A FIXED POINT APPROACH TO THE STABILITY OF DOUBLE JORDAN CENTRALIZERS AND JORDAN MULTIPLIERS ON BANACH ALGEBRAS

M. Eshaghi Gordji¹, A. Bodaghi², C. Park³

We say a functional equation (ξ) is stable if any function g satisfying the equation (ξ) approximately is near to true solution of (ξ), moreover, a functional equation (ξ) is superstable if any function g satisfying the equation (ξ) approximately is a true solution of (ξ). In the present paper, we investigate the stability and the superstability of double centralizers and of multipliers on Banach algebras by using the fixed point methods.

Keywords: Double centralizer; Multiplier; Hyers-Ulam stability.

MSC2000: 39B82, 39B52.

1. Introduction

In this paper, we assume that A is a complex Banach algebra. A linear mapping $L : A \rightarrow A$ is said to be left Jordan centralizer on A if $L(a^2) = L(a)a$ for all $a \in A$. Similarly, a linear mapping $R : A \rightarrow A$ satisfying that $R(a^2) = aR(a)$ for all $a \in A$ is called right Jordan centralizer on A. A double Jordan centralizer on A is a pair $(L, R)$, where $L$ is a left Jordan centralizer, $R$ is a right Jordan centralizer and $aL(b) = R(a)b$ for all $a, b \in A$. For example, $(L_c, R_c)$ is a double Jordan centralizer, where $L_c(a) := ca$ and $R_c(a) := ac$ (see [9] and [11]).

A mapping $T : A \rightarrow A$ is said to be a Jordan multiplier (see [12]) if $aT(a) = T(a)a$ for all $a \in A$. Clearly, if $A_l(A) = \{0\}$ ($A_r(A) = \{0\}$, respectively) then T is a left (right) Jordan centralizer (see [14]).

In 1940, Ulam [26] proposed the following question concerning stability of group homomorphisms: under what condition does there is an additive mapping near an approximately additive mapping? Hyers [10] answered the problem of Ulam for the case where $G_1$ and $G_2$ are Banach spaces. A generalized version of the theorem of Hyers for approximately linear mapping was given by Th.

¹Department of Mathematics, Semnan University, P. O. Box 35195-363, Semnan, Iran; Center of Excellence in Nonlinear Analysis and Applications (CENAA), Semnan University, Semnan, Iran, e-mail: madjid.eshaghi@gmail.com
²Department of Mathematics, Islamic Azad University, Garmsar Branch, Garmsar, Iran, e-mail: abasalt.bodaghi@gmail.com
³Department of Mathematics, Research Institute for Natural Sciences, Hanyang University, Seoul 133-791, South Korea, e-mail: baak@hanyang.ac.kr
M. Rassias [25]. After that, several functional equations have been extensively investigated by a number of authors (for instances, [4]–[7], [13] and [15]–[24]). In 2003, Cadariu and Radu applied the fixed point method to the investigation of the Jensen functional equation [1] (see also [2, 3, 8]). They presented a short and a simple proof (no “direct method”, initiated by Hyers in 1941) for the Hyers-Ulam stability of the Jensen functional equation [1], for the Cauchy functional equation [3] and for some other functional equations [1, 2, 3].

We need the following known fixed point theorem which is useful for our goals.

**Theorem 1.1.** *(The alternative of fixed point [5])* Suppose \((Ω, d)\) be a complete generalized metric space and let \(J : Ω \to Ω\) be a strictly contractive mapping with Lipschitz constant \(L < 1\). Then for each element \(x \in Ω\), either \(d(J^n x, J^{n+1} x) = \infty\) for all \(n \geq 0\), or there exists a natural number \(n_0\) such that:

(*) \(d(J^n x, J^{n+1} x) < \infty\) for all \(n \geq n_0\);
(**) the sequence \(\{J^n x\}\) is convergent to a fixed point \(y^*\) of \(J\);
(***) \(y^*\) is the unique fixed point of \(J\) in the set 
\[ Λ = \{y \in Ω : d(J^{n_0} x, y) < \infty\}; \]

(* ****) \(d(y, y^*) \leq \frac{1}{1-L} d(y, J y)\) for all \(y \in Λ\).

Moreover, we will use the following known lemma in the proof of main results of our paper. We do not remove the proof of lemma. We suppose that \(T^1 := \{z \in \mathbb{C} : |z| = 1\}\).

