ON $(\alpha-\phi)$-MEIR-KEELER CONTRACTIONS ON PARTIAL HAUSDORFF METRIC SPACES

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In this note we introduce the concept of a $(\alpha-\phi)$-Meir-Keeler contraction for multi-valued mappings and we investigate the existence of fixed points of such mappings in a complete partial metric space. Our results generalize, extend and unify several recent fixed point results.

Keywords: Fixed point; strictly α-admissible; ϕ-Meir-Keeler contraction; Partial Hausdorff metric space.

1. Introduction and Preliminaries

Throughout the paper, $\mathbb{N}$ and $\mathbb{N}_0$ denote the set of positive integers and the set of nonnegative integers, respectively. Furthermore, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}^+_0$ represent the set of reals, positive reals and nonnegative reals, respectively.

We recall the notion of a partial metric introduced by Matthews [13].

Definition 1.1. [13] *A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+_0$ such that for all $x, y, z \in X$,*
(p_1) *$x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$;*
(p_2) *$p(x, x) \leq p(x, y)$;*
(p_3) *$p(x, y) = p(y, x)$;*
(p_4) *$p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.*
A pair (X, p) is called a partial metric space.

Remark 1.1. *If $p(x, y) = 0$, then from (p_1) and (p_2), we have $x = y$. The converse may not hold. For example let $X = \mathbb{R}^+_0$ and $p : X \times X \to \mathbb{R}^+_0$ be $p(x, y) = \max\{x, y\}$. Then (X, p) is a partial metric space and $p(x, x) \neq 0$ for all $x \in X \setminus \{0\}$.*

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x, \gamma) : x \in X, \gamma > 0\}$, where $B_p(x, \gamma) = \{y \in X : p(x, y) < p(x, x) + \gamma\}$ for all $x \in X$ and $\gamma > 0$. If p is a partial metric on X, then the function $d_p : X \times X \to \mathbb{R}^+_0$ given by

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

is a metric on X.

Definition 1.2. [13, 5] *Let (X, p) be a partial metric space. Then*
(1) *a sequence $\{x_n\}$ in (X, p) converges to $x \in X$ if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$;*

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(2) *a sequence* $\{x_n\}$ *in* (X,p) *is called a Cauchy sequence if* $\lim_{m,n\to\infty} p(x_m,x_n)$ *exists (and is finite);*
(3) (X,p) *is said to be complete if every Cauchy sequence* $\{x_n\}$ *in* X *converges, with respect to* τ_p, *to a point* $x\in X$ *such that* $p(x,x)=\lim_{m,n\to\infty} p(x_m,x_n)$;
(4) *a subset* A *of a partial metric space* (X,p) *is closed in* (X,p) *if it contains its limit points, that is, if a sequence* $\{x_n\}$ *in* A *converges to some* $x\in X$, *then* $x\in A$.
(5) *a subset* A *of a partial metric space* (X,p) *is bounded in* (X,p) *if there exist* $x_0\in X$ *and* $M\in\mathbb{R}$ *such that for all* $a\in A$, *we have* $a\in B_p(x_0,M)$, *that is,* $p(x_0,a)<p(a,a)+M$.

Remark 1.2. *The limit in a partial metric space may not be unique. For example, consider the sequence* $\{\frac{1}{n^2+2}\}_{n\in\mathbb{N}}$ *in the partial metric space* (X,p) *where* $p(x,y)=\max\{x,y\}$. *Note*

$$p(1,1)=\lim_{n\to\infty} p(1,\frac{1}{n^2+2}) \quad \text{and} \quad p(2,2)=\lim_{n\to\infty} p(2,\frac{1}{n^2+2}).$$

Lemma 1.1. [13, 16] *(1)* $\{x_n\}$ *is a Cauchy sequence in a partial metric space* (X,p) *if and only if it is a Cauchy sequence in the metric space* (X,d_p);

(2) a partial metric space (X,p) *is complete if and only if the metric space* (X,d_p) *is complete. Furthermore,* $\lim_{n\to\infty} d_p(x_n,x)=0$ *if and only if* $p(x,x)=\lim_{n\to\infty} p(x_n,x)=\lim_{n\to\infty} p(x_n,x_m)$.

