THE WEIGHTING METHOD AND MULTIOBJECTIVE PROGRAMMING UNDER NEW CONCEPTS OF GENERALIZED (Φ, ρ)-INVEXITY

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In the paper, the weighting method is used for solving the considered nonconvex vector optimization problem. The equivalence between a weak Pareto solution of the original vector optimization problem and an optimal solution of its corresponding unconstrained scalar optimization problem is established under (Φ,ρ)-invexity. Further, the definition of a differentiable KT-(Φ,ρ)-invex vector optimization problem is introduced and the sufficient optimality conditions are established for such nonconvex differentiable multiobjective programming problems. In order to prove several Mond-Weir duality results for a new class of nonconvex differentiable vector optimization problems, the concept of WD-(Φ,ρ)-invexity is introduced. It turns out that the results presented in the paper are proved also for such nonconvex vector optimization problems in which not all functions constituting them have the fundamental property of many generalized convexity notions, earlier introduced in the literature.

Keywords: multiobjective programming; weighting method; KT-(Φ,ρ)-invexity; WD-(Φ,ρ)-invexity; Pareto optimality; Mond-Weir duality.

1. Introduction

In recent years, attempts are made by several authors to define various classes of nonconvex optimization problems and to study their optimality criteria and duality results. In [25], Martin proposed two weaker notions than invexity introduced by Hanson [14], called KT-invexity and WD-invexity. He proved that every Kuhn-Tucker point is a minimizer of a scalar optimization problem with inequality constraints if and only if this problem is KT-invex. Also, he established that Wolfe weak duality holds if and only if the primal optimization problem is WD-invex. In recent years, several generalizations of Martin's definitions have been introduced to optimization theory in order to weaken the assumption of convexity in establishing optimality and duality results for new classes of nonconvex differentiable optimization problems (see, for example, [1], [2], [3], [4], [5], [6], [13], [16], [17], [18], [19], [20], [21], [22]). Recently, Caristi et al. [8] and later Ferrara and Stefanescu [10] established sufficient optimality conditions and duality results for differentiable scalar and vector optimization problems under (Φ,ρ)-invexity hypotheses.

In the paper, we use the weighting method for solving differentiable vector optimization problems involving (Φ,ρ)-invex functions. Thus, we relate a weak Pareto solution of such nonconvex smooth vector optimization problem to an optimal solution of its corresponding weighting scalar optimization problem constructed in this method. Further, by taking the motivation from Martin [14] and Caristi et al. [8], we generalize the definitions of KT-invexity, WD-invexity and (Φ,ρ)-invexity to the case of differentiable multiobjective programming problems with inequality constraints. Namely, we introduce the concepts of KT-(Φ,ρ)-invexity and WD-(Φ,ρ)-invexity for nonconvex differentiable

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vector optimization problems. We use the introduced notion of KT-(Φ,ρ)-invexity to establish sufficient optimality conditions and WD-(Φ,ρ)-invexity to establish several Mond-Weir duality results for a new class of nonconvex differentiable vector optimization problems. We also prove the equivalence between a weakly efficient solution of the KT-(Φ,ρ)-invex vector optimization problem and a minimizer in its associated scalar optimization problem constructed in the weighting method. It turns out that the sufficient optimality conditions and Mond-Weir duality results are established for such nonconvex vector optimization problems for which other generalized convexity notions existing in the literature may avoid.

2. Scalarization method for a new class of nonconvex differentiable vector optimization problems

For any vectors $x = (x_1, \dots, x_n)^T$, $y = (y_1, \dots, y_n)^T$ in R^n, we define: $x < y$ if and only if $x_i < y_i$ for all i = 1,2,...,n; $x \leqq y$ if and only if $x_i \leqq y_i$ for all i = 1,2,...,n; $x \leq y$ if and only if $x \leqq y$ and $x \neq y$; $x = y$ if and only if $x_i = y_i$ for all i = 1,2,...,n;

We consider the following unconstrained vector optimization problem:

$$f(x) = \left(f_1(x), \dots, f_k(x)\right) \to min \quad \text{s.t.} \quad x \in X, \quad \text{(UVOP)}$$

where $f: X \to R^k$ is a differentiable function on a nonempty open convex set $X \subseteq R^n$.

Definition 2.1. A feasible point $\bar{x}$ is said to be a weak Pareto (weakly efficient, weak minimum) solution for (UVOP) if and only if there exists no $x \in X$ such that $f(x) < f(\bar{x})$.

Definition 2.2. A feasible point $\bar{x}$ is said to be a Pareto (efficient) solution for (UVOP) if and only if there exists no $x \in X$ such that $f(x) \leq f(\bar{x})$.

Definition 2.3. [10] The function $f: X \to R^k$ is said to be $\left(\Phi, \rho_f\right)$-invex at $\bar{x} \in X$ on X if there exist a function $\Phi : X \times X \times R^{n+1} \to R$, where $\Phi(x, \bar{x}, \cdot)$ is convex on R^{n+1}, $\Phi\left(x, \bar{x}, (0, a)\right) \geqq 0$ for all $x \in X$ and any $a \in R_+$, $\rho = (\rho_1, \dots, \rho_k) \in R^k$, such that, the following inequalities

$$f_i(x) - f_i(\bar{x}) \geqq \Phi\left(x, \bar{x}, \left(\nabla f_i(\bar{x}), \rho_{f_i}\right)\right), \quad i = 1,\dots,k, \tag{1}$$

hold for all $x \in X$. If inequalities (1) are satisfied at any point $\bar{x} \in X$, then f is said to be a $\left(\Phi, \rho_f\right)$-invex function on X.

