ON JAGGI TYPE CONTRACTION MAPPINGS

Erdal Karapınar[1]

By a work of Jaggi, it is known that the existence of certain inequalities for continuous maps over metric spaces implies the existence and uniqueness of fixed points. In this paper, we show that if p denotes a partial metric, the existence of a rational form of type

$$p(Tt, Ts) \leq a_1 \frac{p(t, Tt) \cdot p(s, Ts)}{d(t, s)} + a_2 p(t, s)$$

for some a_1 and a_2 with $a_1 + a_2 < 1$ for a continuous map T over a partial metric space leads to the same conclusions, that is, the existence and uniqueness of fixed points.

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1. Introduction

In this paper we aim to study Jaggi's inequalities that imply the existence and uniqueness of fixed points in metric spaces from the view point of partial metric spaces. The notion of partial metric space was introduced by Matthews [46],[47] to handle and to solve "economically" the domain theory problems in the frame of computer science. In this pioneer work, Matthews [46] proved the analog of Banach contraction mapping principle in the context of complete partial metric space, and hence, he built one more bridge between the computer science and mathematics [43]. After than, a number of authors reported several fixed point results to corroborate the relations between the fixed point theory [1]-[67] and computer science [17, 56, 57, 68, 69, 70, 71].

We would like to start with an overview of the subject of partial metrics and partial metric spaces.

Definition 1.1. *Let M be a nonempty set. A function $p : M \times M \to [0, \infty)$ is called a partial metric on M if it satisfies the properties*

(P1) $p(t, t) = p(t, s)$ *and* $p(t, t) = p(s, s)$ *implies* $t = s$,
(P2) $p(t, r) + p(s, s) \leq p(t, s) + p(s, r)$,
(P3) $p(t, t) \leq p(t, s)$,
(P4) $p(t, s) = p(s, t)$

for all $t, s, r \in M$. The pair (M, p) is referred to as a partial metric space (see [46],[47] *for more details on the subject).*

We would like to present a very well-known and elementary example of a partial metric space which is extensively studied in the literature. We shall often abbreviate *partial metric space* by PMS in this text.

[1]Atilim University, Department of Mathematics, 06836, İncek, Ankara, Turkey,
[2]Department of Medical Research, China Medical University, Taichung, TAIWAN
e-mail: erdalkarapinar@yahoo.com

Throughout the manuscript, we denote $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ where $\mathbb{N}$ is the positive integers. Further, $\mathbb{R}$ represent the real numbers and $\mathbb{R}_0^+ := [0, \infty)$.

Example 1.1. *Let us consider the set $M = [0,1]$. Let $p : M \times M \to M$ be the map defined by $p(t,s) = \max\{t,s\}$. Then for any $t,s \in M$ if we have $\max\{t,t\} = \max\{t,s\}$ and $\max\{t,t\} = \max\{s,s\}$, then we obviously get $t = s$ because $\max\{t,t\} = t$ and $\max\{s,s\} = s$. Note that $\max\{t,t\} \leq \max\{t,s\}$ and $\max\{t,s\} = \max\{s,t\}$ hold for any $t,s \in M$. Therefore, the properties (P1), (P3), and (P4)in Definition 1.1 clearly hold. We need to verify (P2).*

Given t, s and r in M, assume that we have $\max\{t,s,r\} = t$. Then we derive that $p(t,r) + p(s,s) = t + s \leq t + \max\{s,r\} = p(t,s) + p(s,r)$. Since the elements in $\{t,s,r\}$ can be renamed so that t becomes the maximum again, the inequality above can be repeated in any case. This shows that the property in $(P2)$ also holds for p. Hence, we conclude that (M,p) is a partial metric space.

It is clear that the partial metric p in Example 1.1 is not a metric on M. In general, the value $p(t,t)$ may not be equal to 0, e.g., $p(3,3) = 3$ in Example 1.1. But it is possible to introduce a metric on M associated with a given partial metric on the same set. In particular, it can be shown that if p is a partial metric on M, then the function $d_p : M \times M \to [0, \infty)$ given by

$$d_p(t,s) = 2p(t,s) - p(t,t) - p(s,s), \tag{1}$$

is a metric on M (see [46]). The relationship between partial metrics and metrics given above plays a crucial role in proving certain statements in the frame of Fixed Point Theory on the context of Partial Metric Spaces.

Example 1.2. *Let us consider the set $M = \{[a,b] : a,b \in \mathbb{R},\ a \leq b\}$ and the map $p : M \times M \to M$ defined by $p([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$. We will leave the details on verifying the fact that (M,p) is a partial metric space to the readers (see [47] for more on this example).*

Let $x = [a,b]$ and $y = [c,d]$. We have the following cases: (1) $b \leq d$ and $a \leq c$, or (2) $b \leq d$ and $c \leq a$, or (3) $d \leq b$ and $a \leq c$, or (4) $d \leq b$ and $c \leq a$. In either case, we calculate that $2p(t,s) - p(t,t) - p(s,s) = |d-b| + |c-a|$. In particular, we obtain that

$$d_p(t,s) = |d-b| + |c-a|.$$

Notice that we have $d_p(t,s) \geq 0$, $d_p(t,s) = d_p(s,t)$, and $d_p(t,s) = 0$ if and only if $t = s$ for every $t,s \in M$. We also have

$$\begin{aligned} d_p(t,s) &= |d-b| + |c-a| \\ &= |d-f+f-b| + |c-e+e-a| \\ &\leq |d-f| + |c-e| + |f-b| + |e-a| \\ &= d_p(t,r) + d_p(z,y) \end{aligned}$$

for any $z = [e,f] \in M$. Therefore, (M,d_p) is a metric space.