Fixed point theory in partial metric spaces were studied by many authors in the literature (see [1, 2, 3, 4, 8, 10, 11, 12, 16, 18, 19] and the reference therein). The authors in [9] proved that some of the fixed point results in a partial metric space are equivalent to related results in the associated metric space. In fact, for a self-mapping T on X, the authors in [9] realized that $M_d^T(x,y)=M_p^T(x,y)$, where $M_\rho^T(x,y)=\max\{\rho(x,y),\rho(x,Tx),\rho(Ty,y),\rho(Tx,y),\rho(x,Ty)\}$ with $\rho=p,d$ where d,p are the metric, the partial metric, respectively. In our paper the recent result in [9] is not applicable.

Let (X,d) be a metric space and let $CB(X)$ denote the collection of all nonempty, closed and bounded subsets of X. For $A,B\in CB(X)$, we define

$$\mathcal{H}(A,B):=\max\{\sup_{a\in A} d(a,B),\sup_{b\in B} d(b,A)\},$$

where $d(x,B):=\inf\{d(x,b):b\in B\}$. In the literature, $\mathcal{H}$ is called the Hausdorff metric induced by the metric d. A multi-valued mapping $T:X\to CB(X)$ is called a contraction if

$$\mathcal{H}(Tx,Ty)\le kd(x,y),$$

for all $x,y\in X$ and $k\in[0,1)$.

Theorem 1.1. [15] *Let* (X,d) *be a complete metric space and* $T:X\to CB(X)$ *be a multi-valued contraction. Then there exists* $x\in X$ *such that* $x\in Tx$.

The authors in [5] considered the notion of a Hausdorff metric in a partial metric space (called the partial Hausdorff metric $\mathcal{H}_p$). Let (X,p) be a partial metric space and let $CB^p(X)$ be the collection of all nonempty, closed and bounded subset of the partial metric space (X,p). Let $\delta_p:CB^p(X)\times CB^p(X)\to\mathbb{R}_0^+$ and $\mathcal{H}_p:CB^p(X)\times CB^p(X)\to\mathbb{R}_0^+$ be mappings such that

$$p(x,A):=\inf\{p(x,a):a\in A\},$$
$$\delta_p(A,B):=\sup\{p(a,B):a\in A\},$$
$$\delta_p(B,A):=\sup\{p(b,A):b\in B\},$$
$$\mathcal{H}_p(A,B)=\max\{\delta_p(A,B),\delta_p(B,A)\},$$

for $A,B\in CB^p(X)$ and $x\in X$ (for more details see [5]). If $p(x,A)=0$, then $d_p(x,A)=0$ where $d_p(x,A)=\inf\{d_p(x,a):a\in A\}$.

Remark 1.3. [5] *Let (X, p) be a partial metric space and A be a nonempty subset of X. Then*

$$a \in \overline{A} \text{ if and only if } p(a, A) = p(a, a).$$

The authors in [5] considered the mappings $\delta_p : CB^p(X) \times CB^p(X) \to \mathbb{R}_0^+$ and $\mathcal{H}_p : CB^p(X) \times CB^p(X) \to \mathbb{R}_0^+$ and proved the following results.

Proposition 1.1. [5] *Let (X, p) be a partial metric space. For $A, B \in CB^p(X)$, the following properties hold:*
(1) $\delta_p(A, A) = \sup\{p(a, a) : a \in A\}$;
(2) $\delta_p(A, A) \leq \delta_p(A, B)$;
(3) $\delta_p(A, B) = 0$ *implies that* $A \subset B$;
(4) $\delta_p(A, B) \leq \delta_p(A, C) + \delta_p(C, B) - \inf_{c \in C} p(c, c)$.

Proposition 1.2. [5] *Let (X, p) be a partial metric space. For $A, B \in CB^p(X)$, the following properties hold:*
(1) $\mathcal{H}_p(A, A) \leq \mathcal{H}_p(A, B)$;
(2) $\mathcal{H}_p(A, B) = \mathcal{H}_p(B, A)$;
(3) $\mathcal{H}_p(A, B) \leq \mathcal{H}_p(A, C) + \mathcal{H}_p(C, B) - \inf_{c \in C} p(c, c)$;
(4) $\mathcal{H}_p(A, B) = 0$ *implies that* $A = B$.

Lemma 1.2. [5] *Let (X, p) be a partial metric space, $A, B \in CB^p(X)$ and $h > 1$. For any $a \in A$, there exists $b = b(a) \in B$ such that*

$$p(a, b) \leq h\mathcal{H}_p(A, B).$$

In 1969, Meir and Keeler [14] introduced the notion of a Meir-Keeler-type contraction in a metric space (X, d).

