UPPER AND LOWER BOUNDS FOR RIEMANN TYPE QUANTUM INTEGRALS OF PREINVEX AND PREINVEX DOMINATED FUNCTIONS

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In this paper, we obtain some new upper and lower bounds for Riemann-type quantum quadratures of preinvex functions of Mititelu type. Some generalized classes of preinvexities and corresponding boundary properties that involve Riemann-type q-integral are approached. The class of preinvex dominated functions is introduced and some quantum Hermite-Hadamard type integral estimations are derived in this framework.

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1. Introduction and preliminaries

A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex, if

$$f((1-t)x+ty) \leq (1-t)f(x)+tf(y), \quad \forall x, y \in I, t \in [0,1].$$

If the above inequality is reversed, then, we say that f is concave on I. A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is convex if and only if it satisfies the following double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2},$$

known as Hermite-Hadamard's inequality, from the names of its authors.

In recent years, numerous new generalizations of convex sets and convex functions have been proposed, see [3, 10, 20]. Many authors gave generalized forms of Hermite-Hadamard's inequality, see [2, 4, 14, 15, 17, 21]. Recently many authors have used the concepts of quantum calculus to obtain more generalized forms of classical inequalities, see [12, 17, 18, 19]. The aim of this article is to establish some new q-analogues of Hermite-Hadamard type inequalities within the class of the preinvex functions. We also discuss some special cases of generalized preinvexity classes.

Hanson [7] has introduced the class of differentiable invex functions, without calling them by this word, in connection with their special global optimum behaviour.

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In the same year, Craven [1] introduced the term "invex" for calling this class of functions, due to their property described as "invariance by convexity". Mititelu [8] defined the concept of invex set, as follows.
Let K be a nonempty set in $\mathbb{R}$. Let $f : K \to \mathbb{R}$ be a continuous function and let $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function.

Definition 1.1 ([8]). *A set $K \in \mathbb{R}$ is said to be invex with respect to η, if*

$$x + t\eta(y, x) \in K, \quad \forall x, y \in K, t \in [0, 1]. \tag{1.1}$$

The concept of invex set K is sometimes referred to as η-connected set.

Remark 1.1. *If $\eta(y, x) = y - x$, then invexity of set K reduces to convexity. Thus, it is true that every convex set is also an invex set with respect to $\eta(y, x) = y - x$, but the converse is not necessarily true. More properties of invex sets are in* [10, 11, 22] *and the references therein.*

Mititelu [9] introduced a general concept of invex and preinvex functions at a point and on open sets. He extended it to arbitrary sets in [10] and discussed the cases in which the two concepts coincide. The following definition is the version of Mititelu's definition that is reduced to the context of this paper. Suppose that K is an invex set with respect to η, $\varrho \in \mathbb{R}$ and $\theta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Definition 1.2 ([9]). *A function $f : K \to \mathbb{R}$ is said to be ρ-preinvex with respect to functions η and θ, if:*

$$f(x + t\eta(y, x)) \leq (1 - t)f(x) + tf(y) - \rho t[\theta(y, x)]^2, \quad \forall x, y \in K, t \in [0, 1]. \tag{1.2}$$

(1) *If $\rho > 0$ then f is called strongly ρ-preinvex with respect to η and θ;*
(2) *If $\rho = 0$ then f is called ρ-preinvex with respect to η;*
(3) *If $\rho < 0$ then f is called weakly ρ-preinvex with respect to η and θ;*
(4) *If the above inequality is strict for all $x, y \in K$, $x \neq y$ and for all $t \in (0, 1)$ then f is called strictly ρ-preinvex with respect to η and θ.*

A function f is said to be preincave if and only if $-f$ is preinvex. For $\eta(y, x) = y - x$ and $\rho = 0$ in Definition 1.2 a preinvex function reduces to a convex function in the classical sense.

