

## USING LAPLACE DECOMPOSITION METHOD TO SOLVE NONLINEAR KLIEN-GORDON EQUATION

Emad K. JARADAT<sup>1</sup>, Amer D. ALOQALI<sup>1</sup>, Wajd ALHABASHNEH<sup>2</sup>

*The nonlinear Klein-Gordon equation used to model many nonlinear phenomena. In quantum field theory, the corresponding Klein-Gordon field characterized by "particles" with rest mass  $m$  and no other structure (e.g., no spin, no electric charge, etc.) Therefore, the Klein-Gordon field is physically the simplest of the relativistic fields that one can study. In this paper, an analytical technique proposed to solve the nonlinear Klein-Gordon equation with high order nonlinearity. The proposed method based on applying the Laplace transform to nonlinear partial differential equation and replacing the nonlinear terms by the Adomian polynomials. This method known as the Laplace decomposition method (LDM). The obtained approximate analytical solution of the equation will be in the form of a summation with easily obtainable terms. An application discussed to illustrate the effectiveness and the performance of the proposed method, which successively provided for finding the solutions of the nonlinear Klein-Gordon equation.*

**Keywords:** nonlinear Klein-Gordon equation, Laplace transform, nonlinear partial differential equation, Laplace decomposition method, Analytical solution.

### 1. Introduction

Nonlinear phenomena, that occurs in an incredible amount of areas of science and engineering such as plasma physics m fluid physics , fluid dynamics , solid state physics , mathematical biology and chemical kinetics , can be modeling by partial differential equations. One of the most important class of all partial differential equations appearing in applied science is that associated with the Klein Gordon. The Klein-Gordon equation represents a relativistic wave equation, related to the Schrodinger equation, which predict the behavior of particles at high energies and velocities comparable to the speed of light. The solutions of linear and nonlinear Klein-Gordon equation play a significant role in many scientific applications. In this paper, we consider the one-dimensional nonlinear Klein-Gordon equation with power nonlinearity:

$$u_{tt}(x, t) = u_{xx} + \lambda u^p \quad (1.1)$$

Subject to initial conditions

---

<sup>1</sup> Associated Prof., Dept. Of physics, Mutah University, Jordan, ejaradat@mutah.edu.jo

<sup>2</sup> Eng., Dept. Of physics, Mutah University, Jordan, emad\_jaradat75@yahoo.com

$$u(x, 0) = f(x) \quad , \quad u_t(x, 0) = k(x) \quad (1.2)$$

and appropriate boundary conditions

$$u(0, t) = g(t) \quad , \quad u_x(0, t) = h(t) \quad (1.3)$$

Where  $u$  is a function of  $x$  and  $t$ , it represents the wave displacement at position  $x$  and time  $t$ ,  $\lambda$  is a physical constant and the term  $\lambda u^p$  represents the nonlinear force which arises in the study of theoretical physics.  $p$  takes the values  $1, 2, 3, \dots$  and the indices  $t$  and  $x$  denote derivatives with respect to these variables. Unless  $p=1$ , the eq.(1.1) is a nonlinear Klein-Gordon equation. The solutions of eq.(1.1) include the motion of quantum scalar or a pseudo-scalar field which is a field whose quanta are spineless particles.

Many powerful methods have improved to solve this type of nonlinear partial differential equations. Sine-Gordon equation solved in [1] by presenting a new form of Bäcklund transformations, which gives a unified method of finding Bäcklund transformations for solutions of certain nonlinear evolution equations. The authors in [2] developed Darboux's idea to solve linear and nonlinear partial differential equations arising in Soliton theory. The extended tanh method was used in [3] to derive abundant solitary wave solutions of nonlinear wave equations. Paper [4] constructed the wave solutions of the approximate equations for long water waves by using a homogeneous balance method.

Nonlinear KGE and its various forms are all well studied constructed and solved in various papers. Authors in [5] implemented a reduced differential transform method (RDTM) for solving the linear and nonlinear Klein-Gordon equations. A method which is known as the homotopy analysis was applied in [6] to obtain the solution of nonlinear fractional Klein-Gordon equation. A numerical method based on collocation points was developed in [7] to solve the nonlinear Klein-Gordon equations by using the Taylor matrix method. In paper [8] the nonlinear one-dimensional Klein-Gordon equation was solved with the help of the cubic B-spline collocation method on the uniform mesh points. In this paper a simple but effective technique will be used to approximate the solution of the nonlinear KGE. The technique known as Laplace decomposition method (LDM) which is a combination of the Laplace transform and the Adomian decomposition method.