Definition 2.4. A feasible point $\bar{x} \in X$ is said to be a vector critical point of the problem (UVOP) if there exists a vector $\lambda \in R^k$ with $\lambda \geq 0$ such that $\nabla f(\bar{x}) = 0$.

Scalar stationary points are those whose vector gradients are zero. For vector optimization problems, vector critical points are those such that there exists a non-negative linear combination of the gradient vectors of each component objective function, valued at this point, equal to zero. Craven [9] established the following result for the problem (UVOP):

Theorem 2.1. Let $\bar{x} \in X$ be a weakly efficient solution to the problem (UVOP). Then there exists vector $\bar{\lambda} \in R^k$ with $\bar{\lambda} \geq 0$ such that $\bar{\lambda}\nabla f(\bar{x}) = 0$, in other words, $\bar{x}$ is a critical point to the problem (UVOP).

Now, we give a condition under which any critical point of the unconstrained vector optimization problem (UVOP) is its weakly efficient solution.

Theorem 2.2. Let $\bar{x} \in X$ be a vector critical point of the unconstrained vector optimization problem (UVOP), that is, there exists $\bar{\lambda} \geq 0$ such that $\bar{\lambda}\nabla f(\bar{x}) = 0$. Further, assume that the objective function f is $\left(\Phi, \rho_f\right)$-invex at $\bar{x} \in X$ on X, where $\sum_{i=1}^{k} \bar{\lambda}_i \rho_{f_i} \geqq 0$. Then $\bar{x}$ is a weakly efficient solution of the problem (UVOP).

Proof Since $\bar{x}$ is a vector critical point of the unconstrained vector optimization problem (UVOP), there exists $\bar{\lambda} \geq 0$ such that $\bar{\lambda}\nabla f(\bar{x}) = 0$. By assumption, f is a $\left(\Phi, \rho_f\right)$-invex function at $\bar{x}$ on X. Then, by Definition 2.3, inequalities (1) are satisfied. Multiplying (1) by $\bar{\lambda}_i$, i=1,…,k, we have

$$\bar{\lambda}_i f_i(x) - \bar{\lambda}_i f_i(\bar{x}) \geqq \bar{\lambda}_i \Phi\left(x, \bar{x}, \left(\nabla f_i(\bar{x}), \rho_{f_i}\right)\right), \quad i = 1,\ldots,k. \tag{2}$$

Let us denote $\bar{\alpha}_i = \sum_{i=1}^{k} \frac{\bar{\lambda}_i}{\sum_{t=1}^{k} \bar{\lambda}_t}$. Note that $0 \leqq \bar{\alpha}_i \leqq 0$, but at least one $\bar{\alpha}_i > 0$ and, moreover, $\sum_{i=1}^{k} \bar{\alpha}_i = 1$. Then, (2) yields

$$\sum_{i=1}^{k} \bar{\alpha}_i f_i(x) - \sum_{i=1}^{k} \bar{\alpha}_i f_i(\bar{x}) \geqq \sum_{i=1}^{k} \bar{\alpha}_i \Phi\left(x, \bar{x}, \left(\nabla f_i(\bar{x}), \rho_{f_i}\right)\right). \tag{3}$$

By definition, $\Phi(x, \bar{x}, \cdot)$ is convex on R^{n+1}. Since $0 \leqq \bar{\alpha}_i \leqq 0$ and, moreover, $\sum_{i=1}^{k} \bar{\alpha}_i = 1$, by the definition of a convex function, we have

$$\Phi\left(x, \bar{x}, \left(\sum_{i=1}^{k} \bar{\alpha}_i \nabla f_i(\bar{x}), \sum_{i=1}^{k} \bar{\alpha}_i \rho_{f_i}\right)\right) \leqq \sum_{i=1}^{k} \bar{\alpha}_i \Phi\left(x, \bar{x}, \left(\nabla f_i(\bar{x}), \rho_{f_i}\right)\right). \tag{4}$$

By (3) and (4), it follows that

$$\sum_{i=1}^{k} \bar{\alpha}_i f_i(x) - \sum_{i=1}^{k} \bar{\alpha}_i f_i(\bar{x}) \geqq \Phi\left(x, \bar{x}, \left(\sum_{i=1}^{k} \bar{\alpha}_i \nabla f_i(\bar{x}), \sum_{i=1}^{k} \bar{\alpha}_i \rho_{f_i}\right)\right). \tag{5}$$

By $\bar{\lambda}\nabla f(\bar{x}) = 0$, (5) gives

$$\sum_{i=1}^{k} \bar{\alpha}_i f_i(x) - \sum_{i=1}^{k} \bar{\alpha}_i f_i(\bar{x}) \geqq \Phi\left(x, \bar{x}, \frac{1}{\sum_{t=1}^{k} \bar{\lambda}_t}\left(0, \sum_{i=1}^{k} \bar{\lambda}_i \rho_{f_i}\right)\right). \tag{6}$$

By definition, $\Phi\left(x, \bar{x}, (0, a)\right) \geqq 0$ for all $x \in X$ and any $a \in R_+$. Hence, hypothesis $\sum_{i=1}^{k} \bar{\lambda}_i \rho_{f_i} \geqq 0$ and (6) yield that the following inequality $\sum_{i=1}^{k} \bar{\alpha}_i f_i(x) \geqq \sum_{i=1}^{k} \bar{\alpha}_i f_i(\bar{x})$ holds for all $x \in X$. Since $\bar{\alpha} \geq 0$, the above inequality implies that $\bar{x}$ is a weakly efficient solution of the problem (UVOP).

