U.P.B. Sci. Bull., Series A, Vol. 77, Iss. 4, 2015 ISSN 1223-7027

$n-$WEAK AMENABILITY FOR LAU PRODUCT OF BANACH ALGEBRAS

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Given Banach algebras A and B, let θ be a character on B. We explain explicitly the derivations from $\theta-$Lau product $A\times_\theta B$ into its $\mathrm{n}^{\mathrm{th}}-$dual $(A\times_\theta B)^{(n)}$ from which we obtain necessary and sufficient conditions for $A\times_\theta B$ to be $n-$weakly amenable.

Keywords: $n-$weak amenability, derivation, module action, Lau product.
MSC2010: 46H25, 47B47

1. INTRODUCTION AND PRELIMINARIES

Let A and B be Banach algebras with $\emptyset\neq\sigma(B)$ (:=the set of all nonzero multiplicative functionals of B) and let $\theta\in\sigma(B)$. Then the ℓ_1-direct product $A\times B$ equipped with the multiplication

$$(a,b)\cdot(c,d)=(ac+\langle\theta,d\rangle a+\langle\theta,b\rangle c,bd),\quad ((a,b),(c,d)\in A\times B)$$

is a Banach algebra which is called the $\theta-$Lau product of A and B and will be denoted by $A\times_\theta B$. This product was introduced by Lau [5] for certain class of Banach algebras and followed by Sangani Monfared [6] for the general case. These products not only induce new examples of Banach algebras which are interesting in its own right but also they are known as a fertile source of (counter)examples in functional analysis and abstract harmonic analysis. A very familiar example, which is of special interest, is the case that $B=\mathbb{C}$ with θ as the identity character i that we get the unitization $A^\sharp=A\times_i\mathbb{C}$ of A. If we include the possibility that $\theta=0$ then we obtain the usual direct product of Banach algebras.

From the homological algebra point of view $A\times_\theta B$ is a strongly splitting Banach algebra extension of B by A, which means that, the quotient $(A\times_\theta B)/A$ is isometrically isomorphic to B. This extension enjoys some properties that are not shared in general by arbitrary strongly splitting extensions. For instance, commutativity does not preserve by a general strongly splitting extension,

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however, $A \times_\theta B$ is commutative in the case where A and B are commutative. Some of the basic properties such as characterizations of bounded approximate identity, topological center and weak amenability for this product are investigated in [6]. Character (inner) amenability of $A \times_\theta B$ is also investigated in [7] and [2].

In this paper, we are mainly concerned with the investigation of $n-$weak amenability of $A \times_\theta B$ (the case $n = 1$ has already investigated in [6]). In this respect we give some necessary and sufficient conditions for $n-$weak amenability of $A \times_\theta B$. The most interesting case occurs when A is unital. In this flexible case we mainly show that, $A \times_\theta B$ is $n-$weakly amenable if and only if both A and B are $n-$weakly amenable.

Before we proceed for the main results we need some preliminaries about $n-$weak amenability. A derivation from a Banach algebra A into a Banach $A-$module X is a bounded linear mapping $D : A \to X$ such that $D(ab) = D(a) \cdot b + a \cdot D(b)$ $(a, b \in A)$. The set of all derivations from A into X is denoted by $Z^1(A, X)$. For each $x \in X$ the derivation $\delta_x : a \mapsto a \cdot x - x \cdot a$ $(a \in A)$ is called an inner derivation. The set of all inner derivations from A to X is denoted by $N^1(A, X)$. The quotient $Z^1(A, X)/N^1(A, X)$ is called the first cohomology group of A with coefficients in X and will be denoted by $H^1(A, X)$. Throughout the paper n is assumed to be a non-negative integer. The $n^{\text{th}}-$ dual $A^{(n)}$ of a Banach algebra A is a Banach $A-$module under the module operations defined inductively by

$\langle m \cdot a, u\rangle = \langle m, a \cdot u\rangle,\ \langle a \cdot m, u\rangle = \langle m, u \cdot a\rangle, \quad (m \in A^{(n)}, u \in A^{(n-1)}, a \in A = A^{(0)})$.