Definition 1.3. [14] *Let (X, p) be a metric space, $f : X \to X$. Then f is called a Meir-Keeler-type contraction whenever for each $\eta > 0$ there exists $\gamma > 0$ such that*

$$\eta \leq d(x, y) < \eta + \gamma \Longrightarrow d(fx, fy) < \eta.$$

Definition 1.4. [17] *Let $f : X \to X$ be a self-mapping of a set X and $\alpha : X \times X \to \mathbb{R}_0^+$. Then f is called α-admissible if*

$$x, y \in X, \ \ \alpha(x, y) \geq 1 \Longrightarrow \alpha(fx, fy) \geq 1.$$

In this paper we introduce the notion of a $(\alpha - \phi)$-Meir-Keeler contraction for a multi-valued mapping $T : X \to CB^p(X)$ in a partial metric space (X, p) and we examine the existence of fixed points of such mappings. The results improve and extend several results in the literature including a recent paper of Chen and Karapınar [6]. In particular we note that the results are new in the metric space situation.

2. Main results

In this section, we state and prove our main result.

Let Ψ be the family of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

i): ψ is nondecreasing;

ii): there exist $k_0 \in \mathbb{N}$ and $a \in (0, 1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that

$$\psi^{k+1}(t) \leq a\psi^k(t) + v_k,$$

for $k \geq k_0$ and any $t \in \mathbb{R}^+$.

In the literature such functions are called (c)-comparison functions (see [25] and also [28, 29, 30]).

Lemma 2.1. *(See e.g. [25]) If $\psi \in \Psi$, then the following hold:*
(i) $(\psi^n(t))_{n\in\mathbb{N}}$ *converges to* 0 *as* $n \to \infty$ *for all* $t \in \mathbb{R}^+$*;*
(ii) $\psi(t) < t$*, for any* $t \in \mathbb{R}^+$;
(iii) ψ *is continuous at* 0*;*
(iv) *the series* $\sum_{k=1}^{\infty} \psi^k(t)$ *converges for any* $t \in \mathbb{R}^+$.

We denote by Φ the class of functions $\phi : (\mathbb{R}_0^+)^4 \to \mathbb{R}_0^+$ satisfying the following conditions:
(ϕ_1) ϕ is an increasing and continuous function in each coordinate;
(ϕ_2) for each ϕ there exists $\psi \in \Psi$ such that $\phi(t,t,t,t) = \psi(t)$ for all $t \in \mathbb{R}_0^+$,

Example 2.1. *Let* $\phi_1, \phi_2, \phi_3, \phi_4 : (\mathbb{R}_0^+)^4 \to \mathbb{R}_0^+$ *be mappings such that*

$$\begin{aligned} \phi_1(t_1,t_2,t_3,t_4) &= \tfrac{k}{4}(t_1+t_2+t_3+t_4), \\ \phi_2(t_1,t_2,t_3,t_4) &= k\max\{t_1,t_2,t_3,t_4\}, \\ \phi_3(t_1,t_2,t_3,t_4) &= \ln(1+\tfrac{k}{4}(t_1+t_2+t_3+t_4)), \end{aligned}$$

where $0 < k < 1$*. Note* $\phi_1, \phi_2, \phi_3 \in \Phi$*. Moreover, the corresponding* (c)*-comparison functions are*

$$\psi_1(t) = kt, \quad \psi_2(t) = kt, \quad \psi_2(t) = \ln(1+t)$$

where $\psi_1, \psi_2, \psi_3 \in \Psi$.

Example 2.2. *Consider a mapping* $\varphi : (\mathbb{R}_0^+)^4 \to \mathbb{R}_0^+$ *which is defined as*

$$\varphi_2(t_1,t_2,t_3,t_4) = k_1t_1 + k_2t_2 + k_3t_3 + k_4t_4,$$

where $0 < k_i, \quad i = 1,2,3,4$ *and* $\sum_{i=1}^4 k_i < 1$*. Hence,* $\varphi \in \Phi$*. Notice also that the corresponding* (c)*-comparison function is*

$$\psi(t) = (k_1+k_2+k_3+k_4)t,$$

where $\psi \in \Psi$.

We now introduce the notion of a $(\alpha-\phi)$-Meir-Keeler contraction with respect to the partial Hausdorff metric $\mathcal{H}_p$ induced by the partial metric.