Remark 1.2. *In this paper function $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is supposed to have the following property:*

$$\eta(y + t_1\eta(x, y), y + t_2\eta(x, y)) = (t_1 - t_2)\eta(x, y), \quad \forall t_1, t_2 \in [0, 1], t_1 \leq t_2. \tag{1.3}$$

In this case the following consequences hold:
(1) *If $t_1 = t_2 = 0$ then (1.3) implies that $\eta(y, y) = 0$ for all $y \in \mathbb{R}$.*
(2) *If $t_1 = 0$ and $t_2 = t > 0$ then $\eta(y, y + t\eta(x, y)) = -t\eta(x, y)$ for all $x, y \in \mathbb{R}$. This is the first requirement of Condition C introduced in* [11].
(3) *If $\eta(x, y) > 0$ for some $(x, y) \in \mathbb{R}$ then $\eta(y, y + t\eta(x, y)) \leq 0$ for all $t \in [0, 1]$. It means that property (1.3) implies that function η has not constant sign on $\mathbb{R} \times \mathbb{R}$.*

Definition 1.3 ([6]). *The Jackson integral of $f(x)$ is defined as*

$$\int f(x)\mathrm{d}_q x = (1-q)x\sum_{j=0}^{\infty} q^j f(q^j x). \tag{1.4}$$

Definition 1.4. *Let $0 < a < b$. The definite q-integral is defined as*

$$\int_0^b f(x)\mathrm{d}_q x = (1-q)b\sum_{j=0}^{\infty} q^j f(q^j b), \tag{1.5}$$

provided the sum converges absolutely. Also,

$$\int_a^b f(x)\mathrm{d}_q x = \int_0^b f(x)\mathrm{d}_q x - \int_0^a f(x)\mathrm{d}_q x. \tag{1.6}$$

In this paper we use the Riemann type q-integrals introduced in [16] as:

$$R_q(f;a,b) = (b-a)(1-q)\sum_{k=0}^{\infty} f(a+(b-a)q^k)q^k. \tag{1.7}$$

This integral was used in [18] as:

$$\frac{2}{b-a}\int_a^b f(x)\mathrm{d}_q^{\mathscr{R}} x$$
$$= (1-q)\sum_{k=0}^{\infty}\left(f\left(\frac{a+b}{2}+q^k\left(\frac{b-a}{2}\right)\right)+f\left(\frac{a+b}{2}-q^k\left(\frac{b-a}{2}\right)\right)\right)q^k. \tag{1.8}$$

By means of the q-Jackson integral, one has

$$\frac{2}{b-a}\int_a^b f(x)\mathrm{d}_q^R x = \int_{-1}^1 f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\mathrm{d}_q t = \int_{-1}^1 f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\mathrm{d}_q t. \tag{1.9}$$

2. Bounds of the Riemann q-integral within classes of preinvex functions

Throughout this section we suppose that $\eta : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $\theta : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are continuous functions, $\rho \in \mathbb{R}$ and $K \subseteq \mathbb{R}$ is an invex set with respect to η. Denote by $\Omega = [-1, 1]$. First of all, we derive a q-analogue of Hermite-Hadamard's inequality for functions that are ρ-preinvex with respect to η and θ.

Theorem 2.1. *Let $f : K \to \mathbb{R}$ be a ρ-preinvex with respect to η and θ. If function η satisfies (1.3), then*

$$f\left(\frac{2a+\eta(b,a)}{2}\right)+\frac{\rho}{2\eta(b,a)}\int_a^{a+\eta(b,a)}[\theta(x,2a-x+\eta(b,a))]^2\mathrm{d}_q^{\mathscr{R}} x$$
$$\le \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\mathrm{d}_q^{\mathscr{R}} x \le \frac{f(a)+f(b)}{2}-\frac{\rho}{2}[\theta(b,a)]^2. \tag{2.1}$$

Proof. Since it is given that f is a ρ-preinvex function with respect to η and θ,

$$f\left(\frac{2x+\eta(y,x)}{2}\right) \leq \frac{f(x)+f(y)}{2} - \frac{\rho}{2}[\theta(y,x)]^2.$$