## 2. Laplace decomposition method

The Adomian decomposition method, introduced by G. Adomian in 1980's [9-11], has proven to be a successive method to find the approximate solutions for a wide class of ordinary differential equations. A powerful technique developed with the help of the Adomian decomposition. The technique known as the Laplace decomposition method, which used to solve nonlinear ordinary, partial differential equations. The method is very well suited to physical problems since it can solve

nonlinear problems without linearization, perturbation or discretization methods, on the other hand it requires less number of calculation work than traditional approaches.

Many papers introduced this method to solve a various nonlinear partial differential equations. Khuri [12] used this method for the approximate solution of a class of a nonlinear ordinary differential equations. Handibag and Karande [13] applied this method for the solution of the linear and nonlinear heat equation. Elgazery [14] exploited this method to solve Falkner-Skan equation. The Laplace decomposition method was employed in [15] to get approximate analytical solutions of the linear and the nonlinear fractional diffusion-wave equations.

The method is based on applying the Laplace transform to a nonlinear differential equation  $Gu=g$ , where  $G$  represents a nonlinear differential operator. the method consists of decomposing the linear part of  $G$  in  $L+R$ , where  $L$  is an operator that have inverse  $L^{-1}$ , and  $R$  is the remaining part. denote the nonlinear term by  $N$ , then the equation in standard form is :

$$Lu + Ru + Nu = g \quad (2.1)$$

Taking the  $L^{-1}$  to both sides

$$L^{-1}Lu = L^{-1}g - L^{-1}Ru - L^{-1}Nu \quad (2.2)$$

The key of this technique is to decompose the nonlinear term  $Nu$  in the equation (2.1) into a particular series of polynomials.

$$Nu = \sum_{n=0}^{\infty} A_n ; A_n \equiv \text{Adomian polynomials.} \quad (2.3)$$

$$A_n(u_0, u_1, u_2, \dots, u_n) = \frac{1}{n!} \left[ \frac{d^n}{d\lambda^n} N \left( \sum_{i=0}^{\infty} \lambda^i u_i \right) \right]_{\lambda=0} \quad (2.4)$$

In the next section we will use this methodology (LDM) to solve eq.(1.1) and obtain the solution of it in  $x$  as well as  $t$  direction by taking the Laplace transform with respect to  $t$  and  $x$ .

### 3. Method of solution

The aim of this section is to discuss the use of the Laplace decomposition method to solve the form [16] of the nonlinear Klein-Gordon equation (1.1). We consider the general form of nonlinear Klein-Gordon equations, with initial conditions (1.2) and (1.3), is given below

$$Lu(x, t) + Ru(x, t) + \lambda u^\rho = 0 \quad (3.1)$$

where  $L = \frac{\partial^2}{\partial t^2}$ ,  $R = \frac{\partial^2}{\partial x^2}$ ,

$Nu$  represents the general nonlinear operator  $\lambda u^\rho$ .

Apply the Laplace transform on both sides of equation (3.1) with respect to  $t$ , we get

$$\begin{aligned} s^2 u(x, s) - s f(x) - k(s) &= \mathcal{L}_t \{u_{xx} + \lambda u^\rho\} \\ u(x, s) &= \frac{1}{s} f(x) + \frac{1}{s^2} k(x) + \frac{1}{s^2} \mathcal{L}_t \{u_{xx} + \lambda u^\rho\} \end{aligned} \quad (3.2)$$

where  $\mathcal{L}_t$  is the laplace transform with respect to  $t$ . Applying inverse Laplace transform on both sides of equation (3.2) with respect to  $t$ , we get

$$u(x, t) = f(x) + k(x)t + \mathcal{L}_t^{-1} \left\{ \frac{1}{s^2} \mathcal{L} \{u_{xx} + \lambda u^\rho\} \right\} \quad (3.3)$$

The Laplace decomposition method (LDM) [17] assumes a series solution of the function  $u(x, t)$  is given by