Definition 2.1. *Let* (X,p) *be a partial metric space,* $\alpha : X \times X \to \mathbb{R}_0^+$ *be a mapping and* $\phi \in \Phi$*. A multi-valued mapping* $T : X \to CB^p(X)$ *is called a* $(\alpha-\phi)$*-Meir-Keeler contraction with respect to the associated partial Hausdorff metric* $\mathcal{H}_p$ *if the following conditions hold:*
(C) *For each* $\eta > 0$*, there exists* $\delta > 0$ *such that*

$$\begin{aligned} \eta \le{}& \phi(p(x,y), p(x,Tx), p(y,Ty), \frac{1}{2}[p(x,Ty)+p(y,Tx)]) < \eta+\delta \\ &\Longrightarrow \alpha(x,y)\mathcal{H}_p(Tx,Ty) < \eta, \end{aligned} \tag{2.2}$$

for all $x, y \in X$.
A multi-valued mapping T *is called a* ϕ*-Meir-Keeler-type contraction if* $\alpha(x,y) = 1$ *for all* $x, y \in X$ *in* (2.2)*, that is,*

$$\begin{aligned} &\eta \le \phi(p(x,y), p(x,Tx), p(y,Ty), \frac{1}{2}[p(x,Ty)+p(y,Tx)]) < \eta+\delta \\ &\Longrightarrow \mathcal{H}_p(Tx,Ty)) < \eta, \end{aligned}$$

for all $x, y \in X$.

Remark 2.1. *Note that if* $T : X \to CB^p(X)$ *is a* $(\alpha-\phi)$*-Meir-Keeler contraction with respect to the associated partial Hausdorff metric* $\mathcal{H}_p$*, then we have*

$$\alpha(x,y)\mathcal{H}_p(Tx,Ty) \le \phi(p(x,y), p(x,Tx), p(y,Ty), \frac{1}{2}[p(x,Ty)+p(y,Tx)]),$$

for all $x, y \in X$*. Notice that if* $\phi(p(x,y), p(x,Tx), p(y,Ty), \frac{1}{2}[p(x,Ty)+p(y,Tx)]) > 0$*, then we have*

$$\alpha(x,y)\mathcal{H}_p(Tx,Ty) < \phi(p(x,y), p(x,Tx), p(y,Ty), \frac{1}{2}[p(x,Ty)+p(y,Tx)]).$$

We now introduce the notion of a strictly α-admissible mapping which is a slight modification of Definition 1.4.

Definition 2.2. *Let* (X,p) *be a partial metric space,* $T : X \to CB^p(X)$ *and* $\alpha : X \times X \to \mathbb{R}_0^+$*. We say that* T *is strictly* α*-admissible if*

$$\alpha(x,y) > 1 \ \text{ implies } \ \alpha(y,z) > 1,$$

for all $x \in X$*,* $y \in Tx$ *and* $z \in Ty$*.*

We now state and prove our main result.

Theorem 2.1. *Let* (X,p) *be a complete partial metric space and* $\alpha : X \times X \to \mathbb{R}_0^+$ *be a mapping. Suppose that a multi-valued mapping* $T : X \to CB^p(X)$ *is a* $(\alpha-\phi)$*-Meir-Keeler contraction with respect to the associated partial Hausdorff metric* $\mathcal{H}_p$*. Also assume that*

(i) T *is strictly* $\alpha-$*admissible;*
(ii) there exists $x_0 \in X$ *such that* $\alpha(x_0,y) > 1$ *for all* $y \in Tx_0$*;*
(iii) if $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n,x_{n+1}) > 1$ *for all* n *and* $x_n \to x \in X$ *as* $n \to \infty$*, then* $\alpha(x_n,x) > 1$ *for all* n*.*

Then T *has a fixed point in* X*, that is, there exists* $x^* \in X$ *such that* $x^* \in Tx^*$*.*

Proof. Let $x_1 \in Tx_0$. Since $T : X \to CB^p(X)$ is a $(\alpha-\phi)$-Meir-Keeler contraction with respect to the associated partial Hausdorff metric $\mathcal{H}_p$, by Remark 2.1, we have that

$$\begin{aligned}
&\alpha(x_0,x_1)\mathcal{H}_p(Tx_0,Tx_1)\\
&\qquad \le \phi(p(x_0,x_1), p(x_0,Tx_0), p(x_1,Tx_1), \frac{1}{2}[p(x_0,Tx_1)+p(x_1,Tx_0)]). \qquad (2.3)
\end{aligned}$$