Example 1.3. *Let us consider the PMS in Example 1.1. We aim to calculate*

$$2\max\{t,s\} - \max\{t,t\} - \max\{s,s\} = 2\max\{t,s\} - x - y.$$

If $\max\{t,s\} = t$, then we get $d_p(t,s) = t - y \geq 0$. If $\max\{t,s\} = y$, then we find that $d_p(t,s) = y - x \geq 0$. Therefore, we derive that $d_p(t,s) = |x-y|$ is the restriction of the usual length metric of $\mathbb{R}$ to $M = [0,1]$.

It is known that each partial metric p on M generates a T_0-topology, a topology where for any distinct $t,s \in M$ there exits an open set U such that $t \in U$ and $y \notin U$ or $x \notin U$ and $y \in U$. This topology has the family of open p-balls

$$\{B_p(x,r) : t \in M, r > 0\}$$

as its base where $B_p(x,r) = \{y \in M : |p(t,s) - p(t,t)| < r\}$.

Example 1.4. *Let (M,p) be the PMS in the Example 1.1. Let t be an element in $[0,1]$ so that $t \neq 1$ and $r > 0$. Then any ball centered at t is given as*

$$B_p(t,\epsilon) = [0, t+r).$$

Because any $s \leq t$ will satisfy the inequality $\max\{s,t\} = t < t+r = \max\{t,t\}$. If $s > t$, then s is in $B_p(t,r)$ if $\max\{t,s\} = s < t+r$ or $t < s < t+r$. If we let $t = 1$, then any ball centered at 1 is

$$B_p(1,r) = [0,1].$$

Therefore, there are two types of open balls in (M,τ_p). Clearly, (M,τ_p) is a T_0-space.

Let us recall some basic topological definitions for partial metric spaces that we shall need in this work (for details see e.g. [46, 47, 2, 24, 30].)

Definition 1.2. *Let (M,p) be a PMS and t and t_0 denote two points in M. Let $\{t_n\}$ be a sequence contained in M.*

(1) *$\{t_n\}$ converges to t, denoted by if and only if for every $\epsilon > 0$ there exists a positive integer $M > 0$ so that $t_n \in B_p(x,\epsilon)$ for every $n > M$, i.e., $p(t,t) = \lim_{n\to\infty} p(x,t_n)$.*
(2) *$\{t_n\}$ is called a Cauchy sequence if and only if $\lim_{n,m\to\infty} p(t_n,t_m)$ exists and is finite.*
(3) *(M,p) is called complete if and only if every Cauchy sequence $\{t_n\}$ in M converges to a point $t \in M$.*
(4) *A mapping $f : (M,p) \to (M,p)$ is said to be continuous at t_0 if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that $f(B_p(t_0,\delta)) \subseteq B_p(f(t_0),\epsilon)$.*
(5) *A mapping $f : (M,p) \to (M,p)$ is said to be continuous if it is continuous at every point $t \in M$.*

Example 1.5. *Consider the PMS given in Example 1.1. Let $t_n = 1/n$ for $n \in \mathbb{N}$. Notice that we have*

$$\begin{aligned} \lim_{n\to\infty} p\left(0, \frac{1}{n}\right) &= \lim_{n\to\infty} \frac{1}{n} \\ &= 0 \\ &= p(0,0). \end{aligned}$$

Therefore, $\{1/n\}_{n=1}^{\infty}$ is a convergent sequence which converges to 0. In fact, this sequence is Cauchy. Note that if we have $m \geq n$, then we get $p(1/m,1/n) = 1/n$ and $\lim_{n,m\to\infty} 1/n = 0$. If we have $n \geq m$, then we get $p(1/m,1/n) = 1/m$ and $\lim_{n,m\to\infty} 1/m = 0$. In either case, we obtain that $\lim_{n,m\to\infty} p(t_n,t_m)$ exists and is finite.

We would like to point out that this example also shows that limit of a convergent sequence in a PMS may not be unique. Indeed, $\{1/n\}_{n=1}^{\infty}$ also converges to 1 because $\lim_{n\to\infty} p(1,1/n) = p(1,1)$.

We shall use the following definition to name the set of limit points of a sequence in a PMS.

Definition 1.3. *Let $\{t_n\}_{n=1}^{\infty}$ be a sequence in a PMS. We will denote the set of limit points, if there is any, by*

$$L(\{t_n\}_{n=1}^{\infty}) = \{t \in M : \lim_{n\to\infty} p(x,t_n) = p(t,t)\}.$$

We would like to emphasize that $L(\{t_n\}_{n=1}^{\infty})$ may be empty. On the other hand, it may be the whole space, e.g., $L(\{1/n\}_{n=1}^{\infty}) = [0,1]$ in Example 1.5.

Example 1.6. *Let (M,p) be the PMS in Example 1.1. Let us consider the map $T : M \to M$ defined by $Tt = t/10$. We aim to show that T is continuous over M.*

Let t_0 be an element in $[0,1)$. For any given $\epsilon > 0$, however small, we let $\delta = 10\epsilon$. Then we see that

$$T(B_p(t_0,\delta)) = T([0,t_0+\delta)) \subseteq \left[0, \frac{t_0}{10}+\epsilon\right) = B_p(Tt_0,\epsilon).$$

Let $t_0 = 1$. For any given $\epsilon > 0$, however small, we let $\delta = \epsilon$. Then we find that $T(B_p(1,\delta)) = T([0,1]) \subseteq [0,1/10+\epsilon) = B_p(T1,\epsilon)$. Thus, T is continuous on $M = [0,1]$.

