U.P.B. Sci. Bull., Series A, Vol. 80, Iss. 1, 2018 ISSN 1223-7027

GRAM MATRIX AND ORTHOGONALITY IN FRAMES

Abolhassan FEREYDOONI[1] and Elnaz OSGOOEI [2]

In this paper, we aim at introducing a criterion that determines if $\{f_i\}_{i\in\mathbb{I}}$ is a Bessel sequence, a frame or a Riesz sequence or not any of these, based on the norms and the inner products of the elements in $\{f_i\}_{i\in\mathbb{I}}$. In the cases of Riesz and Bessel sequences, we introduced a criterion but in the case of a frame, we did not find any answers. This criterion will be shown by $K(\{f_i\}_{i\in\mathbb{I}})$. Using the criterion introduced, some interesting extensions of orthogonality will be presented.

Keywords: Frames, Bessel sequences, Orthonormal basis, Riesz bases, Gram matrix
MSC2010: Primary 42C15, 47A05.

1. Preliminaries

Frames are generalizations of orthonormal bases, but, more than orthonormal bases, they have shown their ability and stability in the representation of functions [1, 4, 10, 11]. The frames have been deeply studied from an abstract point of view. The results of such studies have been used in concrete frames such as Gabor and Wavelet frames which are very important from a practical point of view [2, 9, 5, 8].

An orthonormal set $\{e_n\}$ in a Hilbert space is characterized by a simple relation

$$\langle e_m, e_n\rangle = \delta_{m,n}.$$

In the other words, the Gram matrix is the identity matrix. Moreover, $\{e_n\}$ is an orthonormal basis if $\overline{span}\{e_n\} = \mathcal{H}$. But, for frames the situation is more complicated; *i.e.,* the Gram Matrix has no such a simple form. In what follows we recall the basic notations, concepts and results which are used in the paper. Let $\mathcal{H}$ be a complex Hilbert space and $\mathbb{I}$ be the sequence of natural numbers. The range of an operator is denoted by $\mathcal{R}(.)$. A Bessel sequence for $\mathcal{H}$ is a sequence $\{f_i\}_{i\in\mathbb{I}} \subset \mathcal{H}$ such that there is a positive constant B satisfying

$$\sum_{i\in\mathbb{I}} |\langle f, f_i\rangle|^2 \le B\|f\|^2, \quad f\in\mathcal{H}. \tag{1}$$

Additionally, if for $0 < A < \infty$,

$$A\|f\|^2 \le \sum_{i\in\mathbb{I}} |\langle f, f_i\rangle|^2 \le B\|f\|^2, \quad f\in\mathcal{H}. \tag{2}$$

[1] Department of Basic Sciences, Ilam University, Ilam, Iran. E-mail: fereydooniman@yahoo.com, a.fereydooni@ilam.ac.ir

[2] Faculty of Science, Urmia University of Technology, Urmia, Iran. E-mail: osgooei@yahoo.com; e.osgooei@uut.ac.ir

$\{f_i\}_{i\in\mathbb{I}}$ is a frame. The constants A and B are called lower and upper frame bounds, respectively.

A Riesz basis for $\mathcal{H}$ is a family $\{f_i\}_{i\in\mathbb{I}}$ such that for some constant $0 < A \leqslant B < \infty$,

$$A\sum_{i\in\mathbb{I}}|c_i|^2 \leq \left\|\sum_{i\in\mathbb{I}}c_if_i\right\|^2 \leqslant B\sum_{i\in\mathbb{I}}|c_i|^2, \qquad \{c_i\}_{i\in\mathbb{I}} \in \ell^2.$$

Associated with each Bessel sequence $\{f_i\}_{i\in\mathbb{I}}$ we have three linear and bounded operators, the synthesis operator

$$T : \ell^2(\mathbb{I}) \to \mathcal{H},\;\; T\{c_i\} = \sum_{i\in\mathbb{I}} c_if_i;$$

the analysis operator which is defined by

$$T^* : \mathcal{H} \to \ell^2(\mathbb{I}),\;\; T^*f = \{\langle f, f_i\rangle\}_{i\in\mathbb{I}},$$

and the frame operator

$$S : \mathcal{H} \to \mathcal{H},\;\; Sf = TT^*f = \sum_{i\in\mathbb{I}}\langle f, f_i\rangle f_i.$$