One of the methods used for solving vector optimization problems, the weighting method, relates their weakly efficient solutions to optimal solutions of corresponding scalar problems. In this approach, the following scalar unconstrained optimization problem, the so-called weighting scalar optimization problem, is constructed for the considered multiobjective programming problem $\sum_{i=1}^{k} \lambda_i f_i(x) \to min$ s.t. $x \in X$, (P(λ)) where $\lambda \in R^k$. The following result is well-known in the literature (see, [15]):

Theorem 2.3. A minimizer of the weighting scalar optimization problem (P(λ)) is a weakly efficient solution of the vector optimization problem (UVOP). If all weighting coefficients are positive, that is $\lambda_i > 0$, i = 1,...,k, then an optimal solution of the problem (P(λ)) is an efficient solution of the problem (UVOP).

Now, we prove the converse result for a new class of nonconvex vector optimization problems.

Theorem 2.4. Let the objective function f in the problem (UVOP) be $\left(\Phi, \rho_f\right)$-invex on X. Further, assume that $\bar{x}$ is a weakly efficient solution in problem (UVOP) and the necessary optimality conditions are satisfied at $\bar{x}$ with Lagrange multiplier $\bar{\lambda} \in R^k, \bar{\lambda} \geq$

0 with $\sum_{i=1}^{k} \bar{\lambda}_i \rho_{f_i} \geqq 0$. Then a weakly efficient solution $\bar{x}$ of (UVOP) solves a weighting scalar optimization problem.

Proof The proof is similar to the proof of Theorem 2.2 and, hence, it is omitted.

Corollary 2.1. Let the objective function f in the problem (UVOP) be (Φ, ρ_f)-invex on X with $\rho_{f_i} \geqq 0$, i = 1,…k. Then every weakly efficient solution of the problem (UVOP) solves a weighting scalar optimization problem.

3. KT-(Φ, ρ)-invexity and optimality

In many practical applications, a vector optimization problem has the set of all feasible solutions given by a number of inequality constraints. Therefore, we consider the following constrained vector optimization problem with inequality constraints:

$$f(x) = (f_1(x), \dots, f_k(x)) \to min \text{ s.t. } g_j(x) \leqq 0, \ j = 1, \dots, m, \ x \in X, \qquad \text{(VP)}$$

where $f_i: X \to R$, i ∈ I = {1,...,k} and $g_j: X \to R$, j ∈ J = {1,...m}, are differentiable functions defined on a nonempty open convex set $X \subseteq R^n$. Further, let $D := \{x \in X: \ g_j(x) \leqq 0, \ j = 1, \dots, m\}$ be the set of all feasible solutions of the considered vector optimization problem (VP) and $J(\bar{x}) := \{j \in J: \ g_j(\bar{x}) = 0\}$.

In this section, for the considered multiobjective programming problem (VP), we define a new concept of generalized convexity which is a generalization of a class of (Φ,ρ)-invex functions earlier defined by Caristi et al. [8] and the class of differentiable KT-invex vector optimization problems introduced by Osuna-Gómez et al. [17].

Definition 3.1. Let u ∈ D be given. If there exist a function $\Phi: D \times D \times R^{n+1} \to R$, where $\Phi(x, u, \cdot)$ is convex on R^{n+1}, $\Phi(x, u, (0, a)) \geqq 0$ for all $x \in D$ and any $a \in R_+$, $\rho = (\rho_{f_1}, \dots, \rho_{f_k}, \rho_g, \dots, \rho_{g_m}) \in R^{k+m}$, such that

$$\left.\begin{array}{r} x, u \in D \\ g(x) \leqq 0 \\ g(u) \leqq 0 \end{array}\right] \Rightarrow \left[\begin{array}{l} f_i(x) - f_i(\mathrm{u}) \geqq \Phi\left(x, \mathrm{u}, \left(\nabla f_i(\mathrm{u}), \rho_{f_i}\right)\right), \ i \in I, \\ \quad \Phi\left(x, \mathrm{u}, \left(\nabla g_j(\mathrm{u}), \rho_{g_j}\right)\right) \leqq 0, \quad j \in J(u), \end{array}\right. \qquad (7)$$

then the vector optimization problem (VP) is said to be a vector KT-(Φ,ρ)-invex optimization problem at u ∈ D on D (with respect to Φ, ρ_f and ρ_g). If (7) is satisfied at any point u ∈ D, then the vector optimization problem (VP) is said to be a vector KT-(Φ,ρ)-invex optimization problem on D.

Definition 3.2. Let u ∈ D be given. If there exist a function $\Phi: D \times D \times R^{n+1} \to R$, where $\Phi(x, u, \cdot)$ is convex on R^{n+1}, $\Phi(x, u, (0, a)) \geqq 0$ for all $x \in D$ and any $a \in R_+$, $\rho = (\rho_{f_1}, \dots, \rho_{f_k}, \rho_g, \dots, \rho_{g_m}) \in R^{k+m}$, such that

$$\left.\begin{array}{r} x, u \in D, \ x \neq u \\ g(x) \leqq 0 \\ g(u) \leqq 0 \end{array}\right] \Rightarrow \left[\begin{array}{l} f_i(x) - f_i(\mathrm{u}) > \Phi\left(x, \mathrm{u}, \left(\nabla f_i(\mathrm{u}), \rho_{f_i}\right)\right), \ i \in I, \\ \quad \Phi\left(x, \mathrm{u}, \left(\nabla g_j(\mathrm{u}), \rho_{g_j}\right)\right) \leqq 0, \quad j \in J(u), \end{array}\right. \qquad (8)$$

then the vector optimization problem (VP) is said to be a vector strict KT-(Φ,ρ)-invex optimization problem at u ∈ D on D (with respect to Φ, ρ_f and ρ_g). If the above relation is satisfied at any point u ∈ D, then the vector optimization problem (VP) is said to be a vector strict KT-(Φ,ρ)-invex optimization problem on D.