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t) \quad (3.4)$$

The nonlinear term in equation (3.1) can be decomposed by using Adomian polynomials  $A_n$  [18] which is given by the formula (2.4)

The first five Adomian polynomials for the variable  $Nu=f(u)$  are given by

$$\begin{aligned} A_0 &= f(u_0), \quad A_1 = u_1 f'(u_0), \quad A_2 = u_2 f'(u_0) + \frac{1}{2!} u_1^2 f''(u_0) \\ A_3 &= u_3 f'(u_0) + u_1 u_2 f''(u_0) + \frac{1}{3!} u_1^3 f^{(3)}(u_0) \\ A_4 &= u_4 f'(u_0) + \left( u_1 u_3 + \frac{1}{2!} u_2^2 \right) f''(u_0) + \frac{1}{2!} u_1^2 u_2 f^{(3)}(u_0) + \frac{1}{4!} u_1^4 f^{(4)}(u_0) \end{aligned}$$

Therefore;  $Nu(x, t) = \sum_{m=0}^{\infty} A_m$  (3.5)

We obtain the first few Adomian polynomial components for as.

$Nu(x, t) = \lambda u^\rho$  as

$$\begin{aligned} A_0 &= \lambda u_0^\rho, \quad A_1 = \lambda \rho u_1 u_0^{\rho-1}, \quad A_2 = \lambda \rho u_2 u_0^{\rho-1} + \frac{1}{2!} u_1^2 \rho(\rho-1) u_0^{\rho-2} \\ A_3 &= \lambda \rho u_3 u_0^{\rho-1} + \rho \lambda u_1 u_2 (\rho-1) u_0^{\rho-2} + \frac{1}{3!} \rho \lambda u_1^3 (\rho-1)(\rho-2) u_0^{\rho-3} \end{aligned}$$

and so on ... putting eq. (3.4) and eq. (3.5) in eq. (3.3), we get

$$\sum_{n=0}^{\infty} u_n(x, t) = f(x) + k(x)t + \mathcal{L}^{-1} \left\{ \frac{1}{s^2} \mathcal{L} \left\{ \sum_{n=0}^{\infty} u_{nxx}(x, t) + \sum_{m=0}^{\infty} A_m \right\} \right\} \quad (3.6)$$

Match the both sides of the above equation, we get

$$\begin{aligned} u_0(x, t) &= f(x) + k(x)t, \quad u_1(x, t) = \mathcal{L}^{-1} \left\{ \frac{1}{s^2} \mathcal{L} \{u_{0xx}(x, t) + A_0\} \right\} \\ u_2(x, t) &= \mathcal{L}^{-1} \left\{ \frac{1}{s^2} \mathcal{L} \{u_{1xx}(x, t) + A_1\} \right\}, \quad u_3(x, t) = \mathcal{L}^{-1} \left\{ \frac{1}{s^2} \mathcal{L} \{u_{2xx}(x, t) + A_2\} \right\} \end{aligned}$$

And so on ... In general, the recursive relation is given by

$$\begin{aligned} u_0(x, t) &= f(x) + k(x)t \\ u_{n+1}(x, t) &= \mathcal{L}^{-1} \left\{ \frac{1}{s^2} \mathcal{L} \{u_{nxx}(x, t) + A_n\} \right\} \quad n \geq 0 \end{aligned} \quad (3.7)$$

By using relation (3.7) we can find the components of  $u(x, t)$ ;  $u_0, u_1, \dots, u_n$  for  $n \geq 0$ . Substitute all these values in equation (3.4), we get the solution of eq.(3.1) in  $t$  direction. If we take the Laplace transform of (3.1) with respect to  $x$ , we will get the same solution in  $x$  direction. The recursive equation in  $x$  direction is

$$\begin{aligned} u_0(x, t) &= g(t) + h(t)x, \\ u_{n+1}(x, t) &= \mathcal{L}_x^{-1} \left\{ \frac{1}{s^2} \mathcal{L}_x \{ u_{ntt}(x, t) - A_n \} \right\} \quad n \geq 0 \end{aligned} \quad (3.8)$$

where  $\mathcal{L}_x$  represent the Laplace transform with respect to  $x$ .