Let us change the variables, by letting $x = a + \frac{1+t}{2}\eta(b,a)$ and $y = a + \frac{1-t}{2}\eta(b,a)$. Using (1.3), one gets

$$\eta\left(a+\frac{1-t}{2}\eta(b,a), a+\frac{1+t}{2}\eta(b,a)\right) = -t\eta(b,a).$$

Then the preinvexity inequality becomes

$$\begin{aligned} f\left(\frac{2a+\eta(b,a)}{2}\right) \leq & \frac{1}{2}\left[f\left(a+\frac{1+t}{2}\eta(b,a)\right) + f\left(a+\frac{1-t}{2}\eta(b,a)\right)\right] \\ & - \frac{\rho}{2}\theta^2\left(a+\frac{1-t}{2}\eta(b,a), a+\frac{1+t}{2}\eta(b,a)\right). \end{aligned}$$

q-integrating the above inequality, with respect to t on Ω, one has

$$\begin{aligned} 2f\left(\frac{2a+\eta(b,a)}{2}\right) \leq & \frac{2}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\mathrm{d}_q^{\mathscr{R}}x \\ & - \frac{\rho}{\eta(b,a)} \int_a^{a+\eta(b,a)} \theta^2(x, 2a-x+\eta(b,a))\mathrm{d}_q^{\mathscr{R}}x. \end{aligned} \tag{2.2}$$

On another hand, by the ρ-preinvexity of f with respect to η and θ one can write that

$$f\left(a+\frac{1+t}{2}\eta(b,a)\right) \leq \left(\frac{1-t}{2}\right)f(a) + \left(\frac{1+t}{2}\right)f(b) - \rho\left(\frac{1+t}{2}\right)\theta^2(b,a).$$

q-integrating the above inequality, with respect to t on Ω, one gets

$$\frac{2}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\mathrm{d}_q^{\mathscr{R}}x \leq f(a)+f(b) - \rho\theta^2(b,a). \tag{2.3}$$

The two inequalities (2.2) and (2.3) complete the proof. □

Corollary 2.1. *Let $f : K \to \mathbb{R}$ be a ρ-preinvex with respect to η and $\theta(x,y) = x-y$. If function η satisfies (1.3), then*

$$\begin{aligned} & f\left(\frac{2a+\eta(b,a)}{2}\right) + \frac{\rho}{2[3]_q}[\eta(b,a)]^2 \\ & \leq \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\mathrm{d}_q^{\mathscr{R}}x \leq \frac{f(a)+f(b)}{2} - \frac{\rho}{2}(b-a)^2. \end{aligned} \tag{2.4}$$

If $\rho = 0$ then Theorem 2.1 leads to the following result referring to preinvex functions with respect to η.

Theorem 2.2. *Let $f : K \to \mathbb{R}$ be a preinvex function with respect to η. If function η satisfies (1.3), then*

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \frac{1}{\eta(b,a)} \int\limits_{a}^{a+\eta(b,a)} f(x)\mathrm{d}_q^{\mathscr{R}} x \leq \frac{f(a)+f(b)}{2}. \tag{2.5}$$

Remark 2.1. *Many authors use Condition C from* [11] *to prove inequalities within invexity and preinvexity framework. As one can see, condition (1.3) is useful enough to prove the inequalities in the previous theorems. Condition (1.3) is not equivalent to Condition C, as discussed in* [23]*. In fact, many inequalities derived in literature need (1.3) only (see for example* [**?**, 12, 13, 18, 19]*).*

Remark 2.2. *In the particular case $\eta(b,a) = b - a$, then Theorem 2.2 reduces to Theorem 2.1 from* [18].