In this section, we prove the sufficient optimality conditions for (weak) Pareto optimality in the considered multiobjective programming problem (VP) under assumption that (VP) is vector (strict) KT-(Φ,ρ)-invex. It is well known (see, for example, [11], [15]) that, under a suitable constraint qualification, if $\bar{x} \in D$ is a (weak) Pareto optimal solution in the considered multiobjective programming problem (VP), then the following necessary optimality conditions, known as Karush-Kuhn-Tucker conditions, are satisfied:

Theorem 3.1. (Karush-Kuhn-Tucker necessary optimality conditions). Let $\bar{x} \in D$ be a weak Pareto solution of the problem (VP) and a suitable constraint qualification be satisfied at $\bar{x}$. Then, there exist $\bar{\lambda} \in R^k$ and $\bar{\mu} \in R^m$ such that

$$\sum_{i=1}^{k} \bar{\lambda}_i \nabla f_i(\bar{x}) + \sum_{j=1}^{m} \bar{\mu}_j \nabla g_j(\bar{x}) = 0, \tag{9}$$

$$\bar{\mu}_j g_j(\bar{x}) = 0, \quad j \in J, \tag{10}$$

$$\bar{\lambda} \geq 0, \quad \bar{\mu} \geqq 0. \tag{11}$$

Definition 3.3. The point $(\bar{x}, \bar{\lambda}, \bar{\mu}) \in D \times R^k \times R^m$ is said to be a vector Karush-Kuhn-Tucker point of the considered vector optimization problem (VP), if the conditions (9)-(10) are satisfied at $\bar{x}$ with Lagrange multipliers $\bar{\lambda}$ and $\bar{\mu}$.

Theorem 3.2. Let the considered multiobjective programming problem (VP) be a vector KT-(Φ,ρ)-invex optimization problem on D with respect to Φ, ρ_f and ρ_g. Then, every vector Karush-Kuhn-Tucker point $(\bar{x}, \bar{\lambda}, \bar{\mu}) \in D \times R^k \times R^m$ of the problem (VP) is its weakly efficient solution if $\sum_{i=1}^{k} \bar{\lambda}_i \rho_{f_i} + \sum_{j=1}^{m} \bar{\mu}_j \rho_{g_j} \geqq 0$.

Proof Let the considered vector optimization problem (VP) be a vector KT-(Φ,ρ)-invex optimization problem on D. Further, we assume that $(\bar{x}, \bar{\lambda}, \bar{\mu}) \in D \times R^k \times R^m$ is a vector Karush-Kuhn-Tucker point of the problem (VP). Suppose, contrary to the result, that $\bar{x}$ is not a weakly efficient solution of the problem (VP). Then, by definition, there exists a feasible solution $\tilde{x}$ of the problem (VP) such that $f(\tilde{x}) < f(\bar{x})$. Since the problem (VP) is a vector KT-(Φ,ρ)-invex optimization problem on D, by Definition 3.1 and the Karush-Kuhn-Tucker necessary optimality condition (11), we get

$$\bar{\lambda}_i \Phi\left(\tilde{x}, \bar{x}, \left(\nabla f_i(\bar{x}), \rho_{f_i}\right)\right) \leqq 0, \quad i \in I, \tag{12}$$

$$\bar{\lambda}_{i^*} \Phi\left(\tilde{x}, \bar{x}, \left(\nabla f_{i^*}(\bar{x}), \rho_{f_{i^*}}\right)\right) < 0 \text{ for at least one } i^* \in I, \tag{13}$$

$$\bar{\mu}_j \Phi\left(\tilde{x}, \bar{x}, \left(\nabla g_j(\bar{x}), \rho_{g_j}\right)\right) \leqq 0, \quad j \in J(\bar{x}). \tag{14}$$

Let us denote $\bar{\alpha}_i = \sum_{i=1}^{k} \frac{\bar{\lambda}_i}{\sum_{i=1}^{k} \bar{\lambda}_i + \sum_{j=1}^{m} \bar{\mu}_j}$, i =1,…,k, $\bar{\beta}_j = \sum_{i=1}^{k} \frac{\bar{\mu}_j}{\sum_{i=1}^{k} \bar{\lambda}_i + \sum_{j=1}^{m} \bar{\mu}_j}$, j = 1,…,m. Note that $0 \leqq \bar{\alpha}_i \leqq 0$, but at least one $\bar{\alpha}_i > 0$, $0 \leqq \bar{\beta}_j \leqq 0$, and, moreover, $\sum_{i=1}^{k} \bar{\alpha}_i + \sum_{j=1}^{m} \bar{\beta}_j = 1$. By (12)-(14), it follows that

$$\sum_{i=1}^{k} \bar{\alpha}_i \Phi\left(\tilde{x}, \bar{x}, \left(\nabla f_i(\bar{x}), \rho_{f_i}\right)\right) + \sum_{j=1}^{m} \bar{\beta}_j \Phi\left(\tilde{x}, \bar{x}, \left(\nabla g_j(\bar{x}), \rho_{g_j}\right)\right) < 0. \tag{15}$$