Example 1.7. *Let (M,p) be the PMS in Example 1.1. Let us consider the map $T : M \to M$ defined by $Tt = 0$. Let t_0 be a point in $[0,1)$. For any given $\epsilon > 0$, however small, we let $\delta = \epsilon$. Then we observe that*

$$T(B_p(t_0,\delta)) = T([0,t_0+\delta)) \subseteq [0,\epsilon) = B_p(Tt_0,\epsilon).$$

Let $t_0 = 1$. For any given $\epsilon > 0$, however small, we let $\delta = \epsilon$. Then we find that $T(B_p(1,\delta)) = T([0,1]) \subseteq [0,\epsilon) = B_p(T1,\epsilon)$. Thus, T is continuous on $M = [0,1]$.

We would like to include the statements of a number of lemmas that we shall need directly or indirectly to prove the main results of this paper. We omit the proofs of some of these lemmas as they are either easily accessible in the literature or elementary in the sense that they can be derived directly from Definition 3.1 and/or the relationship between metrics and partial metrics.

Lemma 1.1. *Let (M,p) be a partial metric space and (M,d_p) be the corresponding metric space. Then we have the necessary and sufficient conditions*

(1) *a sequence $\{t_n\}$ is a Cauchy sequence in (M,p) if and only if it is a Cauchy sequence in (M,d_p), and*
(2) *the space (M,p) is complete if and only if the space (M,d_p) is complete.*

In the spirit of the earlier examples, we would like to include basic examples showing an application of the lemma above.

Example 1.8. *Let us set $M = [0,1] \cup [2,3]$ and define a map $p : M \times M \to [0,\infty)$ such that*

$$p(t,s) = \begin{cases} \max\{t,s\} & \text{if } \{t,s\} \cap [2,3] \neq \emptyset, \\ |t-s| & \text{if } \{t,s\} \subset [0,1]. \end{cases}$$

Then (M,p) is a partial metric space (see [47] for details). We aim to show that (M,p) is a complete PMS. By Lemma 1.1, it is enough to show that (M,d_p) is complete.

Let t and s be two elements in M. If we have $\{t,s\} \subset [0,1]$, then we get $d_p(t,s) = 2|t-s|$. Let us assume that $\{t,s\} \cap [2,3] \neq \emptyset$. Then there are two cases to consider: $\max\{t,s\} = t$ or $\max\{t,s\} = s$. If we have the first case, then we see that $d_p(t,s) = 2t-t-s$ or $d_p(t,s) = t-s$. In the second case, we find that $d_p(t,s) = 2s-t-s$ or $d_p(t,s) = s-t$. In other words, we have $d_p(t,s) = |t-s|$ if $\{t,s\} \cap [2,3] \neq \emptyset$. Therefore, we conclude that d_p is essentially the usual length metric on M.

We know that $(\mathbb{R}, |\cdot|)$ is a complete metric space with the induced topology from the length metric. Since M is a closed subset of $\mathbb{R}$, the metric space (M,d_p) is also complete by Lemma 1.1.

Example 1.9. *Let (M,p) be the PMS in Example 1.1. By Example 1.3, we know that d_p is the usual length metric $|.|$. Since $(\mathbb{R}, |.|)$ is complete and $M = [0,1]$ is a closed subset of $\mathbb{R}$, the partial metric space (M,p) is also complete by Lemma 1.1.*

One of the implications of (2) in Lemma 1.1 is the necessary and sufficient condition stated as $\lim_{n\to\infty} d_p(t,t_n) = 0$ if and only if $p(t,t) = \lim_{n\to\infty} p(t,t_n)$ and $p(t,t) = \lim_{n,m\to\infty} p(t_m,t_n)$ for any Cauchy sequence $\{t_n\}$ in (M,p) or (M,d_p) (see e.g. [2, 24, 30]).

Lemma 1.2. *Let (M, p) be a complete PMS and $\{t_n\}_{n=1}^{\infty}$ be a Cauchy sequence in M. If there exists an element $u \in L(\{t_n\}_{n=1}^{\infty})$ such that $p(u, u) = 0$, then we have $u \in L(\{t_{n_j}\}_{j=1}^{\infty})$ for every subsequence $\{t_{n_j}\}_{j=1}^{\infty}$ of $\{t_n\}_{n=1}^{\infty}$.*

Proof. We need to show that $\lim_{j\to\infty} p(t_{n_j}, u) = p(u, u)$. We know that $\{t_n\}_{n=1}^{\infty}$ is Cauchy. For any given $\epsilon >$, there exists a positive integer $M > 0$, however large, so that $|p(t_m, t_n) - p(u, u)| < \epsilon$ for every $m, n > M$. Therefore, we have $|p(t_{n_j}, t_n) - p(u, u)| < \epsilon$ for large enough j so that $n_j > M$. In other words, we observe that $\lim_{n,j\to\infty} p(t_{n_j}, t_n) = p(u, u)$. Similarly, we also observe that $\lim_{n\to\infty} p(t_n, t_n) = p(u, u)$. We use the property (P2) in Definition 3.1 to derive that

$$p(t_{n_j}, u) + p(t_n, t_n) \leq p(t_{n_j}, t_n) + p(t_n, u)$$

for every $n, j \in \mathbb{N}$. We calculate the limit of both sides of the inequality above. We find that

$$\begin{aligned} \lim_{n,j\to\infty} \left(p(t_{n_j}, u) + p(t_n, t_n)\right) &\leq \lim_{n,j\to\infty} \left(p(t_{n_j}, t_n) + p(t_n, u)\right) \\ \lim_{j\to\infty} p(t_{n_j}, u) &\leq 0. \end{aligned}$$

Thus, we conclude that $\lim_{j\to\infty} p(t_{n_j}, u) = p(u, u)$. □

The proof of the following lemma is obvious.