#### 4. Illustrate application

We will illustrate the technique of Laplace decomposition method to solve the nonlinear Klein Gordon equation by setting  $\lambda=1$  in eq. (1.1) and taking the following initial conditions

$$u(x, 0) = e^x, \quad u_t(x, 0) = 0$$

Then, we get the Klein-Gordon equation

$$u_{tt} = u_{xx} + u^\rho; \quad \rho = 1, 2, 3, \dots \quad (4.1)$$

Let us obtain the general solution to eq.(4.1) for any value of  $\rho$  in  $t$  direction. Consider the special case when  $\rho=1$  which represents the linear Klein-Gordon equation. Then the eq.(4.1) will take the form

$$u_{tt} = u_{xx} + u \quad (4.2)$$

We need to find the related Adomian polynomials for eq.(4.2) which simply can be written as

$A_0 = u_0, A_1 = u_1, A_2 = u_2, \dots$  and so on.

Using the recursive relation (3.7), we get

$$\begin{aligned} u_0(x, t) &= e^x \\ u_1(x, t) &= 2e^x \frac{t^2}{2!}, \quad u_2(x, t) = 4e^x \frac{t^4}{4!}, \quad u_3(x, t) = 8e^x \frac{t^6}{6!} \end{aligned}$$

And so on. Putting these individual terms in eq.(3.4) one we get the exact solution in  $t$  direction.

$$u(x, t) = e^x + 2e^x \frac{t^2}{2!} + 4e^x \frac{t^4}{4!} + 8e^x \frac{t^6}{6!} + \dots \quad (4.3)$$

Now, let us take  $\rho = 2, 3, 4, \dots$  then, eq.(4.1) represents the nonlinear Klein-Gordon equation. For  $\rho=2$

$$u_{tt} = u_{xx} + u^2 \quad (4.4)$$

the Adomian polynomials can be found from relation (2.4), we get

$$A_0 = u_0^2, \quad A_1 = 2u_0u_1, \quad A_2 = 2u_0u_2 + u_1^2, \quad A_3 = 2u_0u_3 + 2u_1u_2$$

The same steps we followed in previous case with the help of recursive relation (3.7) can be used to find the terms of  $u(x, t)$ .

$$u_0(x, t) = e^x, \quad u_1(x, t) = (e^x + e^{2x}) \frac{t^2}{2!}, \quad u_2(x, t) = (2e^{3x} + 6e^{2x} + e^x) \frac{t^4}{4!},$$

$$u_3(x, t) = (10e^{4x} + 42e^{3x} + 32e^{2x} + e^x) \frac{t^6}{6!}$$

and so on ... this yields the slution of eq. (4.4)

$$u(x, t) = e^x + (e^x + e^{2x}) \frac{t^2}{2!} + (2e^{3x} + 6e^{2x} + e^x) \frac{t^4}{4!} + (10e^{4x} + 42e^{3x} + 32e^{2x} + e^x) \frac{t^6}{6!} + \dots \quad (4.5)$$

Many values for  $\rho$  must be substituted in eq.(4.1) to enable us writing a general formula for the solution of this form of the nonlinear Klein-Gordon equation and obtaining the solution for high order power of the nonlinear force with the same initial conditions supposed previously. So let us take that  $\rho=3$ , this makes eq.(4.1) takes the form

$$u_{tt} = u_{xx} + u^3 \quad (4.6)$$

In addition, the relation (2.4) helps us to obtain the Adomian polynomials, which are

$$A_0 = u_0^3, A_1 = 3u_0^2 u_1, A_2 = 3u_0^2 u_2 + \frac{6}{2!} u_1^2 u_0, A_3 = 3u_0^2 u_3 + 6u_0 u_1 u_2 + \frac{6}{3!} u_1^3$$

For the solution of this equation in  $t$  direction we use the same relation (3.7) to get the terms of  $u(x, t)$ . The few terms of the solution are

$$u_0(x, t) = e^x, u_1(x, t) = (e^x + e^{3x}) \frac{t^2}{2!}, u_2(x, t) = (3e^{5x} + 12e^{3x} + e^x) \frac{t^4}{4!}$$

$$u_3(x, t) = (27e^{7x} + 147e^{5x} + 129e^{3x} + e^x) \frac{t^6}{6!}$$

The rest of terms are found by the same steps then the summation of all terms gives the solution of eq.(4.6)

$$u(x, t) = e^x + (e^x + e^{3x}) \frac{t^2}{2!} + (3e^{5x} + 12e^{3x} + e^x) \frac{t^4}{4!} + (27e^{7x} + 147e^{5x} + 129e^{3x} + e^x) \frac{t^6}{6!} + \dots \quad (4.7)$$