By Definition 3.1, we have that $\Phi(\tilde{x}, \bar{x}, \cdot)$ is convex on R^{n+1}. Thus, by (15) and convexity of $\Phi(\tilde{x}, \bar{x}, \cdot)$, we obtain

$$\Phi\left(\tilde{x}, \bar{x}, \left(\sum_{i=1}^{k} \bar{\alpha}_i \nabla f_i(\bar{x}) + \sum_{j=1}^{m} \bar{\beta}_j \nabla g_j(\bar{x}), \sum_{i=1}^{k} \bar{\alpha}_i \rho_{f_i} + \sum_{j=1}^{m} \bar{\beta}_j \rho_{g_j}\right)\right) \leqq$$

$$\sum_{i=1}^{k} \bar{\alpha}_i \Phi\left(\tilde{x}, \bar{x}, \left(\nabla f_i(\bar{x}), \rho_{f_i}\right)\right) + \sum_{j=1}^{m} \bar{\beta}_j \Phi\left(\tilde{x}, \bar{x}, \left(\nabla g_j(\bar{x}), \rho_{g_j}\right)\right). \tag{16}$$

Combining (15) and (16), we get that the following inequality

$$\Phi\left(\tilde{x},\bar{x},\left(\sum_{i=1}^{k}\overline{\alpha}_i\nabla f_i(\bar{x})+\sum_{j=1}^{m}\bar{\beta}_j\nabla g_j(\bar{x}),\sum_{i=1}^{k}\overline{\alpha}_i\,\rho_{f_i}+\sum_{j=1}^{m}\bar{\beta}_j\rho_{g_j}\right)\right)<0$$

holds. By the Karush-Kuhn-Tucker necessary optimality condition (9), we have

$$\Phi\left(\tilde{x},\bar{x},\left(0,\sum_{i=1}^{k}\overline{\alpha}_i\,\rho_{f_i}+\sum_{j=1}^{m}\bar{\beta}_j\rho_{g_j}\right)\right)<0. \tag{17}$$

By assumption, it follows that $\sum_{i=1}^{k}\overline{\alpha}_i\,\rho_{f_i}+\sum_{j=1}^{m}\bar{\beta}_j\rho_{g_j}\geqq 0$. As it follows from Definition 3.1, $\Phi(\tilde{x},\bar{x},(0,a))\geqq 0$ for any $a\in R_+$. This implies that the inequality

$$\Phi\left(\tilde{x},\bar{x},\left(0,\sum_{i=1}^{k}\overline{\alpha}_i\,\rho_{f_i}+\sum_{j=1}^{m}\bar{\beta}_j\rho_{g_j}\right)\right)\geqq 0$$

holds, contradicting (17). This completes the proof of this theorem.

Theorem 3.3. Let the considered vector optimization problem (VP) be vector KT-(Φ,ρ)-invex on D, a suitable constraint qualification be satisfied at any weakly efficient solution $\bar{x}$ of the problem (VP) and the Karush-Kuhn-Tucker necessary optimality conditions be at $\bar{x}$ satisfied with Lagrange multipliers $\bar{\lambda}\in R^k$ and $\bar{\mu}\in R^m$. If $\sum_{i=1}^{k}\bar{\lambda}_i\,\rho_{f_i}+\sum_{j=1}^{m}\bar{\mu}_j\rho_{g_j}\geqq 0$, then every weakly efficient solution of the original vector optimization problem (VP) solves a weighting scalar optimization problem.

In order to illustrate the results established in this section, we consider the example of a multiobjective programming problem with KT-(Φ,ρ)-invex functions.

Example 3.1. Consider the following nonconvex multiobjective programming problem

$$\begin{aligned} f(x)&=\left(ln((x_1-1)^2+1)\,,ln((x_2-1)^2+1)\right)\\ g(x)&=1-x_1x_2\leqq 0. \end{aligned}\qquad (VP1)$$

Note that $D=\{(x_1,x_2)\in R^2\colon x_1x_2\geqq 1\}$ and $\bar{x}=(1,1)$ is such a feasible solution at which the Karush-Kuhn-Tucker necessary optimality conditions are satisfied. It can be shown, by Definition 3.1, that (VP1) is KT-(Φ,ρ)-invex at $\bar{x}$ on D, where

$$\begin{aligned}\Phi(\mathrm{x},\bar{x},(\vartheta,\rho))&=\vartheta_1\ln((\mathrm{x}_1-1)^2+1)+\vartheta_2\ln((\mathrm{x}_2-1)^2+1)\\ &\quad+(2^\rho-1)\left(\ln((\mathrm{x}_1-1)^2+1)+\ln((\mathrm{x}_2-1)^2+1)\right),\end{aligned}$$

and ρ is equal to $\rho_{f_1}=0$, $\rho_{f_2}=0$ and $\rho_g=1$, respectively. Note that all hypotheses of Theorem 3.3 are satisfied, then $\bar{x}$ is Pareto optimal to the considered multiobjective programming problem. It is not difficult to show that the constraint function g is not invex on D with respect to any function $\eta: D\times D\to R^2$. This follows from the fact that a stationary point of the constraint function g is not its global minimizer (see [7]). Since not all functions constituting the considered vector optimization problem are invex with respect to the same function η (what is more, some of them are not invex with respect to any η), then the sufficient optimality conditions given in [17] are not applicable in this case. Further, the objective function f and the constraint function g are not (Φ,ρ)-invex at $\bar{x}$ on D with respect to Φ and ρ defined above and, therefore, also the sufficient conditions given in [10] are not applicable in this case. Thus, the optimality conditions established in the paper are applicable for a larger class of nonconvex vector optimization problems than the sufficient optimality conditions established under other generalized convexity notions, even those ones mentioned above.