For a review of the basic results of the frames theory I suggest that the reader study book [1]. If $\{f_i\}_{i\in\mathbb{I}}$ is a Bessel sequence, we can compose the synthesis operator T and its adjoint T^* to obtain the bounded operator

$$T^*T : \ell^2(\mathbb{I}) \to \ell^2(\mathbb{I}),\;\; T^*T\{c_i\}_{i\in\mathbb{I}} = \left\{\left\langle \sum_{j\in\mathbb{I}} c_jf_j, f_i\right\rangle\right\}_{i\in\mathbb{I}}.$$

If $\{e_i\}_{i\in\mathbb{I}}$ is the canonical basis of $\ell^2(\mathbb{I})$, the matrix representation of T^*T is as follows:

$$G := T^*T = \left\{\left\langle T^*Te_j, e_i\right\rangle\right\}_{i,j\in\mathbb{I}} = \left\{\left\langle Te_j, Te_i\right\rangle\right\}_{i,j\in\mathbb{I}} = \left\{\left\langle f_j, f_i\right\rangle\right\}_{i,j\in\mathbb{I}}$$

with the (i,j)-entry $G_{i,j} = \langle f_j, f_i\rangle$. The matrix $G = \{\langle f_j, f_i\rangle\}_{i,j\in\mathbb{I}}$ is called the **Gram matrix** associated with $\{f_i\}_{i\in\mathbb{I}}$.

To recognize that a sequence $\{f_i\}_{i\in\mathbb{I}}$ is a Bessel sequence or a frame we have to check (1)-(2) for all $f \in \mathcal{H}$. Our main goal in this paper is presenting a practical method to diagnose Bessel or Riesz sequences just by considering $\{\langle f_j, f_i\rangle\}_{i,j\in\mathbb{I}}$. In order to construct this new method we need the following results:

Theorem 1.1. [1] *Let $\{f_i\}_{i\in\mathbb{I}}$ be a sequence in $\mathcal{H}$ and let G be the Gram matrix associated to $\{f_i\}_{i\in\mathbb{I}}$. The following statements are satisfied:*

(1) *The Gram matrix G defines a bounded operator from $\ell^2(\mathbb{I})$ into $\ell^2(\mathbb{I})$ if and only if the sequence $\{f_i\}_{i\in\mathbb{I}}$ is a Bessel sequence. In this case, the Gram matrix defines an injective operator from $\mathcal{R}(T^*)$ into $\mathcal{R}(T^*)$ and $\overline{\mathcal{R}(G)} = \mathcal{R}(T^*)$. The operator norm of G is the optimal Bessel bound.*

(2) *The Gram matrix defines a bounded operator from $\mathcal{R}(T^*)$ onto $\mathcal{R}(T^*)$ with bounded inverse if and only if $\{f_i\}_{i\in\mathbb{I}}$ is a frame sequence.*

(3) *The Gram matrix G defines a bounded, invertible operator on $\ell^2(\mathbb{I})$ if and only if $\{f_i\}_{i\in\mathbb{I}}$ is a Riesz sequence.*

Before proceeding we recall some notations. Let $\mathcal{K}$ be a Hilbert space and V,W be linear operators on $\mathcal{K}$. By $V \leq W$, we mean that for every $f \in \mathcal{K}$, $\langle Vf, f\rangle \leq \langle Wf, f\rangle$. We write $A \leq |\lfloor V \rceil| \leq B$ whenever for every $f \in \mathcal{K}$, $A\|f\| \leq \|Vf\| \leq B\|f\|$ (the relation $A < |\lfloor V \rceil| < B$ can be defined in a similar way) [7]. The operator V is positive (non-negative) if $V \geq 0$. When V is positive,

$$\|V\| = \sup_{\|f\|=1} \langle Vf, f\rangle.$$

.

Some parts of the following lemma were proved previously. Here, we prove it completely.

Lemma 1.1. *Let V be a positive linear operator on $\mathcal{K}$ and $0 \leq A \leq B \leq \infty$ $(B \neq 0)$. Consider the following statements:*

(1) $AI \leq V \leqslant BI$.
(2) $I - \frac{1}{B}V \leq \left(1 - \frac{A}{B}\right)I$.
(3) $\left\|I - \frac{1}{B}V\right\| \leq 1 - \frac{A}{B}$.
(4) $A \leq |\lfloor V \rceil| \leq B$.