Lemma 1.3. *Let (M, p) be a complete PMS and $T : M \to M$ be a continuous map. Let $\{t_n\}_{n=1}^{\infty}$ be a Cauchy sequence in M. If there exists an element $u \in L(\{t_n\}_{n=1}^{\infty})$ such that $p(u, u) = 0$, then we have $Tu \in L(\{Tt_n\}_{n=1}^{\infty})$.*

Lemma 1.4. [47, 51]

(*i*) *$\{t_n\}$ is a Cauchy sequence in a partial metric space (M, p) if and only if it is a Cauchy sequence in the metric space (M, d_p);*

(*ii*) *A partial metric space (M, p) is complete if and only if the metric space (M, d_p) is complete. Furthermore, $\lim_{n\to\infty} d_p(t_n, t) = 0$ if and only if $p(t, t) = \lim_{n\to\infty} p(t_n, t) = \lim_{n\to\infty} p(t_n, t_m)$.*

(*iii*) *If $\{t_n\}$ is a convergent sequence in (M, d_p), then it is a convergent sequence in the partial metric space (M, p).*

Lemma 1.5. [2] *Let $\{t_n\}$ and $\{s_n\}$ be two sequences in a partial metric space M such that*

$$\lim_{n\to\infty} p(t_n, t) = \lim_{n\to\infty} p(t_n, t_n) = p(t, t),$$

and

$$\lim_{n\to\infty} p(s_n, s) = \lim_{n\to\infty} p(s_n, s_n) = p(s, s),$$

then $\lim_{n\to\infty} p(t_n, s_n) = p(t, s)$. In particular, $\lim_{n\to\infty} p(t_n, r) = p(t, r)$ for every $r \in M$.

Lemma 1.6. *[See e.g.*[33]*] Let (M, p) be a partial metric space. Then*

(A) *If $p(t, s) = 0$ then $t = s$,*

(B) *If $t \neq s$, then $p(t, s) > 0$.*

Remark 1.1. *In a recent work of Haghi* et al. [16]*, it was realized that*

$$M_d^T(t, s) = M_p^T(t, s), \text{ with}$$

where T is a self-mapping on a non-empty set M, p, q are metric, partial metric, respectively, and

$$M_m^T(t, s) = \max\{m(t, s), m(t, Tt), m(Ts, s), m(Tt, s), m(x, Ty)\}$$

where $m = d, p$. In this note we avoid to use the maximum notation so that the methods used in [16] *can not be applied.*

Definition 1.4. [49] *Let $T : M \to M$ be a mapping and $\alpha : M \times M \to [0, \infty)$ be a function. We say that T is an α-orbital admissible if*

$$\alpha(t, Tt) \geq 1 \Rightarrow \alpha(Tt, T^2t) \geq 1.$$

Let Ψ be the family of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(Ψ_1) ψ is nondecreasing;
(Ψ_2) there exist $k_0 \in \mathbb{N}$ and $a \in (0, 1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that

$$\psi^{k+1}(t) \leq a\psi^k(t) + v_k,$$

for $k \geq k_0$ and any $t \in [0, \infty)$.

Here, these functions are called as (c)-comparison (see e.g. [60]). In [60], Rus proved that if $\psi \in \Psi$, then the following hold:

(a) the sequence $(\psi^n(t))_{n\in\mathbb{N}}$ converges to 0 as $n \to \infty$ for all $t \in [0, \infty)$;
(b) the strict inequality, $\psi(t) < t$, holds for any $t \in [0, \infty)$;
(c) the function ψ is continuous at 0;
(d) the series $\sum_{k=1}^{\infty} \psi^k(t)$ converges for any $t \in [0, \infty)$.

In this paper, we investigate the existence and uniqueness of a fixed point of the operators in the frame of rational contraction (will be called Jaggi Type) in the setting of complete partial metric spaces via auxiliary functions α and $\psi \in \Psi$ defined above.

2. Main Results

We start this section with the following definition.

Definition 2.1. *Let (M, p) be a complete PMS and $T : M \to M$ be a map. Then T will be referred to as a map of $(\alpha - \psi)$-Jaggi type if there exist $\psi \in \Psi$, and nonnegative real numbers a_1, a_2 with $a_1 + a_2 < 1$ so that the inequality*

$$\alpha(t, s)p(Tt, Ts) \leq \psi\left(a_1 \frac{p(t, Tt) \cdot p(s, Ts)}{p(t, s)} + a_2 p(t, s)\right) \tag{2}$$

holds for every distinct $t, s \in M$.

