ j$, where $j: B\longrightarrow A$ is the inclusion map. Then θ_λ is a bounded B −bimodul map. Note that because $\varphi(e_\alpha)$ lies in the center of A, $\pi_B^{**}\circ T_\alpha^{**} = R_\alpha^{**}\circ\pi_A^{**}$, where $R_\alpha: A\longrightarrow B$ is defined by $a\mapsto a\varphi(e_\alpha^2) \quad (a\in A)$. Let $b\in B,\ f\in B^*$. By the iterated limit theorem, we have

$$\lim_\lambda\langle f,\pi_B^{**}\circ\theta_\lambda(b)\rangle = \lim_\alpha\lim_\beta\langle f,\pi_B^{**}\circ T_\alpha^{**}\circ\theta_\beta\circ j(b)\rangle$$
$$= \lim_\alpha\lim_\beta\langle f, R_\alpha^{**}\circ\pi_A^{**}\circ\theta_\beta(b)\rangle$$
$$= \lim_\alpha\langle f, R_\alpha^{**}(\varphi(b))\rangle$$
$$= \lim_\alpha\langle f,\varphi(b)\varphi(e_\alpha^2)\rangle$$
$$= \langle f,\varphi(b)\rangle.$$

Hence, $\pi_B^{**}\circ\theta_\lambda(b)\rightarrow\varphi(b) \quad (b\in B)$.

Corollary 3.2. Let A be a Banach algebra and B be an abstract Segal algebra with respect to A. Suppose that $\varphi\in Hom(A)$ and B contains a net $(e_\alpha)_\alpha$ such that $(e_\alpha)^2$ is an approximate identity for B and $a\varphi(e_\alpha)=\varphi(e_\alpha)a$ for all $a\in A$. If $\varphi(B)\subseteq B$ and A is φ-approximate Helemskii biflat, then B is φ-approximate Helemskii biflat.

Denition 3.4. Let A be a Banach algebra and $\varphi\in Hom(A)$. We say that A is $\varphi-$ pseudo contractible if it has a central φ-approximate diagonal, i.e., a φ-approximate diagonal (m_α) satisfying $\varphi(a)m_\alpha = m_\alpha\varphi(a)$ for all $a\in A$ and all α.

Proposition 3.2. For a Banach algebra A two the following statements are equivalent.

i) A is $\varphi-$ pseudo contractible.

ii) A is $\varphi-$ approximate Helemskii biflat and has a central approximate identity.

Proof. (i)⇒ (ii) Suppose that $(m_\alpha)\subset A\widehat{\otimes} A$ is a central φ-approximate diagonal for A. We define $\theta_\alpha: A\longrightarrow (\hat{A}\otimes A)^{**}$ by $\theta_\alpha(a) := \varphi(a)\cdot m_\alpha$. Then for every $a\in A$, we have

$$\lim_\alpha\pi^{**}\circ\theta_\alpha(a) = \lim_\alpha\pi^{**}(\varphi(a)\cdot m_\alpha)$$
$$= \varphi(a).$$

So $\pi(m_\alpha)$ is a central approximate identity for A.

(ii)⇒ (i) Since A is $\varphi-$ approximate Helemskii biflat, there is a net $\theta_\alpha: A\longrightarrow (A\widehat{\otimes} A)^{**}$ $(\alpha\in\triangle)$ such that $\lim_\alpha\pi^{**}\circ\theta_\alpha(a)=\varphi(a)$ $(a\in A)$. Let $(e_\beta)_{\beta\in I}$ be a central approximate identity for A. Let $E=I\times\triangle^I$ be directed by the product ordering and for each $\lambda = (\beta, \alpha) \in E$ define $m_\lambda = \theta_\alpha(e_\beta)$. Then (m_λ) is a central φ-approximate diagonal for A.

Here, we now give an examples of $\varphi -$approximate Helemskii biflat Banach algebra which is not $\varphi -$ Helemskii biflat.

Example 3.1. Consider the semigroup $\mathbb{N}_\wedge$, with the operation semigroup $m \wedge n = \min\{m, n\},\ m, n \in \mathbb{N}$. We know that $(\delta_n)_{n\in\mathbb{N}}$ is a bounded approximate identity for $l^1(\mathbb{N}_\wedge)$. If we define $m_n = \delta_n \otimes \delta_n$, then $\varphi(a) \cdot m_n - m_n \cdot \varphi(a) \longrightarrow 0$ for all $a \in l^1(\mathbb{N}_\wedge)$ and also $\pi(m_n) \cdot \varphi(a) \longrightarrow \varphi(a)$. Terefore $l^1(\mathbb{N}_\wedge)$ is a $\varphi -$ pseudo amenable ($\varphi \in Hom(\mathbb{N}_\wedge)$) and so by Theorem 3.3, $l^1(\mathbb{N}_\wedge)$ is a $\varphi -$ approximate Helemskii biflat. If we choose $\varphi \in Hom(l^1(\mathbb{N}_\wedge))$ such that φ has a dense range, then $l^1(\mathbb{N}_\wedge)$ is not φ-Helemskii biflat and so $l^1(\mathbb{N}_\wedge)$ is not biflat Banach algebra.

Acknowledgements

I gratefully thank the referees for carefully reading the paper and for the suggestions that greatly improved the presentation of the paper.

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