Of course, A is a Banach $A-$module under its multiplication.

A Banach algebra A is said to be $n-$weakly amenable if $H^1(A, A^{(n)}) = \{0\}$. This notion was initiated and intensively studied in [1]. Trivially, $1-$weak amenability is nothing else than the so-called weak amenability. The basic properties of these notions are extensively discussed in [4].

For brevity of notation we usually identify an element of A with its canonical image in $A^{(2n)}$, as well as an element of A^* with its image in $A^{(2n+1)}$. We usually use $\langle\cdot, \cdot\rangle$ for the duality between a Banach space and its dual and we also use the symbol " $\cdot$ " for the various module operations linking various Banach algebras.

2. **Main Results**

To clarify the relation between $n-$weak amenability of $A \times_\theta B$ and that of A and B we need to characterize the derivations from $A \times_\theta B$ into $(A \times_\theta B)^{(n)}$. One can simply identify the underlying space of $(A \times_\theta B)^{(n)}$ with the Banach space $A^{(n)} \times B^{(n)}$ equipped with the ℓ_1-norm when n is even and $\ell_\infty-$norm when n is odd.

A direct verification reveals that $(A \times_\theta B)-$module operations of $(A \times_\theta B)^{(n)}$ are as follows. For $a \in A,\ b \in B,\ f \in A^{(2n+1)},\ g \in B^{(2n+1)},\ F \in A^{(2n)}$ and $G \in$

$B^{(2n)}$:

$$\begin{aligned}(a,b)\cdot(f,g)&=(a\cdot f+\langle\theta,b\rangle f,\langle f,a\rangle\theta+b\cdot g)\\(f,g)\cdot(a,b)&=(f\cdot a+\langle\theta,b\rangle f,\langle f,a\rangle\theta+g\cdot b)\\(a,b)\cdot(F,G)&=(a\cdot F+\langle\theta,G\rangle a+\langle\theta,b\rangle F,b\cdot G)\\(F,G)\cdot(a,b)&=(F\cdot a+\langle\theta,G\rangle a+\langle\theta,b\rangle F,G\cdot b).\end{aligned}$$

In the following we characterize the derivations from $A\times_\theta B$ into $(A\times_\theta B)^{(n)}$. The next result is devoted to the odd case.

Proposition 2.1. *A bounded linear map $D:A\times_\theta B\to(A\times_\theta B)^{(2n+1)}$ is a derivation if and only if there exist derivations $d_A:A\to A^{(2n+1)},\ d_B:B\to B^{(2n+1)}$ and bounded linear maps $S:A\to B^{(2n+1)},\ T:B\to A^{(2n+1)}$ such that for each $a,c\in A,\ b,d\in B$,*

(i) $D((a,b))=(d_A(a)+T(b),S(a)+d_B(b))$,
(ii) $a\cdot T(b)=T(b)\cdot a=0$,
(iii) $\langle T(b),a\rangle\theta+S(a)\cdot b=\langle\theta,b\rangle S(a)$,
(iv) $S(a)\cdot b=b\cdot S(a)$,
(v) $S(ac)=(\langle d_A(a),c\rangle+\langle d_A(c),a\rangle)\theta$, and
(vi) $T(bd)=\langle\theta,b\rangle T(d)+\langle\theta,d\rangle T(b)$.
In particular, D is inner if and only if $T=0,\ S=0$ and $d_A,\ d_B$ are inner.