Theorem 2.3. *Let $f, g : K \to \mathbb{R}$ be two preinvex functions with respect to η. If function η satisfies (1.3), then*

$$\begin{aligned} &2f\left(\frac{2a+\eta(b,a)}{2}\right) g\left(\frac{2a+\eta(b,a)}{2}\right) - K_1(q)M(a,b) - K_2(q)N(a,b) \\ &\leq \frac{1}{\eta(b,a)} \int\limits_{a}^{a+\eta(b,a)} f(x)g(x)\mathrm{d}_q^{\mathscr{R}} x, \end{aligned}$$

where

$$K_1(q) = \frac{[3]_q - 1}{4[3]_q}, \qquad K_2(q) = \frac{[3]_q + 1}{4[3]_q},$$

$$M(a,b) = f(a)g(a) + f(b)g(b), \tag{2.6}$$

$$N(a,b) = f(a)g(b) + f(b)g(a), \tag{2.7}$$

respectively.

Proof. Since f and g are preinvex functions, then

$$\begin{aligned} &f\left(\frac{2x+\eta(y,x)}{2}\right) g\left(\frac{2x+\eta(y,x)}{2}\right) \\ &\leq \frac{1}{4}\left[f(x)g(x) + f(y)g(y) + f(x)g(y) + f(y)g(x)\right]. \end{aligned}$$

Let $x = a + \frac{1+t}{2}\eta(b,a)$ and $y = a + \frac{1-t}{2}\eta(b,a)$ and keeping in mind that η satisfies (1.3), one computes

$$\begin{aligned} &f\left(\frac{2a+\eta(b,a)}{2}\right) g\left(\frac{2a+\eta(b,a)}{2}\right) \\ &\leq \frac{1}{4}\Bigg[f\Big(a+\frac{1+t}{2}\eta(b,a)\Big)g\Big(a+\frac{1+t}{2}\eta(b,a)\Big) + f\Big(a+\frac{1-t}{2}\eta(b,a)\Big)g\Big(a+\frac{1-t}{2}\eta(b,a)\Big) \\ &\quad + f\Big(a+\frac{1+t}{2}\eta(b,a)\Big)g\Big(a+\frac{1-t}{2}\eta(b,a)\Big) + f\Big(a+\frac{1-t}{2}\eta(b,a)\Big)g\Big(a+\frac{1+t}{2}\eta(b,a)\Big)\Bigg] \end{aligned}$$

$$\leq \frac{1}{4}\left[f\Big(a+\frac{1+t}{2}\eta(b,a)\Big)g\Big(a+\frac{1+t}{2}\eta(b,a)\Big)+f\Big(a+\frac{1-t}{2}\eta(b,a)\Big)g\Big(a+\frac{1-t}{2}\eta(b,a)\Big)\right.$$
$$+\left\{\Big(\frac{1-t}{2}\Big)f(a)+\Big(\frac{1+t}{2}\Big)f(b)\right\}\left\{\Big(\frac{1+t}{2}\Big)g(a)+\Big(\frac{1-t}{2}\Big)g(b)\right\}$$
$$\left.+\left\{\Big(\frac{1+t}{2}\Big)f(a)+\Big(\frac{1-t}{2}\Big)f(b)\right\}\left\{\Big(\frac{1-t}{2}\Big)g(a)+\Big(\frac{1+t}{2}\Big)g(b)\right\}\right]$$
$$=\frac{1}{4}\left[f\Big(a+\frac{1+t}{2}\eta(b,a)\Big)g\Big(a+\frac{1+t}{2}\eta(b,a)\Big)+f\Big(a+\frac{1-t}{2}\eta(b,a)\Big)g\Big(a+\frac{1-t}{2}\eta(b,a)\Big)\right.$$
$$\left.+\frac{1-t^2}{2}\left\{f(a)g(a)+f(b)g(b)\right\}+\frac{1+t^2}{2}\left\{f(a)g(b)+f(b)g(a)\right\}\right].$$

q-integrating above inequality with respect to t over Ω, one gets

$$2f\left(\frac{2a+\eta(b,a)}{2}\right)g\left(\frac{2a+\eta(b,a)}{2}\right)$$
$$\leq\frac{1}{4}\left[\frac{4}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)g(x)\mathrm{d}_q^{\mathscr{R}}x+\left(\frac{[3]_q-1}{[3]_q}\right)M(a,b)+\left(\frac{[3]_q+1}{[3]_q}\right)N(a,b)\right].$$

This completes the proof. □

Note that when $q\to 1$, one gets a previously known result from [2] and [14]. If $\eta(b,a)=b-a$, the following new result for convex functions occurs.