Lemma 2.1. *Let M be a non-empty set. Suppose that $\alpha : M \times M \to \mathbb{R}_0^+$ is a function and $T : M \to M$ is an α-orbital admissible mapping. If there exists $t_0 \in M$ such that $\alpha(t_0, Tt_0) \geq 1$, and $t_n = Tt_{n-1}$ for $n = 0, 1, \ldots$, then, we have*

$$\alpha(t_n, t_{n+1}) \geq 1, \text{ for each } n = 0, 1, \ldots. \tag{3}$$

Proof. On account of the assumptions of the theorem, there exists $t_0 \in M$ such that $\alpha(t_0, Tt_0) \geq 1$. Owing to the fact that T is α-orbital admissible, we find

$$\alpha(t_0, t_1) = \alpha(t_0, Tt_0) \geq 1 \Rightarrow \alpha(Tt_0, Tt_1) = \alpha(t_1, t_2) \geq 1.$$

By iterating the above inequality, we derive that

$$\alpha(t_n, t_{n+1}) = \alpha(Tt_{n-1}, Tt_n) \geq 1, \text{ for each } n = 0, 1, \ldots.$$

□

Theorem 2.1. *Let (M, p) be a complete PMS, $T : M \to M$ be a map of $(\alpha - \psi)$-Jaggi type and there exists $t_0 \in M$ such that $\alpha(t_0, Tt_0) \geq 1$. If the α-orbital admissible mapping T is continuous, then T has a fixed point in M.*

Proof. By the assumption of the theorem, there exists $t_0 \in M$ such that $\alpha(t_0, Tt_0) \geq 1$. So, we can construct a sequence as follows:

$$t_n = Tt_{n-1} \text{ for } n = 0, 1, \ldots, .$$

Due to Lemma 2.1, we have (3).

Without loss of generality, we may assume that

$$p(t_n, t_{n+1}) > 0, \text{ for each } n = 0, 1, \ldots. \tag{4}$$

Indeed, if there exists an k_0 so that $p(t_n, t_{n+1}) = 0$, then, by Lemma 1.6, we have $t_{k_0} = t_{k_0+1} = Tt_{k_0}$. It completes the proof with the observation that $u = t_{k_0}$ is a fixed point of T.

Accordingly, throughout the proof, we have (4). Hence, we can use the inequality (2) for any successive terms $t = t_n$ and $s = t_{n+1}$

$$\begin{aligned} p(t_{n+1}, t_{n+2}) &= p(Tt_n, Tt_{n+1}) \leq \alpha(t_n, t_{n+1}) p(Tt_n, Tt_{n+1}) \\ &\leq \psi\left(a_1 \tfrac{p(t_n, Tt_n) \cdot p(t_{n+1}, Tt_{n+1})}{p(t_n, t_{n+1})} + a_2 p(t_n, t_{n+1})\right) \\ &= \psi\left(a_1 p(t_{n+1}, t_{n+2}) + a_2 p(t_n, t_{n+1})\right) \end{aligned} \tag{5}$$

We need to examine two cases:

(*i*) $p(t_{n+1}, t_{n+2}) > p(t_n, t_{n+1})$,
(*ii*) $p(t_{n+1}, t_{n+2}) \leq p(t_n, t_{n+1})$.

If the first case occurs for some n, then, due to (Ψ_1) together with the fact that $\psi(t) < t$, $\forall t > 0$, the inequality (5) turns into

$$\begin{aligned} p(t_{n+1}, t_{n+2}) &\leq \psi\left([a_1 + a_2] p(t_{n+1}, t_{n+2})\right) \\ &\leq \psi(p(t_{n+1}, t_{n+2})) < p(t_{n+1}, t_{n+2}), \end{aligned} \tag{6}$$

a contradiction. Hence, the second case, $p(t_{n+1}, t_{n+2}) \leq p(t_n, t_{n+1})$, holds true for all $n \in \mathbb{N}_0$. Moreover, the inequality (5) yields that

$$p(t_{n+1}, t_{n+2}) \leq \psi\left([a_1 + a_2] p(t_{n+1}, t_{n+2})\right) \leq \psi\left(p(t_{n+1}, t_{n+2})\right) < p(t_{n+1}, t_{n+2}) \tag{7}$$

Iteratively, we derive that

$$p(t_{n+1}, t_n) \leq \psi^n(p(t_1, t_0)), \text{ for all } n \geq 1. \tag{8}$$

Due to Lemma (8) (i), we find that

$$\lim_{n \to \infty} p(t_{n+1}, t_n) = 0. \tag{9}$$

Again by keeping the expression (8) in the mind, and by using the triangular inequality (P4), for all $k \geq 1$, we have

$$\begin{aligned} p(t_n, t_{n+k}) &\leq p(t_n, t_{n+1}) + \ldots + p(t_{n+k-1}, t_{n+k}) - \sum_{j=1}^{k-1} (p(t_{n+j}, t_{n+j})) \\ &\leq \sum_{j=n}^{n+k-1} \psi^j(p(t_1, t_0)) \\ &\leq \sum_{j=n}^{+\infty} \psi^j(p(t_1, t_0)) \to 0 \text{ as } n \to \infty. \end{aligned}$$

This implies that

$$\lim_{n \to \infty} p(t_n, t_{n+k}) = 0,$$

and hence $\{t_n\}$ is a Cauchy sequence in (M,p). Since (M,p) is complete, there exists $u \in M$ such that

$$\lim_{n\to\infty} p(t_n,u) = 0 = \lim_{n\to\infty} p(t_n,t_{n+k}) = p(u,u). \tag{10}$$

Since T is continuous, by the definition of the continuity, we conclude from (10) that

$$\lim_{n\to\infty} p(t_{n+1},Tu) = \lim_{n\to\infty} p(Tt_n,Tu) = 0. \tag{11}$$

On account of Lemma 1.5 together with (10) and (11), we find that u is a fixed point of T, that is, $Tu = u$.