Proof. Let $D:A\times_\theta B\to(A\times_\theta B)^{(2n+1)}$ be a bounded linear mapping. As $(A\times_\theta B)^{(2n+1)}$ can be identified with $A^{(2n+1)}\times B^{(2n+1)}$, there exist bounded linear mappings $d_1:A\times_\theta B\to A^{(2n+1)}$ and $d_2:A\times_\theta B\to B^{(2n+1)}$ such that $D=(d_1,d_2)$. Let $d_A(a)=d_1((a,0)),d_B(b)=d_2((0,b)),T(b)=d_1((0,b))$ and $S(a)=d_2((a,0)),(a\in A,b\in B)$.
Then trivially $d_A,\ d_B,\ T$ and S are linear mappings satisfying (i). Moreover for every $a,c\in A$ and $b,d\in B$,

$$\begin{aligned}&D((a,b)\cdot(c,d))\\=\ &D\big((ac+\langle\theta,d\rangle a+\langle\theta,b\rangle c,\ bd)\big)\\=\ &\Big(d_A\big(ac+\langle\theta,d\rangle a+\langle\theta,b\rangle c\big)+T(bd),\ S\big(ac+\langle\theta,d\rangle a+\langle\theta,b\rangle c\big)+d_B(bd)\Big).\end{aligned}\tag{1}$$

And

$$\begin{aligned}&D\big((a,b)\big)\cdot(c,d)+(a,b)\cdot D\big((c,d)\big)\\=\ &\Big(d_A(a)+T(b),S(a)+d_B(b)\Big)\cdot(c,d)\\&+(a,b)\cdot\Big(d_A(c)+T(d),\ S(c)+d_B(d)\Big)\end{aligned}$$

$$\begin{aligned}
&= \Big((d_A(a)+T(b))\cdot c+\langle\theta,d\rangle(d_A(a)+T(b)),\\
&\quad \langle d_A(a)+T(b),c\rangle\theta+(S(a)+d_B(b))\cdot d\Big)\\
&\quad +\Big(a\cdot(d_A(c)+T(d))+\langle\theta,b\rangle(d_A(c)+T(d)),\\
&\quad \langle d_A(c)+T(d),a\rangle\theta+b\cdot(S(c)+d_B(d))\Big)\\
&= \Big((d_A(a)+T(b))\cdot c+\langle\theta,d\rangle(d_A(a)+T(b))+a\cdot(d_A(c)+T(d))\\
&\quad +\langle\theta,b\rangle(d_A(c)+T(d)),\langle d_A(a)+T(b),c\rangle\theta+(S(a)+d_B(b))\cdot d\\
&\quad +\langle d_A(c)+T(d),a\rangle\theta+b\cdot(S(c)+d_B(d))\Big).
\end{aligned} \tag{2}$$

Thus D is a derivation if and only if (1) and (2) coincide, from which we get,

$$\begin{aligned}
&d_A(ac)+\langle\theta,d\rangle d_A(a)+\langle\theta,b\rangle d_A(c)+T(bd)\\
=\ &(d_A(a)+T(b))\cdot c+\langle\theta,d\rangle(d_A(a)+T(b))+a\cdot(d_A(c)+T(d))\\
&+\langle\theta,b\rangle(d_A(c)+T(d));
\end{aligned} \tag{3}$$

and

$$\begin{aligned}
&S(ac)+\langle\theta,d\rangle S(a)+\langle\theta,b\rangle S(c)+d_B(bd)\\
=\ &\langle d_A(a)+T(b),c\rangle\theta+(S(a)+d_B(b))\cdot d+\langle d_A(c)+T(d),a\rangle\theta\\
&+b\cdot(S(c)+d_B(d)).
\end{aligned} \tag{4}$$

Therefore D is a derivation if and only if the equations (3) and (4) are satisfied. Now a straightforward verification shows that if d_A and d_B are derivations and the equalities $(ii),(iii),(iv),(v)$ are satisfied then (3) and (4) hold. Applying (3) and (4) for suitable values of a,b,c,d shows that d_A and d_B are derivations and the equalities $(ii),(iii),(iv),(v)$ are also satisfied, as claimed.