Corollary 2.2. *Let $f,g:K\to\mathbb{R}$ be two convex functions. Then*

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)-K_1(q)M(a,b)-K_2(q)N(a,b)\leq\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}_q^{\mathscr{R}}x,$$

where $M(a,b)$ and $N(a,b)$ are given by (2.6) and (2.7) respectively and

$$K_1(q)=\frac{[3]_q-1}{4[3]_q},\qquad K_2(q)=\frac{[3]_q+1}{4[3]_q}.$$

Theorem 2.4. *If $f,g:K\to\mathbb{R}$ are two preinvex functions with respect to η, then*

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)g(x)\mathrm{d}_q^{\mathscr{R}}x\leq K_3(q)M(a,b)+K_4(q)N(a,b),$$

where $M(a,b)$ and $N(a,b)$ are given by (2.6) and (2.7) respectively and

$$K_3(q)=\frac{1}{2}\left(1+\frac{1}{[3]_q}\right),\qquad K_4(q)=\frac{[3]_q-1}{2[3]_q}.$$

Proof. Since f and g are two preinvex functions with respect to η, then

$$f\Big(a+\frac{1+t}{2}\eta(b,a)\Big)g\Big(a+\frac{1+t}{2}\eta(b,a)\Big)$$
$$\leq\frac{1}{4}\left[(1-t)^2f(a)g(a)+(1+t)^2f(b)g(b)+(1-t^2)\{f(a)g(b)+f(b)g(a)\}\right].$$

q-integrating both sides of above inequality with respect to t over Ω, one obtains

$$\begin{aligned}\frac{2}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)g(x)\mathrm{d}_q^{\mathscr{R}}x \\ \leq \frac{1}{2}\left(1+\frac{1}{[3]_q}\right)(f(a)g(a)+f(b)g(b)) \\ + \left(\frac{[3]_q-1}{2[3]_q}\right)\{f(a)g(b)+f(b)g(a)\}.\end{aligned}$$

This completes the proof. □

If $q \to 1$ in Theorem 2.4, then the previously known result from [14] is obtained. If $\eta(b,a) = b-a$ in Theorem 2.4, then following new result holds.

Corollary 2.3. *Let $f, g : K_\eta \to \mathbb{R}$ be two convex functions, then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}_q^{\mathscr{R}}x \leq K_3(q)M(a,b) + K_4(q)N(a,b),$$

where all the constants are defined in Theorem 2.4.

3. Cases of generalized preinvexity properties

Many authors have extended the concept of preinvexity in various directions that are approached in nowadays literature. In this section we derive results on the boundaries of the Riemann q-integrals of some recently identified classes of extended preinvexities.

Definition 3.1 ([13]). *Let $h : J \to \mathbb{R}$ where $(0,1) \subseteq J$ is an interval in $\mathbb{R}$. Let K be an invex set with respect to η. A function $f : K \to \mathbb{R}$ is called h-preinvex with respect to η, if*

$$f(x + t\eta(y,x)) \leq h(1-t)f(x) + h(t)f(y), \quad x, y \in K, t \in (0,1). \tag{3.1}$$

Remark 3.1. *For various suitable choices of function h, we have other classes of preinvex functions. More details are in [13]. Here we focus on the case $h(t) = t^s$, with $s \in (0,1]$, which lead us to the s-invexity with respect to η of a function f, defined by means of the inequality*

$$f(x + t\eta(y,x)) \leq (1-t)^s f(x) + t^s f(y), \quad \forall x, y \in K, t \in [0,1]. \tag{3.2}$$

Definition 3.2 ([21]). *. Let K be an invex set with respect to η and $\alpha \in (0,1]$. A function $f : K \to \mathbb{R}$ is said to be α-preinvex with respect to η if for every $x, y \in K$ and $t \in [0,1]$,*

$$f(x + t\eta(y,x)) \leq t^\alpha f(x) + (1-t^\alpha)f(y). \tag{3.3}$$

Using essentially the analysis of Theorem 2.3 and Theorem 2.4 one can obtain new bounds for other classes of preinvex functions, such as s-preinvex functions, h-preinvex functions and α-preinvex functions etc. The next result is a more generalized form of the previous theorems.