**Lemma 1.1.** Let \(n_0 \in \mathbb{N}\) be a positive integer number and let \(X, Y\) be linear vector spaces on \(\mathbb{C}\). Suppose that \(f : X \to Y\) is an additive mapping. Then \(f\) is \(\mathbb{C}\)-linear if and only if \(f(\lambda x) = \lambda f(x)\) for all \(x \in X\) and \(\lambda \) in \(T^1_{\frac{1}{n_0}} := \{e^{iθ} : 0 \leq θ \leq \frac{2π}{n_0}\}\).

**Proof.** Suppose that \(f\) is additive and \(f(\lambda x) = \lambda f(x)\) for all \(x \in X\) and all \(\lambda \) in \(T^1_{\frac{1}{n_0}}\). Now, let \(µ \in T^1\). Then we have \(µ = e^{iθ}\) such that \(0 \leq θ \leq 2π\). We set \(µ_1 = e^{\frac{θ}{n_0}}\),

thus \(µ_1\) belongs to \(T^1_{\frac{1}{n_0}}\) and

\[ f(µ_1 x) = f(µ_1^{n_0} x) = µ_1^{n_0} f(x) = µ f(x) \]

for all \(x \in X\). If \(µ\) belongs to \(nT^1 = \{nz : z \in T^1\}\) then by additivity of \(f\), \(f(µ x) = µ f(x)\) for all \(x \in X\) and \(µ\) in \(nT^1\). If \(t \in (0, \infty)\) then by archimedean property there exists a natural number \(n\) such that the point \((t, 0)\) lies in the interior of circle with center at origin and radius \(n\) in \(\mathbb{C}\). Put \(t_1 := t + \sqrt{n^2 - t^2}\) \(i \in nT^1\).
and

\[ t_2 := t - \sqrt{n^2 - t^2} \quad i \in nT. \]

We have \( t = \frac{t_1 + t_2}{2} \) and

\[ f(tx) = f\left(\frac{t_1 + t_2}{2}x\right) = \frac{t_1 + t_2}{2}f(x) = tf(x) \]

for all \( x \) in \( X \).

If \( \mu \in \mathbb{C} \), then

\[ \mu = |\mu|e^{i\mu_1}, \]

therefore

\[ f(\mu x) = f(|\mu|e^{i\mu_1}x) = |\mu|e^{i\mu_1}f(x) = \mu f(x) \]

for all \( x \) in \( X \). In the other words \( f \) is \( \mathbb{C} \)-linear. The converse is clear. \( \square \)

From now on, we suppose that \( n_0 \in \mathbb{N} \) is a positive integer number, and that

\[ T_{n_0}^1 := \{e^{i\theta}; 0 \leq \theta \leq \frac{2\pi}{n_0}\}. \]

2. Stability of double centralizers

For every \( x, y \in A \), we put \( x^0 - y^0 = 0, x^0y = y \) and also denote \( a \times A \times \ldots \times A \) by \( A^a \). We establish the Hyers-Ulam stability of double Jordan centralizers as follows.

**Theorem 2.1.** Let \( f_j : A \to A \) be mappings with \( f_j(0) = 0 \) \((j = 0, 1)\), and let \( \varphi : A^4 \to [0, \infty) \) be a function such that

\[ \|f_j(\lambda x + \lambda y + z^2) - \lambda f_j(x) - \lambda f_j(y) - [j(zf_j(z))^{1-j} + (1-j)(f_j(z)z)^{1-j}] \]

\[ +f_{1-j}(a) - af_j(a)\| \leq \varphi(x, y, z, a) \] (1)

for all \( \lambda \in T_{n_0}^1 \). If

\[ \lim_{n \to \infty} \frac{\varphi(2^n x, 2^n y, 2^n z, 2^n a)}{2^n} = 0, \] (2)

and there exists a constant \( K \) in which \( 0 < K < 1 \) such that

\[ \psi(2x) \leq 2K\psi(x) \] (3)

for all \( x \in A \), then there exists a unique double Jordan centralizer \((L, R)\) on \( A \) satisfying

\[ \|f_0(x) - L(x)\| \leq \frac{\psi(x)}{2(1-K)}, \] (4)

\[ \|f_1(x) - R(x)\| \leq \frac{\psi(x)}{2(1-K)}. \] (5)
for all $x \in A$, where $\psi(x) = \varphi(x, x, 0, 0)$.