□

The continuity condition can be relaxed in Theorem 2.1 by replacing a suitable condition like the given below:

Definition 2.2. *Let $s \geq 1$. We say that a PMS (M,p) is* regular *if $\{t_n\}$ is a sequence in M such that $\alpha(t_n,t_{n+1}) \geq 1$ for each n and $p_n \to t \in M$ as $n \to \infty$, then there is a subsequence $\{t_{n(k)}\}$ of $\{t_n\}$ such that $\alpha(t_{n(k)},t) \geq 1$ for each k.*

Theorem 2.2. *Let (M,p) be a regular complete PMS, $T : M \to M$ be a map of $(\alpha-\psi)$-Jaggi type and there exists $t_0 \in M$ such that $\alpha(t_0,Tt_0) \geq 1$. If T is the α-orbital admissible mapping, then T has a fixed point in M.*

Proof. Following the proof of Theorem 2.1, we know that the sequence $\{t_n\}$ defined by $t_{n+1} = Tt_n$ for all $n \in \mathbb{N}_0$, converges for some $u \in M$. From Lemma 2.1 and Definition 2.2, there exists a subsequence $\{t_{n(k)}\}$ of $\{t_n\}$ such that $\alpha(t_{n(k)},u) \geq 1$ for all k. Employing (2), for all k, we get that

$$\begin{aligned} p(t_{n(k)+1},Tu) = p(Tt_{n(k)},Tu) \quad &\leq \alpha(t_{n(k)},u)p(Tt_{n(k)},Tu) \\ &\leq \psi\left(a_1 \frac{p(Tt_{n(k)},t_{n(k)})p(Tu,u)}{p(t_{n(k)},u)} + a_2 p(t_{n(k)},u)\right) \\ &< a_1 \frac{p(Tt_{n(k)},t_{n(k)})p(Tu,u)}{p(t_{n(k)},u)} + a_2 p(t_{n(k)},u). \end{aligned} \tag{12}$$

Letting $k \to \infty$ in the above equality, we have $p(u,Tu) = 0$, that is, $u = Tu$. □

The assure the uniqueness of the fixed point, we will consider the following hypothesis

(U) for all $t \neq s \in M$, there exists $r \in M$ such that $\alpha(t,r) \geq 1$, $\alpha(s,r) \geq 1$ and $\alpha(r,Tr) \geq 1$

Theorem 2.3. *The fixed point t^* of T, in Theorem 2.1 (resp. Theorem 2.2), is unique, if assume an additional condition (U).*

Proof. By taking $t = t_0$ we derive that $\alpha(t_0,Tt_0) \geq 1$, by hypotheses of Theorem 2.1 (resp. Theorem 2.2) so we obtain that t^* is a point fixed of T, where $t^* = \lim_{n\to\infty} t_n = \lim_{n\to\infty} T^n t$. We shall show that T has a unique fixed point. Suppose, on the contrary, that t^* and s^* are two fixed points of T such that $t^* \neq s^*$. Then, from (U) there exists $z \in M$ such that $\alpha(t^*,r) \geq 1$, $\alpha(s^*,r) \geq 1$ and $\alpha(z,Tr) \geq 1$. Since T is a α-orbital admissible, we get that $\alpha(t^*,T^n r) \geq 1$ and $\alpha(s^*,T^n r) \geq 1$. Hence, from (2) we have

$$\begin{aligned} d(t^*,T^{n+1}r) \quad &= d(Tt^*,T(T^n r)) \leq \alpha(t^*,T^n r)d(Tt^*,T(T^n r)) \\ &\leq \psi\left(a_1 \frac{p(t^*,Tt^*)\cdot p(T^n r,T(T^n r))}{p(t^*,T^n r)} + a_2 p(t^*,T^n r)\right) \end{aligned} \tag{13}$$

This imply that

$$d(t^*,T^{n+1}r)a_2 d(t^*,T^n r) \leq d(t^*,T^n r)$$

By Theorem 2.2 we deduce that the sequence $T^n r$ converges to a fixed point r^* of T. Letting $n \to \infty$ in the above inequality, we get $d(t_*, r^*) < d(t_*, r^*)$. This implies that $d(t^*, r^*) = 0$ so $t^* = r^*$. Similarly, we get $s^* = r^*$. Hence, $t^* = s^*$, which is a contradiction. □

3. **Consequences**

In this section we shall present some consequences of our main theorems and illustrate their applications as examples. First, we would like to give a definition.

Definition 3.1. *Let (M, p) be a complete PMS, $T : M \to M$ be a map and $\psi \in \Psi$. Then T will be referred to as a map of ψ-Jaggi type if there exist a_1 and a_2 in $[0, 1)$ with $a_1 + a_2 < 1$ so that the inequality*

$$p(Tt, Ts) \leq \psi \left(a_1 \frac{p(t, Tt) \cdot p(s, Ts)}{p(t, s)} + a_2 p(t, s) \right) \tag{14}$$

holds for every distinct $t, s \in M$.

Theorem 3.1. *Let (M, p) be a complete PMS and $T : M \to M$ be a map of ψ-Jaggi type. If T is continuous, then T has a unique fixed point in M.*

Proof. For the existence of a fixed point, it is sufficient to take $\alpha(t, s) = 1$ for all $t, s \in M$. Then, Theorem 2.1 turns to be Theorem 3.1. Uniqueness follows from Theorem 2.3, since the condition (U) is fulfilled due to fact that $\alpha(t, s) = 1$ for all $t, s \in M$. □

Theorem 3.2. *Let (M, p) be a complete PMS and $T : M \to M$ be a map of Jaggi type. Let T be a continuous self-map defined on a complete partial metric space (M, p). Further, let T satisfy the following condition:*

$$p(Tt, Ts) \leq \alpha_1 \frac{p(t, Tt).p(s, Ts)}{p(t, s)} + \alpha_2 d(t, s) \tag{15}$$

for all distinct $t, s \in M$,and for some $\alpha_1, \alpha_2 \in [0, 1)$ with $\alpha_1 + \alpha_2 < 1$. Then T has a unique fixed point in M.