Let $\alpha(x_0,x_1) = k_0$. Note $k_0 > 1$. Now Lemma 1.2 with $h = \sqrt{k_0}$ implies that there exists $x_2 \in Tx_1$ with

$$p(x_1,x_2) \le \sqrt{k_0}\mathcal{H}_p(Tx_0,Tx_1). \qquad (2.4)$$

Using (2.3) and (2.4), we get

$$\begin{aligned}
p(x_1,x_2) &< \frac{1}{\sqrt{k_0}}\phi(p(x_0,x_1), p(x_0,Tx_0), p(x_1,Tx_1), \frac{1}{2}[p(x_0,Tx_1)+p(x_1,Tx_0)])\\
&\le \frac{1}{\sqrt{k_0}}\phi(p(x_0,x_1), p(x_0,x_1), p(x_1,x_2), \frac{1}{2}[p(x_0,x_2)+p(x_1,x_1)])\\
&\le \frac{1}{\sqrt{k_0}}\phi(p(x_0,x_1), p(x_0,x_1), p(x_1,x_2), \frac{1}{2}[p(x_0,x_1)+p(x_1,x_2)]). \qquad (2.5)
\end{aligned}$$

If $p(x_0,x_1) \le p(x_1,x_2)$, then, by (ϕ_1) we have

$$\begin{aligned}
p(x_1,x_2) &< \frac{1}{\sqrt{k_0}}\phi(p(x_0,x_1), p(x_0,x_1), p(x_1,x_2), \frac{1}{2}[p(x_0,x_1)+p(x_1,x_2)])\\
&\le \frac{1}{\sqrt{k_0}}\phi(p(x_1,x_2), p(x_1,x_2), p(x_1,x_2), p(x_1,x_2)).
\end{aligned}$$

By (ϕ_1) there exists $\psi \in \Psi$ so that

$$\phi(p(x_1,x_2), p(x_1,x_2), p(x_1,x_2), p(x_1,x_2)) = \psi(p(x_1,x_2)).$$

Combining the observations above, together with Lemma 2.1(ii), we find that

$$p(x_1, x_2) \le \frac{1}{\sqrt{k_0}}\psi(p(x_1, x_2)) < p(x_1, x_2),$$

which is a contradiction. Hence, we have $p(x_1, x_2) < p(x_0, x_1)$. Due to (ψ_1), we derive that

$$\begin{aligned} p(x_1, x_2) &< \frac{1}{\sqrt{k_0}}\phi(p(x_0, x_1), p(x_0, x_1), p(x_1, x_2), \frac{1}{2}[p(x_0, x_1) + p(x_1, x_2)]) \\ &\le \frac{1}{\sqrt{k_0}}\phi(p(x_0, x_1), p(x_0, x_1), p(x_0, x_1), p(x_0, x_1)) \\ &= \frac{1}{\sqrt{k_0}}\psi(p(x_0, x_1)) \\ &\le \frac{1}{\sqrt{k_0}}p(x_0, x_1). \end{aligned} \tag{2.6}$$

Let $\alpha(x_1, x_2) = k_1$. Note $k_1 > 1$. Now Lemma 1.2 with $h = \sqrt{k_1}$ implies that there exists $x_3 \in Tx_2$ with

$$p(x_2, x_3) \le \sqrt{k_1}\mathcal{H}_p(Tx_1, Tx_2). \tag{2.7}$$

Using (2.3) and (2.7), we obtain

$$\begin{aligned} p(x_2, x_3) &< \frac{1}{\sqrt{k_1}}\phi(p(x_1, x_2), p(x_1, Tx_1), p(x_2, Tx_2), \frac{1}{2}[p(x_1, Tx_2) + p(x_2, Tx_1)]) \\ &\le \frac{1}{\sqrt{k_1}}\phi(p(x_1, x_2), p(x_1, x_2), p(x_2, x_3), \frac{1}{2}[p(x_1, x_3) + p(x_2, x_2)]) \\ &\le \frac{1}{\sqrt{k_1}}\phi(p(x_1, x_2), p(x_1, x_2), p(x_2, x_3), \frac{1}{2}[p(x_1, x_2) + p(x_2, x_3)]). \end{aligned}$$