Proof. We consider the set $\Omega$ as follows:

$$\Omega = \{ h : A \rightarrow A | h(0) = 0 \}.$$  

We also define the generalized metric on $\Omega$:

$$d(g, h) := \inf \{ C \in [0, \infty] : \| g(x) - h(x) \| \leq C \psi(x) \text{ for all } x \in A \}.$$  

One can show that $(\Omega, d)$ is a complete metric space. Now, we define a mapping $J : \Omega \rightarrow \Omega$ by

$$Jh(x) = \frac{1}{2}h(2x)$$  

for all $x \in A$. Given $g, h \in \Omega$, let $C \in [0, \infty]$ be an arbitrary constant with $d(g, h) \leq C$, i.e.,

$$\| g(x) - h(x) \| \leq C \psi(x)$$  

for all $x \in A$. Substituting $x$ by $2x$ in the inequality (7) and using from (3) and (6), we have

$$\| Jg(x) - Jh(x) \| \leq \frac{1}{2} \| g(2x) - h(2x) \| \leq \frac{1}{2} C \psi(2x) \leq CK \psi(x)$$  

for all $x \in A$, and so $d(Jg, Jh) \leq CK$. This shows that $J$ is strictly contractive on $\Omega$. Hence we can conclude that

$$d(Jg, Jh) \leq Kd(g, h)$$  

for all $g, h \in \Omega$. Now, we prove that for all $h \in \Omega$, $d(Jh, h) < \infty$. Letting $j = 0, \lambda = 1, x = y, z = a = 0$ in (1), we obtain $\| f_0(2x) - 2f_0(x) \| \leq \psi(x)$ for all $x \in A$. Thus

$$\| \frac{1}{2}f_0(2x) - f_0(x) \| \leq \frac{1}{2} \psi(x)$$  

for all $x \in A$.

It follows from (8) that $d(Jf, f) \leq \frac{1}{2}$. By Theorem 1.1, the sequence $\{ J^n f_0 \}$ converges to a unique fixed point $L : A \rightarrow A$ in the set $\Omega_1 = \{ h \in \Omega, d(f, h) < \infty \}$, i.e.,

$$\lim_{n \rightarrow \infty} \frac{f_0(2^n x)}{2^n} = L(x)$$  

for all $x \in A$, and so

$$d(f_0, L) \leq \frac{1}{1 - K}d(Jf_0, L) \leq \frac{1}{2(1 - K)}.$$  

Thus the inequality (4) holds for all $x \in A$. Now, replace $2^n x$ and $2^n y$ by $x$ and $y$, respectively, and put $z = a = 0$ in (1). If we divide both sides of the
resulting inequality by $2^n$, and letting $n$ tend to infinity, then it follows from (1), (2) and (9) that
\[ L(\lambda x) = \lambda L(x), \]
for all $x \in A$ and all $\lambda \in \mathbb{T}_1^n$. It follows from Lemma 1.1 that $L$ is $\mathbb{C}$–linear. It is routine to show that $L(a^2) = L(a)^2$ from (1), and so it is a left Jordan centralizer on $A$.

According the above argument, one can show that there exists a unique mapping $R : A \to A$ which is a point of $T$ such that
\[ \lim_{n \to \infty} \frac{f_1(2^n x)}{2^n} = R(x) \] (10)
for all $x \in A$. Indeed, $R$ belongs to the set $\Omega_1$. If we put $i = 0, x = y = z = 0$ and substitute $a$ by $2^n a$ in (1) and we divide the both sides of the obtained inequality by $2^n$, then we get
\[ \|a f_0(2^n a) - f_1(2^n a)a\| \leq \varphi(0, 0, 0, 2^n a). \]
Passing to the limit as $n \to \infty$, and from (2) we conclude that $aL(a) = R(a)a$ for all $a \in A$. 

**Corollary 2.1.** Let $r$ and $\theta$ be nonnegative real numbers such that $r < 1$, and let $f_j : A \to A$ be mappings with $f_j(0) = 0$ ($j = 0, 1$) such that
\[ \|f_j(\lambda x + \lambda y + z^2) - \lambda f_j(x) - \lambda f_j(y) - [j(z f_j(z))^j + (1 - j)(f_j(z)z)^{1-j}] \]
\[ -af_j(a) + f_{1-j}(a)a\| \leq \theta((\|x\|^r + \|y\|^r + \|z\|^r + \|a\|^r) \] (11)
for all $\lambda \in \mathbb{T}_1^n$ and all $x, y, z, a \in A$. Then there exists a unique double centralizer $(L, R)$ on $A$ satisfying
\[ \|f_0(x) - L(x)\| \leq \frac{\theta}{2 - 2^r}, \]
\[ \|f_1(x) - R(x)\| \leq \frac{\theta}{2 - 2^r} \]
for all $x, y \in A$.