Proof. It is sufficient to set $\psi(t) = kt$, $k \in [0, 1)$ in Theorem 3.1 with $\alpha_i = ka_i$, $i = 1, 2$. □

The following is an example where Theorem 3.2 is applicable to conclude that there exists a unique fixed point.

Example 3.1. *Let M be the set of real numbers in $[0, 1]$ and $p : M \to M$ be the partial metric defined by $p(t, s) = \max\{t, s\}$. Let us define the map $T : M \to M$ so that $Tx = t/10$.*

We claim that T is a map of Jaggi type. Let $a_1 = 1/5$ and $a_2 = 1/5$. Let t and s be two distinct points in M. Assume that $\max\{t, s\} = t$. Since t and s are distinct, we get $x \neq 0$. Then we have

$$p(Tt, Ts) = \frac{t}{10} \leq \frac{s + t}{5} = \frac{\frac{1}{5} \cdot t \cdot s}{t} + \frac{1}{5} \cdot t = a_1 \frac{p(t, Tt).p(s, Ts)}{p(t, s)} + a_2 p(t, s).$$

Notice that $p(t, Tt) = t$ and $p(s, Ts) = s$. Since the elements in $\{t, s\}$ can be renamed so that the maximum becomes t again, the inequality above can be repeated in any case. Therefore, T is a map of Jaggi type. By Theorem 3.2, we conclude that there exists a unique fixed point. It is straightforward to observe that the point 0 is the fixed point of T.

3.1. Consequences in the setting of metric spaces endowed with a partial order

In the subsection, we list some more consequences of our main result in the setting of metric spaces endowed with partial orders. This trend was initiated by [54, 48] and take great attention of several authors who has worked on the metric fixed point theory and its applications. We shall prove that Theorem 2.3 conclude various existing fixed point results on a metric space endowed with a partial order. For this purpose, we, first, recollect some basic concepts.

Definition 3.2. *For a partially ordered non-empty set $(M, \preceq)$, the self-mapping $T : M \to M$ T is called nondecreasing with respect to $\preceq$ if*

$$t, s \in M,\ t \preceq s \Longrightarrow Tt \preceq Ts.$$

Definition 3.3. *A sequence $\{t_n\}$ in a partially ordered set $(M, \preceq)$ is called nondecreasing with respect to $\preceq$, if $t_n \preceq t_{n+1}$ for all n.*

Definition 3.4. *Let $(M, \preceq)$ be a partially ordered set and d be a metric on M. We say that $(M, p, \preceq)$ is regular if for every nondecreasing sequence $\{t_n\} \subset M$ such that $t_n \to x \in M$ as $n \to \infty$, there exists a subsequence $\{t_{n(k)}\}$ of $\{t_n\}$ such that $t_{n(k)} \preceq x$ for all k.*

Suppose that $(M, \preceq)$ is a partially ordered set and d be a metric on M. We say that $(M, \preceq)$ have a property of (S) if it fulfills the following condition

(S) for all $s, t \in M$ there exists $r \in M$ such that $t \preceq r$ and $s \preceq r$,

For the simplicity, we shall use the notation $(M, p, \preceq)$ to represent the partially ordered set $(M, \preceq)$ equipped with a metric d. The triple $(M, p, \preceq)$ is called metric spaces endowed with a partial order.

Theorem 3.3. *Let $(M, p, \preceq)$ be a partial metric spaces endowed with a partial order, where (M, p) is complete. Let $T : M \to M$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists a function $\psi \in \Psi$ such that*

$$d(Tt, Ts) \leq \psi(R(t, s), \tag{16}$$

for all $t, s \in M$ with $t \succeq s$, where

$$R(t, s) = a_1 \frac{p(t, Tt) \cdot p(s, Ts)}{p(t, s)} + a_2 p(t, s). \tag{17}$$

Suppose also that the following conditions hold:

(i) *there exists $t_0 \in M$ such that $t_0 \preceq Tt_0$;*

(ii) *T is continuous or $(M, \preceq, d)$ is regular.*

Then, T has a fixed point. Moreover, if $(M, \preceq)$ have a property of (S), the observed fixed point is unique.

Proof. Let $\alpha : M \times M \to [0, \infty)$ be a mapping such that

$$\alpha(t, s) = \begin{cases} 1 \text{ if } x \preceq y \text{ or } x \succeq y, \\ 0 \text{ otherwise.} \end{cases}$$

It straightforward that $T : M \to M$ is an $(\alpha$-$\psi)$-Jaggi type, that is,

$$\alpha(t, s) d(Tt, Ts) \leq \psi(R(t, s)),$$

for all $t, s \in M$. From condition (i), the definition of α yields that $\alpha(t_0, Tt_0) \geq 1$.

Moreover, for all $t, s \in M$, from the monotone property of T, we have

$$\alpha(t, s) \geq 1 \Longrightarrow x \succeq y \text{ or } x \preceq y \Longrightarrow Tx \succeq Ty \text{ or } Tx \preceq Ty \Longrightarrow \alpha(Tt, Ts) \geq 1.$$

Consequently, T is $\alpha-$orbital admissible.