Theorem 3.1. *Let $h : J \to \mathbb{R}$ where $(0,1) \subseteq J$ is an interval in $\mathbb{R}$. Let $f : K_\eta \to \mathbb{R}$ be a h-preinvex function with respect to η. If function η satisfies (1.3) then, for $h\left(\frac{1}{2}\right) \neq 0$, one has*

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x) \mathrm{d}_q^{\mathscr{R}} x \leq \frac{f(a)+f(b)}{2} \int_0^1 h(t) \mathrm{d}_q^{\mathscr{R}} t. \quad (3.4)$$

Proof. We start by using the hypothesis that f is a h-preinvex function. As in the above proofs we change the variables by letting $x = a + \frac{1+t}{2}\eta(b,a)$ and $y = a + \frac{1-t}{2}\eta(b,a)$. By means of (1.3), one gets

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \leq h\left(\frac{1}{2}\right)\left[f\left(a + \frac{1+t}{2}\eta(b,a)\right) + f\left(a + \frac{1-t}{2}\eta(b,a)\right)\right].$$

q-integrating above inequality, with respect to t on Ω, the following inequality holds

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \leq h\left(\frac{1}{2}\right)\frac{2}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x) \mathrm{d}_q^{\mathscr{R}} x. \quad (3.5)$$

On another hand,

$$f\left(a + \frac{1+t}{2}\eta(b,a)\right) \leq h\left(\frac{1-t}{2}\right) f(a) + h\left(\frac{1+t}{2}\right) f(b).$$

q-integrating above inequality, with respect to t on Ω one obtains

$$\frac{2}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x) \mathrm{d}_q^{\mathscr{R}} x \leq (f(a)+f(b)) \int_0^1 h(t) \mathrm{d}_q^{\mathscr{R}} t. \quad (3.6)$$

The two inequalities (3.5) and (3.6) complete the proof. □

If $\eta(b,a) = b - a$ in Theorem 3.1 then one has Theorem 2.4 from [18]. If $h(t) = t^s$ in Theorem 3.1 then one gets the following result involving functions that are s-preinvex with respect to η.

Theorem 3.2. *Suppose that $s \in (0,1]$. Let $f : K \to \mathbb{R}$ be a s-preinvex function with respect to η. If function η satisfies (1.3), then*

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x) \mathrm{d}_q^{\mathscr{R}} x \leq \vartheta(q)(f(a)+f(b)),$$

where

$$\vartheta(q) := \frac{(2+q)^s + q^s}{2^{s+1}[2]_q^s}, \qquad [2]_q = 1 + q.$$

Proof. The proof follows the same steps as the general case from Theorem 3.1. □

Note that, if $\eta(b,a) = b - a$, then Theorem 3.2 reduces to Theorem 2.2 from [18]. Also if $s = 1$ in Theorem 3.2, then one gets Theorem 2.2.

Theorem 3.3. *Let $f : K_\eta \to \mathbb{R}$ be α-preinvex function. If function η satisfies (1.3) then the following inequality holds*

$$\frac{1}{2}f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\mathrm{d}_q^{\mathscr{R}}x \leq \varphi(q)f(a) + (1-\varphi(q))f(b),$$

where

$$\varphi(q) := \frac{(2+q)^\alpha + q^\alpha}{2^{\alpha+1}[2]_q^\alpha}.$$

Proof. Using the hypothesis that f is a α-preinvex function with respect to η, by letting $x = a + \frac{1+t}{2}\eta(b,a)$ and $y = a + \frac{1-t}{2}\eta(b,a)$ and using (1.3), one gets