For the last part, first note that if $f\in A^{(2n+1)}$ and $g\in B^{(2n+1)}$ then for each $a\in A, b\in B$, $\delta_{(f,g)}(a,b)=(\delta_f(a),\delta_g(b))$. Now if $D=\delta_{(f,g)}$ for some $f\in A^{(2n+1)}, g\in B^{(2n+1)}$, then $(d_A(a),S(a))=D((a,0))=\delta_{(f,g)}(a,0)=(\delta_f(a),0)$ for each $a\in A$. It follows that $d_A=\delta_f$ and $S=0$. Similarly $d_B=\delta_g$ and $T=0$. Moreover if $S=0,T=0,d_A=\delta_f$ and $d_B=\delta_g$ then $D=\delta_{(f,g)}$; and this completes the proof. □

The next result, which is devoted to the even case, needs a similar proof as Proposition 2.1.

Proposition 2.2. *A bounded linear map $D: A\times_\theta B\to(A\times_\theta B)^{(2n)}$ is a derivation if and only if there exist a derivation $d_B: B\to B^{(2n)}$ and bounded linear maps $S: A\to B^{(2n)}$, $T: B\to A^{(2n)}$ and $R: A\to A^{(2n)}$ such that for each $a,c\in A$, $b,d\in B$,*

(i) $D((a,b))=(R(a)+T(b),S(a)+d_B(b))$,

(ii) $S(ac) = 0$,
(iii) $R(ac) = a \cdot R(c) + R(a) \cdot c + \langle \theta, S(c) \rangle a + \langle \theta, S(a) \rangle c$,
(iv) $b \cdot S(a) = S(a) \cdot b = \langle \theta, b \rangle S(a)$,
(v) $T(b) \cdot a = a \cdot T(b) = -\langle \theta, d_B(b) \rangle a$, *and*
(vi) $T(bd) = \langle \theta, b \rangle T(d) + \langle \theta, d \rangle T(b)$.

In particular, D *is inner if and only if* $S = 0$, $T = 0$ *and* R, d_B *are inner derivations.*

To study $(2n+1)-$weak amenability of $A \times_\theta B$ we need the next convenience. A derivation $d : A \to A^{(2n+1)}$ is said to be $(2n+1)-$cyclic if $\langle d(a), c \rangle + \langle d(c), a \rangle = 0, (a, c \in A)$. We say that A is $(2n+1)-$cyclicly weakly amenable if every $(2n+1)-$cyclic derivation from A into $A^{(2n+1)}$ is inner.

Proposition 2.3. *If* $A \times_\theta B$ *is* $(2n+1)-$*weakly amenable then* B *is* $(2n+1)-$*weakly amenable and* A *is* $(2n+1)-$*cyclicly weakly amenable.*

Proof. Let $d_A : A \to A^{(2n+1)}$ be a $(2n+1)-$cyclic derivation and let $d_B : B \to B^{(2n+1)}$ be a derivation. Define $D : A \times_\theta B \to (A \times_\theta B)^{(2n+1)}$ by $D = (d_A, d_B)$. That $D = (d_A, d_B)$ is a derivation follows trivially from Proposition 2.1 with the hypotheses that d_B is a derivation and d_A is a $(2n+1)-$cyclic derivation. Now $(2n+1)-$weak amenability of $A \times_\theta B$ implies that $D = \delta_{(f,g)}$ for some $f \in A^{(2n+1)}$ and $g \in B^{(2n+1)}$. It follows that $d_A = \delta_f$ and $d_B = \delta_g$. □

Applying the latter proposition for the case $n = 0$ we obtain the next result.

Corollary 2.1 ([6, Theorem 2.11])**.** *Let* $A \times_\theta B$ *be weakly amenable then* B *is weakly amenable and* A *is cyclicly weakly amenable.*

In the next result we provide some conditions under which $(2n+1)-$weakly amenablility of A and B implies that of $A \times_\theta B$. It is worthwhile mentioning that a more general result has been proved in [3, Theorem 4.1], by an approach slightly different from ours . The proof, however, contains a gap and, to our knowledge, we have not able to fix it.