4. WD-(Φ,ρ)-invexity and duality

In this section, for the considered multiobjective programming problem (VP), consider the following dual problem in the sense of Mond-Weir:

$$f(y) \to min \;\; \text{s.t.} \;\; \sum_{i=1}^{k} \lambda_i \nabla f_i(y) + \sum_{j=1}^{m} \mu_j \nabla g_j(y) = 0,$$
$$\mu_j g_j(y) = 0, j = 1, \dots, m, \;\; \lambda \in R^k, \lambda \geq 0, \mu \in R^m, \mu \geqq 0, x \in X. \quad \text{(VD)}$$

Let Ω be the set of all feasible solutions in problem (VD). Further, denote Y = {y ∈ X : (y,λ,μ) ∈ Ω}. In order to prove several duality results between the considered vector optimization problem (VP) and its vector dual problem in the sense of Mond-Weir (VD), we now introduce the definition of WD-(Φ,ρ)-invexity on a nonempty subset of S containing the set D∪Y. Let S be a nonempty subset of X such that D∪Y ⊆ S and u ∈ S be an arbitrary point.

Definition 4.1. Let u ∈ S be given. If there exist a function $\Phi : S \times S \times R^{n+1} \to R$, where $\Phi(x, u, \cdot)$ is convex on R^{n+1}, $\Phi\big(x, u, (0, a)\big) \geqq 0$ for all $x \in S$ and any $a \in R_+$, $\rho = \big(\rho_{f_1}, \dots, \rho_{f_k}, \rho_g, \dots, \rho_{g_m}\big) \in R^{k+m}$, such that

$$\left.\begin{matrix} x \in S \\ u \in S \\ g(x) \leqq 0 \end{matrix}\right] \Rightarrow \left[\begin{matrix} f_i(x) - f_i(u) \geqq \Phi\Big(x, u, \big(\nabla f_i(u), \rho_{f_i}\big)\Big), \;\; i \in I, \\ -g_j(u) - \Phi\Big(x, u, \Big(\nabla g_j(u), \rho_{g_j}\Big)\Big) \leqq 0, \;\; j \in J, \end{matrix}\right. \quad (18)$$

then the vector optimization problem (VP) is said to be a vector WD-(Φ,ρ)-invex optimization problem at u ∈ S on S (with respect to Φ, ρ_f and ρ_g). If (18) is satisfied at any point u ∈ S, then the vector optimization problem (VP) is said to be a vector WD-(Φ,ρ)-invex optimization problem on S.

Definition 4.2. Let u ∈ S be given. If there exist a function $\Phi : S \times S \times R^{n+1} \to R$, where $\Phi(x, u, \cdot)$ is convex on R^{n+1}, $\Phi\big(x, u, (0, a)\big) \geqq 0$ for all $x \in S$ and any $a \in R_+$, $\rho = \big(\rho_{f_1}, \dots, \rho_{f_k}, \rho_g, \dots, \rho_{g_m}\big) \in R^{k+m}$, such that

$$\left.\begin{matrix} x, u \in S, \;\; x \neq u \\ g(x) \leqq 0 \end{matrix}\right] \Rightarrow \left[\begin{matrix} f_i(x) - f_i(u) > \Phi\Big(x, u, \big(\nabla f_i(u), \rho_{f_i}\big)\Big), \;\; i \in I, \\ -g_j(u) - \Phi\Big(x, u, \Big(\nabla g_j(u), \rho_{g_j}\Big)\Big) \leqq 0, \;\; j \in J, \end{matrix}\right. \quad (19)$$

then the vector optimization problem (VP) is said to be a vector strict WD-(Φ,ρ)-invex optimization problem at u ∈ S on S (with respect to Φ, ρ_f and ρ_g). If (19) is satisfied at any point u ∈ S, then the vector optimization problem (VP) is said to be a vector strict WD-(Φ,ρ)-invex optimization problem on S.

Theorem 4.1. (Weak duality). Let x and (y,λ,μ) be any feasible solutions of the vector optimization problem (VP) and its vector Mond-Weir dual problem (VD), respectively. Further, assume that problem (VP) is WD-(Φ,ρ)-invex on D∪Y with respect to Φ, ρ_f and ρ_g. If $\sum_{i=1}^{k} \lambda_i \rho_{f_i} + \sum_{j=1}^{m} \mu_j \rho_{g_j} \geqq 0$, then f(x) ≮ f(y).