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \frac{1}{2^\alpha}f\left(a + \frac{1+t}{2}\eta(b,a)\right) + \left(1 - \frac{1}{2^\alpha}\right)f\left(a + \frac{1-t}{2}\eta(b,a)\right).$$

q-integrating above inequality, with respect to t on Ω, we have

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \frac{2}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\mathrm{d}_q^{\mathscr{R}}x. \tag{3.7}$$

Referring to the upper bound, the α-preinvexity gives

$$f\left(a + \frac{1+t}{2}\eta(b,a)\right) \leq \left(\frac{1+t}{2}\right)^\alpha f(a) + \left(1 - \left(\frac{1+t}{2}\right)^\alpha\right)f(b).$$

q-integrating above inequality, with respect to t on Ω, one has

$$\frac{2}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\mathrm{d}_q^{\mathscr{R}}x \leq \varphi(q)f(a) + (1-\varphi(q))f(b). \tag{3.8}$$

The two inequalities (3.7) and (3.8) complete the proof. □

4. **Preinvex dominated functions**

Throughout this section we assume that $\eta : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $\theta : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are continuous functions, $\rho \in \mathbb{R}$ and $K \subseteq \mathbb{R}$ is an invex set with respect to η. Denote by $\Omega = [-1, 1]$. Let us suppose that function $g : K \to \mathbb{R}$ is a ρ-preinvex function with respect to η and θ. In this section, we consider a new class of functions, which are called as ρ-preinvex with respect to η and θ dominated functions by control function g.

Definition 4.1. *Function $f : K \to \mathbb{R}$ is said to be a ρ-preinvex with respect to η and θ dominated function by g, if*

$$\begin{aligned} &|(1-t)f(x) + tf(y) - f(x+t\eta(y,x))| \\ &\leq (1-t)g(x) + tg(y) - g(x+t\eta(y,x)) - \rho t[\theta(y,x)]^2, \end{aligned} \tag{4.1}$$

for all $x, y \in K$ and $t \in [0,1]$.

Remark 4.1. *Note that when $\eta(y,x) = y - x$ and either $\rho = 0$ or $\theta \equiv 0$, then Definition 4.1 reduces to the definition of g-convex dominated functions, considered in* [5].

We now derive some more bounds of the Riemann-type q-integrals within the class of ρ-preinvex with respect to η and θ dominated function by g.

Theorem 4.1. *Let $g : K \to \mathbb{R}$ be a ρ-preinvex function with respect to η and θ, and $f : K \to \mathbb{R}$ be a ρ-preinvex with respect to η and θ dominated function by g. If function η satisfies (1.3) then*

I. $\left|\frac{1}{\eta(b,a)}\int\limits_a^{a+\eta(b,a)} f(x)\mathrm{d}_q^{\mathscr{R}}x - f\left(\frac{2a+\eta(b,a)}{2}\right)\right|$

$\leq \frac{1}{\eta(b,a)}\int\limits_a^{a+\eta(b,a)} g(x)\mathrm{d}_q^{\mathscr{R}}x - g\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{\rho}{2\eta(b,a)}\int_a^{a+\eta(b,a)}\theta^2(a, 2a-x+\eta(b,a))\mathrm{d}_q^{\mathscr{R}}x,$

II. $\left|\frac{f(a)+f(b)}{2} - \frac{1}{\eta(b,a)}\int\limits_a^{a+\eta(b,a)} f(x)\mathrm{d}_q^{\mathscr{R}}x\right| \leq \frac{g(a)+g(b)}{2} - \frac{1}{\eta(b,a)}\int\limits_a^{a+\eta(b,a)} g(x)\mathrm{d}_q^{\mathscr{R}}x - \rho[\theta(b,a)]^2$

Proof. **I.** Using the hypothesis of the theorem and Definition 4.1 we have

$$\left|\frac{f(x)+f(y)}{2} - f\left(\frac{2x+\eta(y,x)}{2}\right)\right| \leq \frac{g(x)+g(y)}{2} - g\left(\frac{2x+\eta(y,x)}{2}\right) - \frac{\rho}{2}[\theta(y,x)]^2.$$