Proposition 2.4. *Let* A *and* B *be* $(2n+1)-$*weakly amenable. If either* $\overline{A \cdot A^{(2n)}} = A^{(2n)}$ *or* $\overline{A^{(2n)} \cdot A} = A^{(2n)}$ *then* $A \times_\theta B$ *is* $(2n+1)-$*weakly amenable.*

Proof. Let $D : A \times_\theta B \to (A \times_\theta B)^{(2n+1)}$ be a derivation. By Proposition 2.1 there exist derivations $d_A : A \to A^{(2n+1)}$, $d_B : B \to B^{(2n+1)}$ and bounded linear maps $S : A \to B^{(2n+1)}$, $T : B \to A^{(2n+1)}$ such that $D((a, b)) = (d_A(a) + T(b), S(a) + d_B(b)), (a \in A, b \in B)$. Since A and B are $(2n+1)-$weakly amenable, $d_A = \delta_f$ and $d_B = \delta_g$ for some $f \in A^{(2n+1)}, g \in B^{(2n+1)}$, and so by Proposition 2.1 (v),

$$S(ac) = (\langle d_A(a), c \rangle + \langle d_A(c), a \rangle)\theta = (\langle \delta_f(a), c \rangle + \langle \delta_f(c), a \rangle)\theta = 0 \quad (a, c \in A).$$

It follows that S vanishes on A^2. Furthermore, $(2n+1)-$weakly amenability of A implies the weak amenability of A and so $\overline{A^2} = A$, (see [1, Propositions

1.2, 1.3]). We thus have $S=0$. By Proposition 2.1 (ii), for every $a\in A, b\in B$ we have $a\cdot T(b)=T(b)\cdot a=0$. This yields that

$$\langle T(b), a\cdot F\rangle=\langle T(b)\cdot a, F\rangle=0 \text{ and } \langle T(b), F\cdot a\rangle=\langle a\cdot T(b), F\rangle=0 \quad (F\in A^{(2n)}).$$

That is, $T(b)$ vanishes on both $A^{(2n)}\cdot A$ and $A\cdot A^{(2n)}$. These together with either of the identities $\overline{A\cdot A^{(2n)}}=A^{(2n)}$ or $\overline{A^{(2n)}\cdot A}=A^{(2n)}$ imply that $T=0$. Therefore $D=\delta_{(f,g)}$, as required. □

As a rapid consequence of Proposition 2.4 we get:

Corollary 2.2. *If A and B are weakly amenable then $A\times_\theta B$ is weakly amenable.*

Let $\theta\in\sigma(B)$. A derivation $d: B\to B^{(2n)}$ is said to be $\theta_{(2n)}-$null if $\langle\theta, d(b)\rangle=0$, $(b\in B)$; (for the case $n=0$ this means that $d(B)\subseteq ker\theta$). A Banach algebra B is said to be $\theta_{(2n)}-$null weakly amenable if every $\theta_{(2n)}-$null derivation is inner. As an analogous to Proposition 2.3 for the even case we present the next result.

Proposition 2.5. *If $A\times_\theta B$ is $(2n)-$weakly amenable then A is $(2n)-$weakly amenable and B is $\theta_{(2n)}-$null weakly amenable.*

Proof. Let $d_A: A\to A^{(2n)}$ be a derivation and let $d_B: B\to B^{(2n)}$ be a $\theta_{(2n)}-$null derivation. Define $D: A\times_\theta B\to (A\times_\theta B)^{(2n)}$ by $D((a,b))=(d_A(a), d_B(b))$. Applying Proposition 2.2 to D shows that D is a derivation (note that, the hypotheses that d_A is a derivation and d_B is $\theta_{(2n)}-$null is necessary for (iii) and (v) of Proposition 2.2 to hold). So there exists $(F,G)\in (A\times_\theta B)^{(2n)}$ such that $D=\delta_{(F,G)}=(\delta_F,\delta_G)$. It follows that $d_A=\delta_F$ and $d_B=\delta_G$. □

In this stage we would like to state that unfortunately we do not know an analogous to Proposition 2.4 for the even case, in general. However, we establish such a result in Theorem 3.1, which is heavily based on the hypothesis that A is unital.