For a last step, we examine the following cases: If T is continuous, the existence of a fixed point follows from Theorem 2.1. Suppose now that $(M, \preceq, d)$ is regular. Let $\{t_n\}$ be a sequence in M such that $\alpha(t_n, t_{n+1}) \geq 1$ for all n and $t_n \to x \in M$ as $n \to \infty$. Due to regularity, there is a subsequence $\{t_{n(k)}\}$ of $\{t_n\}$ such that $t_{n(k)} \preceq x$ for all k. Hence, we have $\alpha(t_{n(k)}, x) \geq 1$ for all k. So, the existence of a fixed point follows from Theorem 2.2.

For the uniqueness, let $t, s \in M$. By assumption (S) of the theorem, there exists $r \in M$ such that $t \preceq r$ and $s \preceq r$, which yields that $\alpha(x, r) \geq 1$ and $\alpha(y, r) \geq 1$. Consequently, we conclude the uniqueness of the fixed point by Theorem 2.3. □

3.2. Consequences in the setting of the cyclic contractive mappings

In this subsection, we shall consider some consequences of our main results in setting of cyclic mapping. Studying the existence and uniqueness of a fixed point of certain cyclic contractive mappings turns to be one of the most exciting trend in the last decade. This direction was initiated by Kirk *et.al.* [44]and followed by a number of authors (see e.g. [62, 41, 42] and the related references therein).

Here, we shall indicate that our main result, Theorem 2.3, infer a fixed point theorems for cyclic contractive mappings.

Theorem 3.4. *Suppose that $\{A_i\}_{i=1}^2$ are nonempty closed subsets of a complete partial metric space (M, p) and $T : N \to N$ is a given mapping with $N = A_1 \cup A_2$. If the the following conditions are fulfilled*

(I) *$T(A_1) \subseteq A_2$ and $T(A_2) \subseteq A_1$;*

(II) *there exists a function $\psi \in \Psi$ such that*

$$d(Tt, Ts) \leq \psi(R(t, s)), \text{ for all } (t, s) \in A_1 \times A_2,$$

where $R(t, s)$ is defined as in (17),

then, T has a unique fixed point that belongs to $A_1 \cap A_2$.

Proof. We shall divide the proof in 5 steps. Step 1. the pair (N, p) forms a complete metric space since A_1 and A_2 are closed subsets of (M, p).

Step 2. We shall prove that T is a Jaggi-$(\alpha - \psi)$ type. For this purpose, we specify the mapping $\alpha : N \times N \to [0, \infty)$ as

$$\alpha(t, s) = \begin{cases} 1 \text{ if } (t, s) \in (A_1 \times A_2) \cup (A_2 \times A_1), \\ 0 \text{ otherwise.} \end{cases}$$

Regarding (II) and α, we are able to write

$$\alpha(t, s) d(Tt, Ts) \leq \psi(M(t, s)),$$

for all $t, s \in Y$. In other words, T is an Jaggi-$(\alpha - \psi)$ type.

Step 3. We assert that T is $\alpha-$admissible. Suppose that $(t, s) \in N \times N$ with $\alpha(t, s) \geq 1$. For the case $(t, s) \in A_1 \times A_2$, from (I), $(Tt, Ts) \in A_2 \times A_1$, which yields that $\alpha(Tt, Ts) \geq 1$. For the other case, $(t, s) \in A_2 \times A_1$, again from (I), $(Tt, Ts) \in A_1 \times A_2$, which implies that $\alpha(Tt, Ts) \geq 1$. So, we find that $\alpha(Tt, Ts) \geq 1$ whenever $\alpha(t, s) \geq 1$.

Step 4. Note that for any $a \in A_1$, from (I), we get $(a, Ta) \in A_1 \times A_2$, and thus $\alpha(a, Ta) \geq 1$.

Step 5. We claim that (M, p) is regular. Suppose that $\{t_n\}$ is a sequence in M such that $\alpha(t_n, t_{n+1}) \geq 1$ for all n and $t_n \to t \in M$ as $n \to \infty$. On account of the definition of the α mapping, we find

$$(t_n, t_{n+1}) \in (A_1 \times A_2) \cup (A_2 \times A_1), \text{ for all } n.$$

Since $(A_1 \times A_2) \cup (A_2 \times A_1)$ is a closed set with respect to the Euclidean metric, we derive that

$$(t,t) \in (A_1 \times A_2) \cup (A_2 \times A_1),$$

which yields that $t \in A_1 \cap A_2$. Consequently, we have $\alpha(t_n, t) \geq 1$ for all n.

To finalize the proof, suppose that $t, s \in \mathrm{Fix}(T)$. From (I), we find that $t, s \in A_1 \cap A_2$. As a result, we deduce that $\alpha(t, r) \geq 1$ and $\alpha(s, r) \geq 1$, for any $r \in Y$. Thus, condition (U) is fulfilled.

Thus, all the hypotheses of Theorem 2.3 are fulfilled that guarantees the existence and uniqueness of a fixed point of T in $A_1 \cap A_2$ (from (I)). □

3.3. Conclusion and Final Remarks

It is clear that the given consequences of our main results in this paper is not complete. One can give more consequences by re-considering the auxiliary functions α, ψ and even by changing the abstract space with a standard metric. Notice that all fixed point theorems in partial metric spaces are valid in metric spaces. It is clear that the converse is not true.

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