If $p(x_1, x_2) \le p(x_2, x_3)$, then by (ϕ_1) and (ϕ_2), we get

$$\begin{aligned} p(x_2, x_3) &< \frac{1}{\sqrt{k_1}}\phi(p(x_1, x_2), p(x_1, x_2), p(x_2, x_3), \frac{1}{2}[p(x_1, x_2) + p(x_2, x_3)]) \\ &\le \frac{1}{\sqrt{k_1}}\phi(p(x_2, x_3), p(x_2, x_3), p(x_2, x_3), p(x_2, x_3)) \\ &\le \frac{1}{\sqrt{k_1}}\psi(p(x_2, x_3)) < p(x_2, x_3), \end{aligned}$$

a contradiction. Thus, $p(x_2, x_3) < p(x_1, x_2)$ and hence, we have

$$\begin{aligned} p(x_2, x_3) &< \frac{1}{\sqrt{k_1}}\phi(p(x_1, x_2), p(x_1, x_2), p(x_2, x_3), \frac{1}{2}[p(x_1, x_2) + p(x_2, x_3)]) \\ &\le \frac{1}{\sqrt{k_1}}\phi(p(x_1, x_2), p(x_1, x_2), p(x_1, x_2), p(x_1, x_2)) \\ &\le \frac{1}{\sqrt{k_1}}\psi(p(x_1, x_2)) \le \psi(p(x_1, x_2)) \\ &\le \psi^2(p(x_0, x_1)) < p(x_0, x_1). \end{aligned} \tag{2.8}$$

Recursively, we obtain an iterative sequence $\{x_n\} \in X$ as follows:

$$x_n \in Tx_{n-1} \text{ for all } n \in \mathbb{N}.$$

Moreover, we put

$$\alpha(x_n, x_{n+1}) = k_n > 1, \text{ for all } n \in \mathbb{N}_0. \tag{2.9}$$

Since $T : X \to CB^p(X)$ is a $(\alpha - \phi)$-Meir-Keeler contraction with respect to the associated partial Hausdorff metric $\mathcal{H}_p$, again by Remark 2.1, we have

$$\begin{aligned} &\alpha(x_n, x_{n+1})\mathcal{H}_p(Tx_n, Tx_{n+1}) \\ \leq &\phi(p(x_n, x_{n+1}), p(x_n, Tx_n), p(x_{n+1}, Tx_{n+1}), \frac{1}{2}[p(x_n, Tx_{n+1}) + p(x_{n+1}, Tx_n)]) \end{aligned} \tag{2.10}$$

for all $n \in \mathbb{N}_0$. From Lemma 1.2 with $h = \sqrt{k_n}$, we obtain $x_{n+2} \in Tx_{n+1}$ such that

$$p(x_{n+1}, x_{n+2}) \leq \sqrt{k_n}\mathcal{H}_p(Tx_n, Tx_{n+1}), \quad n \in \mathbb{N}_0. \tag{2.11}$$

Using (2.10) and (2.11), we observe that

$$\begin{aligned} &p(x_{n+1}, x_{n+2}) \\ &< \frac{1}{\sqrt{k_n}}\phi(p(x_n, x_{n+1}), p(x_n, Tx_n), p(x_{n+1}, Tx_{n+1}), \frac{1}{2}[p(x_n, Tx_{n+1}) + p(x_{n+1}, Tx_n)]) \\ &\leq \frac{1}{\sqrt{k_n}}\phi(p(x_n, x_{n+1}), p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2}), \frac{1}{2}[p(x_n, x_{n+2}) + p(x_{n+1}, x_{n+1})]) \\ &\leq \frac{1}{\sqrt{k_n}}\phi(p(x_n, x_{n+1}), p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2}), \frac{1}{2}[p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2})]). \end{aligned}$$

If $p(x_n, x_{n+1}) \leq p(x_{n+1}, x_{n+2})$, then

$$\begin{aligned} p(x_{n+1}, x_{n+2}) &< \frac{1}{\sqrt{k_n}}\phi(p(x_{n+1}, x_{n+2}), p(x_{n+1}, x_{n+2}), p(x_{n+1}, x_{n+2}), p(x_{n+1}, x_{n+2})) \\ &\leq \psi(p(x_{n+1}, x_{n+2})) < p(x_{n+1}, x_{n+2}), \end{aligned}$$

a contradiction. Hence, we conclude that $p(x_n, x_{n+1}) > p(x_{n+1}, x_{n+2})$. Therefore, we have

$$p(x_{n+1}, x_{n+2}) \leq \frac{1}{\sqrt{k_n}}\psi(p(x_n, x_{n+1})) \leq \psi(p(x_n, x_{n+1})) < p(x_n, x_{n+1}). \tag{2.12}$$