3. The case that A is unital

Here we turn our attention to the case where A has an identity. In this case the characterizations of derivations $D: A\times_\theta B\to (A\times_\theta B)^{(n)}$ presented in Propositions 2.1 and 2.2 can be considerably simplified. Indeed, a direct verification summarizes the Propositions 2.1 and 2.2 in the line of the following result.

Proposition 3.1. *Let A be unital with the identity 1_A. Then*

(i) every derivation $D: A\times_\theta B\to (A\times_\theta B)^{(2n+1)}$ is in the form of $D=(d_A, S+d_B)$, where $d_A: A\to A^{(2n+1)}$, $d_B: B\to B^{(2n+1)}$ are derivations and $S(a)=\langle d_A(a), 1_A\rangle\theta$ $(a\in A)$.

(ii) Every derivation $D : A \times_\theta B \to (A \times_\theta B)^{(2n)}$ *is in the form of* $D = (d_A + T, d_B)$ *where* $d_A : A \to A^{(2n)}$, $d_B : B \to B^{(2n)}$ *are derivations and* $T(b) = -\langle \theta, d_B(b)\rangle 1_A \quad (b \in B)$.

As a consequence of the latter characterizations of derivations we give the next result concerning to the $n-$weak amenability of $A \times_\theta B$ which actually provides a unified approach for improving the Propositions 2.3, 2.4 and 3.1 in the case where A is unital.

Theorem 3.1. *If* A *is unital then* $A \times_\theta B$ *is* $n-$*weakly amenable if and only if both* A *and* B *are* $n-$*weakly amenable.*

Proof. We only prove the odd case, the even case needs a similar argument. By Proposition 3.1 every derivation $D : A \times_\theta B \to (A \times_\theta B)^{(2n+1)}$ is in the form of

$$D((a,b)) = (d_A(a), S(a) + d_B(b)) \quad (a \in A, b \in B),$$

where $d_A : A \to A^{(2n+1)}$ and $d_B : B \to B^{(2n+1)}$ are derivations and $S : A \to B^{(2n+1)}$ satisfies $S(a) = \langle d_A(a), 1_A\rangle\theta$. It is easy to check that if d_A is inner then $S = 0$; indeed, if $d_A = \delta_f$ for some $f \in A^{(2n+1)}$ then $\langle d_A(a), 1_A\rangle = \langle f, a1_A - 1_A a\rangle = 0$. Therefore $D = \delta_{(f,g)} = (\delta_f, \delta_g)$, for some $f \in A^{(2n+1)}$ and $g \in B^{(2n+1)}$ if and only if $d_A = \delta_f$ and $d_B = \delta_g$. In other words, $A \times_\theta B$ is $(2n+1)-$weakly amenable if and only if both A and B are $(2n+1)-$weakly amenable. □

In the case when A is unital we can say more about $A \times_\theta B$. Indeed, there is a close relation between the first cohomology groups $H^1(A \times_\theta B, (A \times_\theta B)^{(n)})$, $H^1(A, A^{(n)})$ and $H^1(B, B^{(n)})$. More precisely, a standard argument based on the characterizations of derivations in Proposition 3.1 implies that:

$$H^1(A \times_\theta B, (A \times_\theta B)^{(n)}) \cong H^1(A, A^{(n)}) \oplus H^1(B, B^{(n)}).$$

One now can also derive Theorem 3.1 as an immediate consequence of this identification.

Remark 3.1. *The results in the last section, especially Theorem 3.1, are based on the case that* A *is unital. What happens if the requirement that* A *be unital is replaced by "*A *possessing a bounded approximate identity"? To the best of our knowledge, however, no example was yet known whether this fails if one considers the case "*A *has a bounded approximate identity" instead.*

Acknowledgment

The authors would like to thank the anonymous referee for his/her helpful comments. This research was supported by a grant from Ferdowsi University of Mashhad (No. MP90217VIS).

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