**Proof.** By putting $K = 2^{r-1}$ and
\[ \varphi(x, y, z, w, a) = \theta((\|x\|^r + \|y\|^r + \|z\|^r + \|a\|^r) \]
for all $x, y, z, a \in A$ in Theorem 2.1, we obtain the desired result. 

In the following corollary, we prove the superstability of double centralizers under some conditions.
Corollary 2.2. Let \( r \) and \( \theta \) be nonnegative real numbers such that \( r < \frac{1}{6} \), and let \( f_j : A \to A \) be mappings with \( f_j(0) = 0 \) \((j = 0, 1)\) such that

\[
\|f_j(\lambda x + \lambda y + z) - \lambda f_j(x) - \lambda f_j(y) - [j(zf_j(z))^j + (1 - j)(f_j(z)^1 - j)]
- af_j(a) + f_{1-j}(a)a\| \leq \theta(\|x\|^r \|y\|^r \|z\|^r \|a\|^r)
\]

for all \( \lambda \in \mathbb{T}_1 \) and all \( x, y, z, a \in A \). Then \((f_0, f_1)\) is a double centralizer.

Proof. It follows from Theorem 2.1 by taking

\[
\varphi(x, y, z) = \theta(\|x\|^r \|y\|^r \|z\|^r \|a\|^r)
\]

for all \( x, y, z, a \in A \). \( \square \)

3. Stability Of multipliers

In this section, we investigate the Hyers-Ulam stability and the super-stability of multipliers on Banach algebras. First we need the next theorem which is the main key to investigation of the stability and superstability.

Theorem 3.1. let \( f : A \to A \) be a mapping with \( f(0) = 0 \) and let \( \phi : A^3 \to [0, \infty) \) be a function such that

\[
\|f(\lambda x + \lambda y) - \lambda f(x) - \lambda f(y) - f(z)z + zf(z)\| \leq \phi(x, y, z)
\]

for all \( \lambda \in \mathbb{T}_{1/n} \) and all \( x, y, z \in A \). If

\[
\lim_{n \to \infty} \frac{\varphi(2^n x, 2^n y, 2^n z)}{2^n} = 0,
\]

and there exists a constant \( K, 0 < K < 1 \), such that

\[
\psi(2x) \leq 2K \psi(x)
\]

for all \( x \in A \), then there exists a unique multiplier \( T \) on \( A \) satisfying

\[
\|f(x) - L(x)\| \leq \frac{1}{2(1-K)} \psi(x)
\]

for all \( x \in A \), where \( \psi(x) = \varphi(x, x, 0) \).

Proof. First, similar to the proof of Theorem 2.1, we Consider the set \( \Omega := \{h : A \to A \mid h(0) = 0\} \) and introduce the generalized metric \( d \) on \( \Omega \) as follows:

\[
d(g, h) := \inf\{C \in \mathbb{R}^+ : \|g(x) - h(x)\| \leq C \psi(x) \text{ for all } x \in A\}.
\]

Again, similar to the proof of Theorem 2.1, the space \( \Omega \) equipped to the metric \( d \) is complete. Now we define a mapping \( \mathcal{J} : \Omega \to \Omega \) by

\[
\mathcal{J}(h)(x) = \frac{1}{2} h(2x)
\]
A fixed point approach to the stability of

for all $x \in A$. By the same reasoning as in the proof of Theorem 2.1, $\mathcal{J}$ is strictly contractive on $\Omega$. Putting $\lambda = 1, x = y$ and $z = 0$ in (13), we obtain

$$\|f(2x) - 2f(x)\| \leq \psi(x)$$

for all $x \in A$. So

$$\|\frac{1}{2}f(2x) - f(x)\| \leq \frac{1}{2}\psi(x) \quad (17)$$

for all $x \in A$. The inequality (17) shows that $d(Jf, f) \leq \frac{1}{2}$. By Theorem 1.1, $J$ has a unique fixed point in the set $\Omega_1 := \{h \in \Omega : d(f, h) < \infty\}$. Let $T$ be the fixed point of $J$. Then $T$ is the unique mapping with $T(2x) = 2T(x)$ for all $x \in A$ such that there exists $C \in (0, \infty)$ such that $\|T(x) - f(x)\| \leq K\psi(x)$ for all $x \in A$. On the other hand, we have