Let $x = a+\frac{1+t}{2}\eta(b,a)$ and $y = a+\frac{1-t}{2}\eta(b,a)$ and keeping in my mind that η satisfies (1.3), one gets

$$\left|\frac{f\left(a+\frac{1+t}{2}\eta(b,a)\right)+f\left(a+\frac{1-t}{2}\eta(b,a)\right)}{2} - f\left(\frac{2a+\eta(b,a)}{2}\right)\right|$$

$$\leq \frac{g\left(a+\frac{1+t}{2}\eta(b,a)\right)+g\left(a+\frac{1-t}{2}\eta(b,a)\right)}{2} - g\left(\frac{2a+\eta(b,a)}{2}\right)$$

$$-\frac{\rho}{2}[\theta(a+\frac{1-t}{2}\eta(b,a), a+\frac{1+t}{2}\eta(b,a))]^2.$$

Now q-integrating both sides of above inequality with respect to t over $\Omega = [-1,1]$, after a convenient substitution one derives

$$\left|\frac{1}{\eta(b,a)}\int\limits_a^{a+\eta(b,a)} f(x)\mathrm{d}_q^{\mathscr{R}}x - f\left(\frac{2a+\eta(b,a)}{2}\right)\right|$$

$$\leq \frac{1}{\eta(b,a)}\int\limits_a^{a+\eta(b,a)} g(x)\mathrm{d}_q^{\mathscr{R}}x - g\left(\frac{2a+\eta(b,a)}{2}\right)$$

$$-\frac{\rho}{2\eta(b,a)}\int_a^{a+\eta(b,a)}\theta^2(a, 2a-x+\eta(b,a))\mathrm{d}_q^{\mathscr{R}}x.$$

II. Now we prove the second part of the theorem. Again by Definition 4.1 one has

$$\left|\left(\frac{1-t}{2}\right)f(a) + \left(\frac{1+t}{2}\right)f(b) - f\left(a+\frac{1-t}{2}\eta(b,a)\right)\right|$$

$$\leq \left(\frac{1-t}{2}\right)g(a) + \left(\frac{1+t}{2}\right)g(b) - g\left(a+\frac{1-t}{2}\eta(b,a)\right) - \frac{\rho(1+t)}{2}[\theta(b,a)]^2.$$

q-integrating both sides of the above inequality with respect to t over Ω, one derives

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\eta(b,a)}\int\limits_a^{a+\eta(b,a)} f(x)\mathrm{d}_q^{\mathscr{R}}x\right|$$

$$\leq \frac{g(a)+g(b)}{2}-\frac{1}{\eta(b,a)}\int\limits_a^{a+\eta(b,a)} g(x)\mathrm{d}_q^{\mathscr{R}}x-\rho[\theta(b,a)]^2\int_{-1}^{1}\frac{1+t}{2}\mathrm{d}_q^{\mathscr{R}}x,$$

which gives the required inequality. □

Remark 4.2. *For $\eta(y,x)=y-x$ and $\rho=0$ in Theorem 4.1 one gets the following new quantum Hermite-Hadamard type inequality for g-convex dominated functions:*

I. $\left|\frac{1}{b-a}\int\limits_a^b f(x)\mathrm{d}_q^{\mathscr{R}}x-f\left(\frac{a+b}{2}\right)\right|\leq \frac{1}{b-a}\int\limits_a^b g(x)\mathrm{d}_q^{\mathscr{R}}x-g\left(\frac{a+b}{2}\right),$

II. $\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int\limits_a^b f(x)\mathrm{d}_q^{\mathscr{R}}x\right|\leq \frac{g(a)+g(b)}{2}-\frac{1}{b-a}\int\limits_a^b g(x)\mathrm{d}_q^{\mathscr{R}}x.$

Remark 4.3. *For $\eta(y,x)=y-x$ and $\rho=0$ in Theorem 4.1 as $q\to 1$ the above derived inequalities reduce to the case of convex dominated functions in the classical sense, which is discussed in* [5].

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