Proof Suppose, contrary to the result, that f(x) < f(y). By the feasibility of (y,λ,μ) to the problem (VD), it follows that

$$\sum_{i=1}^{k} \lambda_i f_i(x) < \sum_{i=1}^{k} \lambda_i f_i(y). \quad (20)$$

By assumption, the vector optimization problem (VP) is WD-(Φ,ρ)-invex on D∪Y with respect to Φ, ρ_f and ρ_g. Therefore, by Definition 4.1, the inequality

$$\sum_{i=1}^{k}\lambda_i f_i(x)-\sum_{i=1}^{k}\lambda_i f_i(y)\geqq\sum_{i=1}^{k}\lambda_i\Phi\left(x,\mathrm{y},\left(\nabla f_i(\mathrm{y}),\rho_{f_i}\right)\right)\tag{21}$$

holds. Hence, (20) and (21) yield

$$\sum_{i=1}^{k}\lambda_i\Phi\left(x,\mathrm{y},\left(\nabla f_i(\mathrm{y}),\rho_{f_i}\right)\right)<0.\tag{22}$$

Using Definition 4.1 again together with $(y,\lambda,\mu)\in\Omega$, we get

$$-\sum_{j=1}^{m}\mu_j g_j(\mathrm{u})\geqq\sum_{j=1}^{m}\mu_j\Phi\left(x,\mathrm{u},\left(\nabla g_j(\mathrm{u}),\rho_{g_j}\right)\right).$$

Thus, the second constraint of problem (VD) implies

$$\sum_{j=1}^{m}\mu_j\Phi\left(x,\mathrm{u},\left(\nabla g_j(\mathrm{u}),\rho_{g_j}\right)\right)\leqq 0.\tag{23}$$

Combining (22) and (23), we obtain

$$\sum_{i=1}^{k}\lambda_i\Phi\left(x,\mathrm{y},\left(\nabla f_i(\mathrm{y}),\rho_{f_i}\right)\right)+\sum_{j=1}^{m}\mu_j\Phi\left(x,\mathrm{u},\left(\nabla g_j(\mathrm{u}),\rho_{g_j}\right)\right)<0.\tag{24}$$

Let us denote $\alpha_i=\sum_{i=1}^{k}\frac{\lambda_i}{\sum_{i=1}^{k}\lambda_i+\sum_{j=1}^{m}\mu_j}$, i =1,…,k, $\beta_j=\sum_{i=1}^{k}\frac{\mu_j}{\sum_{i=1}^{k}\lambda_i+\sum_{j=1}^{m}\mu_j}$, j = 1,…,m. Note that $0\leqq\alpha_i\leqq 0$, but at least one $\alpha_i>0$, $0\leqq\beta_j\leqq 0$, and, moreover, $\sum_{i=1}^{k}\alpha_i+\sum_{j=1}^{m}\beta_j=1$. By (24), it follows that

$$\sum_{i=1}^{k}\alpha_i\Phi\left(\mathrm{x},\mathrm{y},\left(\nabla f_i(\mathrm{y}),\rho_{f_i}\right)\right)+\sum_{j=1}^{m}\beta_j\,\Phi\left(\mathrm{x},\mathrm{y},\left(\nabla g_j(\mathrm{y}),\rho_{g_j}\right)\right)<0.\tag{25}$$

By Definition 4.1, we have that $\Phi(x,\mathrm{y},\cdot)$ is convex on R^{n+1}. Hence, by (25) and convexity of $\Phi(x,\mathrm{y},\cdot)$, we get

$$\Phi\left(x,\mathrm{y},\left(\sum_{i=1}^{k}\alpha_i\nabla f_i(\mathrm{y})+\sum_{j=1}^{m}\beta_j\nabla g_j(y),\sum_{i=1}^{k}\alpha_i\rho_{f_i}+\sum_{j=1}^{m}\beta_j\rho_{g_j}\right)\right)\leqq$$
$$\sum_{i=1}^{k}\alpha_i\Phi\left(\mathrm{x},\mathrm{y},\left(\nabla f_i(\mathrm{y}),\rho_{f_i}\right)\right)+\sum_{j=1}^{m}\beta_j\,\Phi\left(\mathrm{x},\mathrm{y},\left(\nabla g_j(\mathrm{y}),\rho_{g_j}\right)\right).\tag{26}$$

Thus, (25) and (26) yield that the following inequality

$$\Phi\left(x,\mathrm{y},\left(\sum_{i=1}^{k}\alpha_i\nabla f_i(\mathrm{y})+\sum_{j=1}^{m}\beta_j\nabla g_j(y),\sum_{i=1}^{k}\alpha_i\rho_{f_i}+\sum_{j=1}^{m}\beta_j\rho_{g_j}\right)\right)<0$$

holds. By the first constraint of (VD), the inequality above implies

$$\Phi\left(x,\mathrm{y},\frac{1}{\sum_{i=1}^{k}\lambda_i+\sum_{j=1}^{m}\mu_j}\left(0,\sum_{i=1}^{k}\lambda_i\rho_{f_i}+\sum_{j=1}^{m}\mu_j\rho_{g_j}\right)\right)<0.\tag{27}$$

By assumption, $\sum_{i=1}^{k}\lambda_i\rho_{f_i}+\sum_{j=1}^{m}\mu_j\rho_{g_j}\geqq 0$. Then, since from Definition 3.1, $\Phi\left(\tilde{x},\bar{x},(0,a)\right)\geqq 0$ for any $a\in R_+$, then, by assumption, the following inequality

$$\Phi\left(x,\mathrm{y},\frac{1}{\sum_{i=1}^{k}\lambda_i+\sum_{j=1}^{m}\mu_j}\left(0,\sum_{i=1}^{k}\lambda_i\rho_{f_i}+\sum_{j=1}^{m}\mu_j\rho_{g_j}\right)\right)\geqq 0$$

holds, contradicts (27). This completes the proof of this theorem.