From all obtained solutions, the terms of the solution  $u(x, t)$  for the nonlinear equation when  $\rho = 4$  may be expected. The results indicate that the solution for  $\rho =$

4 will be in terms of  $e^x, e^{4x}, e^{7x}, e^{10x}$  and  $t^2, t^4, t^6$ , and so on ... so, let us find it when  $\rho = 4$  for the equation

$$u_{tt} = u_{xx} + u^4 \quad (4.8)$$

The related polynomials for eq.(4.8) are

$$\begin{aligned} A_0 &= u_0^4, A_1 = 4u_0^3 u_1, A_2 = 4u_0^3 u_2 + \frac{12}{2!} u_1^2 u_0^2 \\ A_3 &= 4u_0^3 u_3 + 12u_0^2 u_1 u_2 + \frac{24}{3!} u_1^3 u_0 \end{aligned}$$

The verification of our expectation can be done by getting  $u_0, u_1, u_2, \dots$  from relation (3.7).

$$u_0(x, t) = e^x, u_1(x, t) = (e^x + e^{4x}) \frac{t^2}{2!}, u_2(x, t) = (4e^{7x} + 20e^{4x} + e^x) \frac{t^4}{4!}$$

$$u_3(x, t) = (34e^{10x} + 312e^{7x} + 342e^{4x} + e^x) \frac{t^6}{6!}$$

So, the terms of solution of eq.(4.8) include what we expected and the exact solution of it is

$$\begin{aligned} u(x, t) &= e^x + (e^x + e^{4x}) \frac{t^2}{2!} + (4e^{7x} + 20e^{4x} + e^x) \frac{t^4}{4!} \\ &+ (34e^{10x} + 312e^{7x} + 342e^{4x} + e^x) \frac{t^6}{6!} + \dots \quad 4.9) \end{aligned}$$

If we generalize these solutions we can get the solution for higher order of  $\rho$ . From equations (4.3), (4.5), (4.7), (4.9) we can write a general formula for the terms of the solution for eq.(4.1) and for any value of  $\rho$  with the same initial conditions.

$$\begin{aligned} u_0(x, t) &= e^x, u_1(x, t) = (e^x + e^{\rho x}) \frac{t^2}{2!}, u_2(x, t) = (e^x + ae^{\rho x} + be^{(2\rho-1)x}) \frac{t^4}{4!} \\ u_3(x, t) &= (e^x + ce^{\rho x} + de^{(2\rho-1)x} + me^{(3\rho-2)x}) \frac{t^6}{6!} \end{aligned}$$

And so on, ... where  $a, b, c, d, m$  are constants, in this manner we get the solutions.

Similarly, a solution of the nonlinear KGE can be obtained in  $x$  direction with given initial boundary conditions if we apply a Laplace transform with respect to  $x$  ( $L_x^{-1}$ ) on both sides of eq.(3.1) and representing the nonlinear term by Adomian polynomials which can be evaluated from relation (2.4). then, the obtained solution can be written in the form of summation in  $x$  direction.

Some space-time graphs for different cases of  $\rho$  and the estimate solutions plotted for different values of  $\rho$ . Since Klein-Gordon equation describes the quantum amplitude for finding a point particle in various places and describes the relativistic waves function, we can show from these figures the effect of the nonlinear force power on the behavior of the particles.