Keeping the expression (2.6), (2.8) and (2.12) in mind, we obtain

$$\begin{aligned} p(x_{n+1}, x_{n+2}) &\leq \frac{1}{\sqrt{k_n}}\psi(p(x_n, x_{n+1}))) \\ &\leq \psi(p(x_n, x_{n+1})) \\ &\leq \psi^2(p(x_{n-1}, x_n)) \\ &\leq \cdots \\ &\leq \psi^n(p(x_0, x_1)). \end{aligned} \tag{2.13}$$

Taking Lemma 2.1(i) into account and by letting $n \to \infty$ in the inequality above we find that

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0. \tag{2.16}$$

From the property (p_2) of a partial metric and using (2.16), we have

$$\lim_{n\to\infty} p(x_n, x_n) = 0. \tag{2.17}$$

Now (2.15) together with the property (p_4) of a partial metric, for any $m \in \mathbb{N}$, we have

$$\begin{aligned} p(x_n, x_{n+m}) &\leq \sum_{i=1}^{m} p(x_{n+i-1}, x_{n+i}) - \sum_{i=1}^{m-1} p(x_{n+i}, x_{n+i}) \\ &\leq \sum_{p=n}^{n+m-1} \psi^n(p(x_1, x_0)) - \sum_{i=1}^{m-1} p(x_{n+i}, x_{n+i}) \\ &\leq \sum_{p=n}^{n+m-1} \psi^n(p(x_1, x_0)) \\ &\leq \sum_{p=n}^{+\infty} \psi^n(p(x_1, x_0)) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus,

$$\lim_{n\to\infty} p(x_n, x_{n+m}) = 0.$$

From the definition of d_p, we obtain that for any $m \in \mathbb{N}$,

$$\lim_{n\to\infty} d_p(x_n, x_{n+m}) \leq \lim_{n\to\infty} 2p(x_n, x_{n+m}) = 0. \tag{2.19}$$

Hence, we conclude that $\{x_n\}$ is a Cauchy sequence in (X, d_p). Since (X, p) is complete, from Lemma 1.1, (X, d_p) is a complete metric space. Therefore, $\{x_n\}$ converges to some $x^* \in X$ with respect to the associated partial metric, and, hence,

$$p(x^*, x^*) = \lim_{n\to\infty} p(x_n, x^*) = \lim_{n\to\infty} p(x_n, x_m) = 0. \tag{2.20}$$

Now we claim $p(x^*, Tx^*) = 0$. Suppose, on the contrary, that $p(x^*, Tx^*) > 0$. Since $T : X \to CB^p(X)$ is a $(\alpha - \phi)$-Meir-Keeler contraction with respect to associated the partial Hausdorff metric $\mathcal{H}_p$, by Remark 2.1, we have

$$\begin{aligned} \alpha(x_n, x^*)\mathcal{H}_p(Tx_n, Tx^*) &< \phi(p(x_n, x^*), p(x_n, Tx_n), p(x^*, Tx^*), \tfrac{1}{2}[p(x_n, Tx^*) + p(x^*, Tx_n)]) \\ &\leq \phi(p(x_n, x^*), p(x_n, x_{n+1}), p(x^*, Tx^*), \tfrac{1}{2}[p(x_n, Tx^*) + p(x^*, x_{n+1})]). \end{aligned}$$

Letting $n \to \infty$ in the inequality above (note (iii) in the statement of Theorem 2.1 and (2.20)), we have

$$\begin{aligned} \lim_{n\to\infty} \mathcal{H}_p(Tx_n, Tx^*) &\leq \lim_{n\to\infty} \alpha(x_n, x^*)\mathcal{H}_p(Tx_n, Tx^*) \\ &\leq \lim_{n\to\infty} \phi(p(x_n, x^*), p(x_n, x_{n+1}), p(x^*, Tx^*), \frac{1}{2}[p(x_n, Tx^*) + p(x^*, x_{n+1})]) \\ &= \phi(0, 0, p(x^*, Tx^*), \frac{1}{2}p(x^*, Tx^*) \\ &\leq \phi(p(x^*, Tx^*), p(x^*, Tx^*), p(x^*, Tx^*), p(x^*, Tx^*) \\ &= \psi(p(x^*, Tx^*)) \\ &< p(x^*, Tx^*); \end{aligned} \tag{2.21}$$