Then (1), (2) and (3) are equivalent and imply (4). Moreover, (4) yields that $\frac{A^2}{B} \leq V \leqslant B$.

Proof. When $B = \infty$, define $\frac{1}{B}V = 0$ and $\frac{A}{B} = 0$, hence all assertions are valid. We prove the statements when $B < \infty$. Obviously, for $f \in \mathcal{H}$,

$$\left\langle \left(I - \frac{1}{B}V\right) f, f \right\rangle \leq \sup_{\|g\|=1} \left\langle \left(I - \frac{1}{B}V\right) g, g \right\rangle.$$

Since V is positive,

$$\sup_{\|f\|=1} \left\langle \left(I - \frac{1}{B}V\right) f, f \right\rangle = \left\|I - \frac{1}{B}V\right\|. \tag{3}$$

Condition (2) is

$$\langle f, f\rangle - \frac{1}{B}\langle Vf, f\rangle \leq \langle f, f\rangle - \frac{A}{B}\langle f, f\rangle,$$

and condition (1) is

$$A\langle f, f\rangle \leq \langle Vf, f\rangle \leq B\langle f, f\rangle.$$

Since V is positive a simple calculation proves the equivalence of two recent relations and henceforth the equivalence of (1) and (2) follows. Clearly (2) $\longrightarrow$ (3), and when (3) is satisfied, (3) proves (2).
($1 \longrightarrow 4$) $V \leq BI$ implies that $\|V\| \leq B$, so for all $f \in \mathcal{H}$, $\|Vf\| \leq B\|f\|$.

Now, by the way of contradiction we assume that there is a $g \in \mathcal{H}$ such that $\|Vg\| < A\|g\|$. Let $\|g\| = 1$, so $\|Vg\| < A$ and thus $\sup_{\|f\|=1} |\langle Vg, f\rangle| < A$. Letting $f = g$ we get $\langle Vg, g\rangle < A$; this contradicts (1).
Now, assume that (4) is satisfied. We know that for a positive operator V, $\|Vf\|^2 \leq \|V\|\langle Vf, f\rangle$, [3]. Hence

$$A^2\langle f, f\rangle = A^2\|f\|^2 \leq \|Vf\|^2 \leq \|V\|\langle Vf, f\rangle.$$

Since $\|V\| \leq B$,

$$\frac{A^2}{B}\langle f,f\rangle \leq \frac{A^2}{\|V\|}\|f\|^2 \leq \langle Vf,f\rangle.$$

On the other hand,

$$\langle Vf,f\rangle \leq \|Vf\|\|f\| \leq B\|f\|^2 = B\langle f,f\rangle.$$

Two recent inequalities show that $\frac{A^2}{B} \leq V \leqslant B$. □

Two special cases of the above lemma are deduced in the following prepositions when V is bounded or invertible.

Proposition 1.1. *Let V be a nonzero positive linear operator on $\mathcal{K}$, the following statements are equivalent:*

(1) *V is bounded with bound B.*
(2) $V \leq BI$, $\quad 0 < B < \infty$.
(3) $I - \frac{1}{B}V \leq I$, $\quad 0 < B < \infty$.
(4) $\|I - \frac{1}{B}V\| \leq 1$, $\quad 0 < B < \infty$.

Proof. Use Lemma 1.1 with $A = 0$ and $\|V\| = B < \infty$. □

Proposition 1.2. *Let V be a bounded positive linear operator V on $\mathcal{K}$. The following statements are equivalent:*

(1) *V is invertible.*
(2) $AI \leq V \leq BI$, *for constants* $0 < A \leq B < \infty$.
(3) $\|I - \frac{1}{B}V\| \leq 1 - \frac{A}{B}$, *for constants* $0 < A \leq B < \infty$.
(4) $0 < I - \frac{1}{B}V < \left(1 - \frac{A}{B}\right)I$, *for constants* $0 < A \leq B < \infty$.