Theorem 4.2. (Weak duality). Let x and (y,λ,μ) be any feasible solutions of (VP) and (VD), respectively. Further, assume that problem (VP) is strict WD-(Φ,ρ)-invex on $D\cup Y$ with respect to Φ, ρ_f and ρ_g. If $\sum_{i=1}^{k}\lambda_i\rho_{f_i}+\sum_{j=1}^{m}\mu_j\rho_{g_j}\geqq 0$, then $f(x)\nleq f(y)$.

Theorem 4.3. (Strong duality). Let $\bar{x}\in D$ be a (weak) Pareto solution of the vector optimization problem (VP) and the suitable constraint qualification be satisfied at $\bar{x}$. Then, there exist $\bar{\lambda}\in R^k$ and $\bar{\mu}\in R^m$ such that $\left(\bar{x},\bar{\lambda},\bar{\mu}\right)$ is feasible for (VD) and the objective functions of (VP) and (VD) are equal to these points. If all hypotheses of the weak duality theorem (Theorem 4.2 or Theorem 4.3) are satisfied, then $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a (weakly) efficient solution of a maximum type for (VD)

Proof By assumption, $\bar{x} \in D$ is a (weak) Pareto solution of (VP) and the suitable constraint qualification is satisfied at $\bar{x}$. Then, there exist Lagrange multipliers $\bar{\lambda} \in R^k$ and $\bar{\mu} \in R^m$ such that the Karush-Kuhn-Tucker necessary optimality conditions (9)-(11) are satisfied at $\bar{x}$. Then, the feasibility of $(\bar{x}, \bar{\lambda}, \bar{\mu})$ in (VD) follows directly from these necessary optimality conditions. Hence, the objective functions of problems (VP) and (VD) are equal at $\bar{x}$ and $(\bar{x}, \bar{\lambda}, \bar{\mu})$ are equal at these points. Thus, (weak) efficiency of $(\bar{x}, \bar{\lambda}, \bar{\mu})$ in (VD) follows directly from the weak duality theorem (Theorem 4.2 or Theorem 4.3).

Theorem 4.4. (Converse duality). Let $(\bar{y}, \bar{\lambda}, \bar{\mu})$ be a (weakly) efficient of a maximum type to the vector Mond-Weir dual problem (VD) such that $\bar{y} \in D$. Further, assume that the considered multiobjective programming problem (VP) is (strict) WD-(Φ,ρ)-invex at $\bar{y}$ on D∪Y with respect to Φ, ρ_f and ρ_g. If $\sum_{i=1}^{k} \bar{\lambda}_i \rho_{f_i} + \sum_{j=1}^{m} \bar{\mu}_j \rho_{g_j} \geqq 0$, then $\bar{y}$ is a weak Pareto solution (Pareto solution) of the considered multiobjective programming problem (VP).

Proof Proof of this theorem follows directly from the weak duality theorem (Theorem 4.2 or Theorem 4.3).

Theorem 4.5. (Restricted converse duality): Let $(\bar{y}, \bar{\lambda}, \bar{\mu})$ be feasible to Mond-Weir vector dual problem (VD). Further, assume that the considered multiobjective programming problem (VP) is (strict) WD-(Φ,ρ)-invex at $\bar{y}$ on D∪Y with respect to Φ, ρ_f and ρ_g with $\sum_{i=1}^{k} \bar{\lambda}_i \rho_{f_i} + \sum_{j=1}^{m} \bar{\mu}_j \rho_{g_j} \geqq 0$. If there exists $\bar{x} \in D$ such that $f(\bar{x}) = f(\bar{y})$, then $\bar{x}$ is a (weak) Pareto solution of the problem (VP).

5. Conclusions

In the paper, the scalarization method, that is, the weighting method, has been used for solving a new class of nonconvex differentiable vector optimization problems. It has been established that a weakly efficient solution of an unconstrained smooth vector optimization problem in which the objective function is (Φ,ρ)-invex is related to an optimal solution of its corresponding weighting scalar optimization problem constructed in this method. Further, we have established the same result in the case when the weighting method has been used for solving the KT-(Φ,ρ)-invex constrained vector optimization problem. Hence, the weighting method has been used for a larger class of nonconvex differentiable vector optimization problems, in comparison to other similar results, previously established in the literature under other generalized convexity notions.

Further, new classes of nonconvex multiobjective programming problems have been defined in the paper. By introducing the concepts of KT-(Φ,ρ)-invexity and WD-(Φ,ρ)-invexity, we have generalized notions of generalized convexity introduced by Martin [14] for scalar optimization problems to new classes of nonconvex differentiable vector optimization problems. The definition of a KT-(Φ,ρ)-invex vector optimization problem introduced in the paper unifies many classes of generalized convex optimization problems, earlier defined in optimization theory. Therefore, the sufficient optimality conditions established in the paper are applicable also for such nonconvex vector optimization problems for which other generalized convexity notions may avoid in proving such a result.

It has been shown that there exists such a nonconvex vector optimization problem for which we are not in a position to prove the sufficient optimality conditions under many other generalized convexity notions, previously defined in the literature. However, KT-(Φ,ρ)-invexity is useful in proving this result for nonconvex multiobjective programming problems of such a type. In order to prove several duality results in the sense of Mond-Weir, the concept of WD-(Φ,ρ)-invexity has been introduced. Hence, also duality results have been proved for a larger class of nonconvex vector optimization problems, in comparison to those ones established in the literature under other concepts of generalized convexity.

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