Hence,

$$\lim_{n\to\infty} \mathcal{H}_p(Tx_n, Tx^*) < p(x^*, Tx^*). \tag{2.23}$$

Now $x_{n+1} \in Tx_n$, so

$$p(x_{n+1}, Tx^*) \leq \delta_p(Tx_n, Tx^*) \leq \mathcal{H}_p(Tx_n, Tx^*). \tag{2.24}$$

From the weakened triangle inequality, together with the inequality (2.24) we have

$$\begin{aligned} p(x^*, Tx^*) &\leq p(x^*, x_{n+1}) + p(x_{n+1}, Tx^*) - p(x_{n+1}, x_{n+1}) \\ &\leq p(x^*, x_{n+1}) + p(x_{n+1}, Tx^*) \\ &\leq p(x^*, x_{n+1}) + \mathcal{H}_p(Tx_n, Tx^*). \end{aligned}$$

Keeping the inequality (2.23) in mind and letting $n \to \infty$ in the inequality above, we derive that

$$p(x^*, Tx^*) \leq \lim_{n\to\infty} p(x^*, x_{n+1}) + \lim_{n\to\infty} \mathcal{H}_p(Tx_n, Tx^*) < p(x^*, Tx^*),$$

which is a contradiction. As a result we have

$$p(x^*, Tx^*) = 0.$$

Now from (2.20), $p(x^*, x^*) = 0$, so we have

$$p(x^*, x^*) = 0 = p(x^*, Tx^*).$$

This implies $x^* \in Tx^*$ from Remark 1.3. □

Example 2.3. *Let $X = \{0, 1, 2\}$ be endowed with the partial metric $p : X \times X \to \mathbb{R}^+$ defined by*

$$p(x, y) = \frac{1}{2}|x - y| + \max\{x, y\} \text{ for all } x, y \in X.$$

Then (X, p) is a complete partial metric space, and we have

$$p(0, 0) = 0; \ p(1, 1) = 1; \ p(2, 2) = 2;$$

$$p(0, 1) = p(1, 0) = \frac{3}{2}; \ p(0, 2) = p(2, 0) = 3; \ p(1, 2) = \frac{5}{2}.$$

We next define $T : X \to CB(X)$ by

$$T(0) = T(1) = \{0\} \text{ and } T(2) = \{0, 1\}.$$

Then we have

(1) *if $x, y \in \{0, 1\}$, then $\mathcal{H}_p(T(x), T(y)) = \mathcal{H}_p(\{0\}, \{0\}) = 0$;*

(2) *if $x \in \{0, 1\}$ and $y = 2$; then $\mathcal{H}_p(T(0), T(2)) = \mathcal{H}_p(T(1), T(2)) = \mathcal{H}_p(\{0\}, \{0, 1\}) = \frac{3}{2}$;*

(3) *if $x = y = 2$; then $\mathcal{H}_p(T(2), T(2)) = \mathcal{H}_p(\{0, 1\}, \{0, 1\}) = sup\{p(x, x) : x \in \{0, 1\}\} = 1$;*

(4) *$p(0, T(0)) = 0$, $p(1, T(1)) = p(1, \{0\}) = \frac{3}{2}$, $p(2, T(2)) = p(2, \{0, 1\}) = \frac{5}{2}$;*

(5) *$p(0, T(1)) = 0$, $p(1, T(0)) = \frac{3}{2}$, $p(0, T(2)) = 0$, $p(2, T(0)) = 3$, $p(1, T(2)) = 1$, $p(2, T(1)) = 3$.*

Now, we put $\phi(t_1, t_2, t_3, t_4) = \frac{2}{3}\max\{t_1, t_2, t_3, t_4\}$. Then all of the hypotheses of Theorem 2.1 are satisfied. Note $x = 0$ is the unique fixed point of T.

3. Conclusion

One could list several corollaries using Example 2.1. In addition choosing α in a suitable way (see for example [17]) one can list several corollaries . Finally since each metric space is a partial metric space, we see that the analog of Theorem 2.1 in a metric space is new.

Competing interests

The authors declare that they have no competing interests.

Authors contributions

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