Proof. A positive linear operator is invertible if and only if it is bounded below. By Lemma 1.1(4$\longrightarrow$1), (2) is satisfied. The other equivalences are clear from Lemma 1.1. □

2. Characterization of Bessel and Riesz sequences using $\{\langle f_j, f_i\rangle\}_{i,j\in\mathbb{I}}$

In this section, we introduce criterion K and state our principal theorem in which the quantity $K(\{f_i\}_{i\in\mathbb{I}})$ determines if $\{f_i\}_{i\in\mathbb{I}}$ is a Bessel or a Riesz sequence.

Lemma 2.1. [1] *Let $M = \{M_{i,j}\}_{i,j\in\mathbb{I}}$ be a matrix with $M_{i,j} = \overline{M_{j,i}}$ for all $i,j \in \mathbb{I}$, and for which there exists a constant $0 < K < \infty$ such that*

$$\sup_{i\in\mathbb{I}} \sum_{j\in\mathbb{I}} |M_{i,j}| < K.$$

Then M defines a bounded operator on $\ell^2(\mathbb{I})$ of norm at most K.

Now we define the criterion $K(\{f_i\}_{i\in\mathbb{I}})$.

Definition 2.1. *Let $\{f_i\}_{i\in\mathbb{I}} \subseteq \mathcal{H}$ and $\sup_{i\in\mathbb{I}}\|f_i\|^2 = B < \infty$. We define*

$$k(f_i) = 1 - \frac{1}{B}\|f_i\|^2 + \sum_{j\in\mathbb{I},\ j\neq i}\left|\frac{\langle f_j, f_i\rangle}{B}\right|, \quad i \in \mathbb{I},$$

and

$$K(\{f_i\}_{i\in\mathbb{I}}) = \sup_{i\in\mathbb{I}} k(f_i).$$

Theorem 2.1. *Let* $\{f_i\}_{i\in\mathbb{I}} \subseteq \mathcal{H}$ *and* $B = \sup_{i\in\mathbb{I}} \|f_i\|^2 < \infty$. *Then the following statements hold:*

(1) *If* $K(\{f_i\}_{i\in\mathbb{I}}) < \infty$, *then* $\{f_i\}_{i\in\mathbb{I}}$ *is a Bessel sequence.*
(2) *If* $K(\{f_i\}_{i\in\mathbb{I}}) < 1$, *then* $\{f_i\}_{i\in\mathbb{I}}$ *is a Riesz sequence.*

Proof. By a simple calculation, for $i \in \mathbb{I}$,

$$\sum_{j\in\mathbb{I}} \left|\delta_{i,j} - \frac{1}{B}\langle f_j, f_i\rangle\right| = \left|1 - \frac{1}{B}\|f_i\|^2\right| + \sum_{j\in\mathbb{I}, j\neq i} \left|\frac{\langle f_j, f_i\rangle}{B}\right| = k(f_i).$$

Let $M = I - \frac{1}{B}G$. Considering $M_{i,j} = \delta_{i,j} - \frac{1}{B}\langle f_j, f_i\rangle$, we see that

$$\begin{aligned}\sup_{i\in\mathbb{I}} \sum_{j\in\mathbb{I}} |M_{i,j}| &= \sup_{i\in\mathbb{I}} \left(\left|1 - \frac{1}{B}\|f_i\|^2\right| + \sum_{j\in\mathbb{I},\ j\neq i} \left|\frac{\langle f_j, f_i\rangle}{B}\right| \right) \\ &= \sup_{i\in\mathbb{I}} k(f_i) = K\left(\{f_i\}_{i\in\mathbb{I}}\right).\end{aligned}$$

Since $K := K\left(\{f_i\}_{i\in\mathbb{I}}\right) < \infty$, Lemma 2.1 implies that $M = I - \frac{1}{B}G$ is a bounded operator with bound K. So $G = B(I - M)$ is bounded and then by Theorem 1.1(1), $\{f_i\}_{i\in\mathbb{I}}$ is a Bessel sequence.
(2) Since $\left\|I - \frac{1}{B}G\right\| \leq K < 1$, G is an invertible operator and $\{f_i\}_{i\in\mathbb{I}}$ is a Riesz sequence by Theorem 1.1(2) □