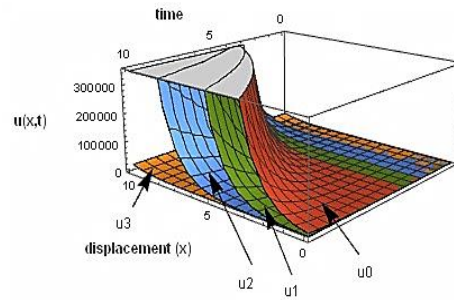


Fig. 1. Space-time graph of the solution terms  $u_0$ ,  $u_1$ ,  $u_2$  and  $u_3$  for linear equation

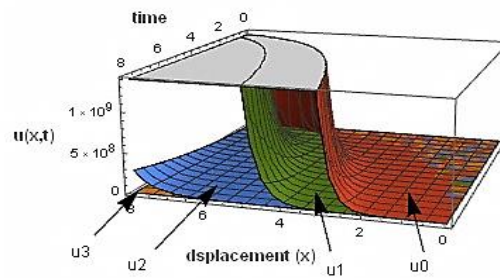


Fig. 2. Space-time graph of solution terms  $u_0$ ,  $u_1$ ,  $u_2$  and  $u_3$  for  $\rho=2$ .

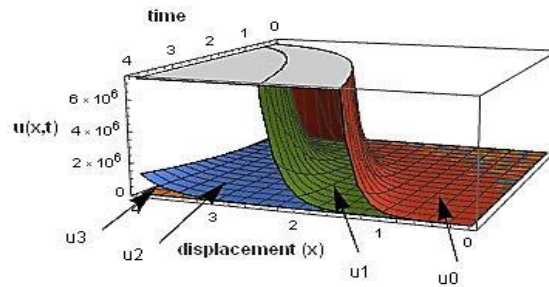


Fig. 3. Space-time graph of the solution terms  $u_0$ ,  $u_1$ ,  $u_2$  and  $u_3$  for  $\rho=3$

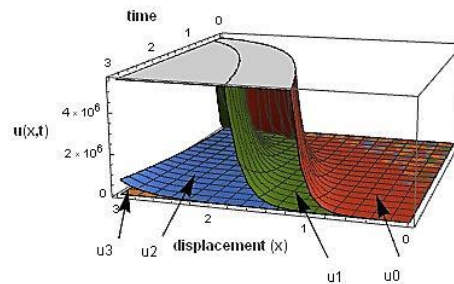


Fig. 4. Space-time graph of the solution terms  $u_0$ ,  $u_1$ ,  $u_2$  and  $u_3$  for  $\rho=4$



For a free particle nonlinear Klein-Gordon equation, it has a nonlinearity in the mass term (proposed in our paper) which, in contrast to what happens in the standard linear case, is proportional to the power of the wave function. Fig.5 Shows how the power of the nonlinear term effects on the propagation of the waves and the behavior of the particles. On the other hand, Fig. 6 Shows the same behavior for the waves at considerable time  $t = 2s$ .

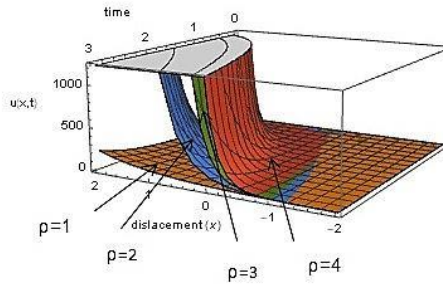


Fig.5. Wave forms of different powers nonlinearity

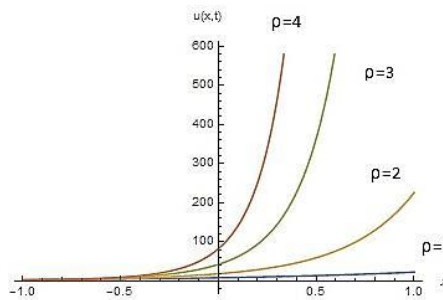


Fig. 6. The approximate solution of different powers for Klein-Gordon eq.,  $-2 \leq x \leq 2$ ,  $0 \leq t \leq 3$ .  
Nonlinearity for Klein-Gordon equation,  $-2 \leq x \leq 2$ ,  $0 \leq t \leq 3$

## 5. Conclusion

Analytical solutions enable researchers to study and construct the effect of different variables or parameters on the function under study in easy way. The Laplace decomposition method is considered a powerful tool of the analytical methods which capable of handling linear /nonlinear partial differential equations. In this paper, this method has successfully applied to nonlinear Klein-Gordon equation with high order nonlinearity. The implementation LDM is simple as finite difference methods. The solutions obtained in the previous section demonstrate the effectiveness and the success of this technique, more exactly, we have solved different first , second , third and fourth of nonlinearity equation to explain the efficiency of this method to get the general solution for high order nonlinearity. It noted that the tool found the solutions without any discretization or restrictive assumptions. The scheme described in this paper is expected to be further employee to solve most of the nonlinear problems in science.