$$\lim_{n \to \infty} d(\mathcal{J}^n(f), h) = 0.$$ 

Thus

$$\lim_{n \to \infty} \frac{1}{2^n} f(2^n x) = h(x) \quad (18)$$

for all $x \in A$. Hence

$$d(f, T) \leq \frac{1}{1-K} d(T, \mathcal{J}(f)) \leq \frac{1}{2(1-L)}.$$ \hspace{1cm} (19)

This implies the inequality (16). From (13), (14) and (18) we obtain

$$\|T(x + y) - T(x) - T(y)\| = \lim_{n \to \infty} \frac{1}{2^n}\|f(2^n(x + y)) + f(2^n(x)) - f(2^n y)\|$$

$$\leq \lim_{n \to \infty} \frac{1}{2^n} \phi(2^n x, 2^n y, 0) = 0$$

for all $x, y \in A$. So $T(x + y) = T(x) + T(y)$ for all $x, y \in A$. Thus $T$ is Cauchy additive. If we put $y = x, z = 0$ in (13), we can conclude that

$$\|2\lambda f(x) - f(2\lambda x)\| \leq \psi(x)$$

for all $x \in A$. It follows that

$$\|T(2\lambda x) - 2\lambda T(x)\| = \lim_{n \to \infty} \frac{1}{2^n}\|f(2\lambda 2^n x) - 2\lambda f(2^n x)\|$$

$$\leq \lim_{n \to \infty} \frac{1}{2^n} \phi(2^n x, 2^n x, 0) = 0$$

for all $\lambda \in \mathbb{T}$ and all $x \in A$. So $T(\lambda x) = \lambda T(x)$ for all $\lambda \in \mathbb{T}$ and all $x \in A$. Linearity of $T$ follows from the proof of Theorem 2.1. If we substitute $z$ by
2^n z in (13), and put \( x = y = 0 \) and we divide the both sides of the obtained inequality by \( 2^n \), we get

\[
\| z \frac{f(2^n z)}{2^n} - \frac{f(2^n z)}{2^n} z \| \leq \frac{\phi(0, 0, 2^n z)}{2^n}.
\]

Passing to the limit as \( n \to \infty \), and from (16) we conclude that \( z T(z) = T(z) z \) for all \( z \in A \).

**Corollary 3.1.** Let \( r \) and \( \theta \) be nonnegative real numbers such that \( r < 1 \), and let \( f : A \to A \) be a mapping with \( f(0) = 0 \) such that

\[
\| f(\lambda x + \lambda y) - \lambda f(x) - f(y) z + z f(z) \| \leq \theta(\|x\|^r + \|y\|^r + \|z\|^r)
\]

for all \( \lambda \in T^1_{\frac{1}{2^n}} \) and all \( x, y, z \in A \). Then there exists a unique multiplier \( T \) on \( A \) satisfying

\[
\| f(x) - T(x) \| \leq \frac{\theta}{2 - 2^r}
\]

for all \( x \in A \).

**Proof.** The proof follows from Theorem 3.1 by taking

\[
\phi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)
\]

for all \( x, y, z \in A \) and by putting \( K = 2^{r-1} \). \( \Box \)

Now, we have the following result for the superstability of multipliers.

**Corollary 3.2.** Let \( r \) and \( \theta \) be nonnegative real numbers such that \( r < \frac{1}{4} \), and let \( f : A \to A \) be a mapping with \( f(0) = 0 \) such that

\[
\| f(\lambda x + \lambda y) - \lambda f(x) - f(y) z + z f(z) \| \leq \theta(\|x\|^r \|y\|^r \|z\|^r)
\]

for all \( \lambda \in T^1_{\frac{1}{2^n}} \) and all \( x, y, z \in A \). Then \( f \) is a multiplier on \( A \).

**Proof.** It follows from Theorem 3 by taking

\[
\phi(x, y, z) = \theta(\|x\|^r \|y\|^r \|z\|^r)
\]

for all \( x, y, z \in A \). \( \Box \)

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