## REFERENCES

- [1] *R. Hirota*, "A new form of b $\bar{c}$ ocklund transformations and its relation to the inverse scattering problem," *Progress of Theoretical Physics*, vol. 52, no. 5, pp. 1498–1512, 1974.
- [2] *V. B. Matveev and M. A. Salle*, *Darboux Transformations and Solitons*, Springer, Berlin, Germany, 1991.
- [3] *A.-M. Wazwaz*, "The tanh method for compact and noncompact solutions for variants of the KdVBurgerand the K(n, n) Burger equations," *Physica D*, vol. 213, no. 2, pp. 147–151, 2006.
- [4] *M. Wang, Y. Zhou, and Z. Li*, "Application of a homogeneous balance method to exact solutions of nonlinear equations in mathematical physics," *Physics Letters, Section A*, vol. 216, no. 1–5, pp. 67–75, 1996.
- [5] *Yildiray Keskin , Sema Servi and Galip Oturanç ,* " Reduced Differential Transform Method for Solving Klein Gordon Equations," *Proceedings of the World Congress on Engineering 2011 Vol I WCE 2011*, July 6 - 8, 2011, London, U.K..
- [6] *Muhammet Kurulay ,* "Solving the fractional nonlinear Klein-Gordon equation by means of the homotopy analysis method," *A springer open journal , Kurulay Advances in Difference Equations 2012*, 2012:187.
- [7] *Younghae Do and Bongsoo Jang ,* " Nonlinear Klein-Gordon and Schrodinger Equations by the Projected Differential Transform Method," *Hindawi Publishing Corporation , Abstract and Applied Analysis ,Volume 2012*, Article ID 150527, 15 pages.
- [8] *J. Rashidinia , M. Ghasemi , R. Jalilian ,* "Numerical solution of the nonlinear Klein\_Gordon equation," *Journal of Computational and Applied Mathematics* 233 (2010) 1866\_1878 .
- [9] *G. Adomian*, *Nonlinear Stochastic Operator Equations*. 1986.
- [10] *G. Adomian*, *Solving frontier problems of physics: the decomposition method*, Vol. 60, Springer Science & Business Media, 2013.
- [11] *G. Adomian*, A review of the decomposition method and some recent results for nonlinear equations, *Mathematical and Computer Modelling* 13 (7) (1990) 17-43.
- [12] *Khuri, S.A.*, 2001. A Laplace decomposition algorithm applied to class of nonlinear differential equations. *J. Math. Appl.*, 4: 141-155.
- [13] *Sujit Handibag and B. D. Karande .* Application of Laplace Decomposition Method to Solve Linear and Nonlinear Heat Equation. *International Journal of Applied Physics and Mathematics*, Vol. 2, No. 5, September 2012.
- [14] *Nasser, S. Elgazery*, 2008. Numerical solution for the Falkner-Skan equation. *Chaos, Solitons and Fractals* 35: 733-746.
- [15] *H. Jafari , C.M. Khalique , M. Nazari ,* " Application of the Laplace decomposition method for solving linear and nonlinear fractional diffusion–wave equations," *Applied Mathematics Letters* 24 (2011) 1799–1805.
- [16] *M. A. Rego-Monteiro and F. D. Nobre .* Nonlinear quantum equations: Classical field theory. Centro Brasileiro de Pesquisas Fisicas and National Institute of Science and Technology for Complex Systems, Rua Xavier Sigaud 150, 22290-180 Rio de Janeiro - RJ, Brazil . *Journal of mathematical physics* 54, 103302 (2013).
- [17] *El-Wakil, S.A. and M.A. Abdou*, 2007. New applications of Adomian decomposition method, *Chaos, Solitons and fractals*, 33: 513-515.
- [18] *Wazwaz, A.W.*, 2002. Anew technique for calculating Adomian polynomials for nonlinear polynomials. *Appl. Math. Comput.*, 111: 33-51.