We say that $\{f_i\}_{i\in\mathbb{I}}$ has the equal norm $B > 0$ if $\|f_i\| = B$ for all $i \in \mathbb{I}$, and is normal if $\|f_i\| = 1$ for all $i \in \mathbb{I}$. When $\{f_i\}_{i\in\mathbb{I}}$ has an equal norm the theorem above gives a simple criterion based on $\langle f_j, f_i\rangle$s and $\|f_i\|$s. When $\{f_i\}_{i\in\mathbb{I}}$ has the equal norm $\sqrt{B}$,

$$K\left(\{f_i\}_{i\in\mathbb{I}}\right) = \sup_{i\in\mathbb{I}} \left(\left|1 - \frac{1}{B}\|f_i\|^2\right| + \sum_{j\in\mathbb{I},\ j\neq i} \left|\frac{\langle f_j, f_i\rangle}{B}\right| \right) = \frac{1}{B} \sup_{i\in\mathbb{I}} \sum_{j\in\mathbb{I}, j\neq i} \left|\langle f_j, f_i\rangle\right|.$$

From the above relation we obtain

Corollary 2.1. *Let* $\{f_i\}_{i\in\mathbb{I}} \subseteq \mathcal{H}$ *has the equal norm* $\sqrt{B}$. *Then*

(1) *If*

$$\sup_{i\in\mathbb{I}} \sum_{j\in\mathbb{I}, j\neq i} \left|\langle f_j, f_i\rangle\right| < \infty,$$

then $\{f_i\}_{i\in\mathbb{I}}$ *is a Bessel sequence.*

(2) *If*

$$\sup_{i\in\mathbb{I}} \sum_{j\in\mathbb{I}, j\neq i} \left|\langle f_j, f_i\rangle\right| < B,$$

then $\{f_i\}_{i\in\mathbb{I}}$ *is a Riesz sequence.*

A criterion for determining Riesz sequences is given in the following theorem.

Theorem 2.2. *Let $\{f_i\}_{i\in\mathbb{I}} \subseteq \mathcal{H}$ be a Bessel sequence. There exists a constant $0<\varepsilon<1$ such that*

$$\sum_{j\in\mathbb{I},\ j\neq i} |\langle f_j, f_i\rangle| \leq \|f_i\|^2-\varepsilon,\ \ i\in\mathbb{I}, \tag{4}$$

if and only if $K(\{f_i\}_{i\in\mathbb{I}})<1$.

Proof. Since $\{f_i\}_{i\in\mathbb{I}}$ is a Bessel sequence, $B=\sup_{i\in\mathbb{I}}\|f_i\|^2<\infty$. Without loss of generality we rewrite (4) in the form of

$$\sum_{j\in\mathbb{I},\ j\neq i} |\langle f_j, f_i\rangle| \leq \|f_i\|^2-B\varepsilon,\ \ i\in\mathbb{I}. \tag{5}$$

This is equivalent to

$$K(\{f_i\}_{i\in\mathbb{I}})=1-\frac{1}{B}\|f_i\|^2+\frac{1}{B}\sum_{j\in\mathbb{I},j\neq i}|\langle f_j, f_i\rangle| \leq 1-\varepsilon,\ \ i\in\mathbb{I}. \tag{6}$$

The above statement holds if and only if $K(\{f_i\}_{i\in\mathbb{I}})<1$.

□

The next corollary presents a simple condition for $\{f_i\}_{i\in\mathbb{I}}$ being a Riesz sequence.

Corollary 2.2. *Let $\{f_i\}_{i\in\mathbb{I}} \subseteq \mathcal{H}$ be a sequence such that $B=\sup_{i\in\mathbb{I}}\|f_i\|^2<\infty$. If*

$$\sup_{i\in\mathbb{I}}\sum_{j\in\mathbb{I},\ j\neq i}|\langle f_j, f_i\rangle| < \inf_{i\in\mathbb{I}}\|f_i\|^2 \tag{7}$$

then $K(\{f_i\}_{i\in\mathbb{I}})<1$. Also, every orthogonal set is a Riesz sequence if and only if $0<\inf_{i\in\mathbb{I}}\|f_i\|^2\leq\sup_{i\in\mathbb{I}}\|f_i\|^2<\infty$.

Proof. We know that

$$\sup_{i\in\mathbb{I}}\left(1-\frac{1}{B}\|f_i\|^2\right)=1-\frac{1}{B}\inf_{i\in\mathbb{I}}\|f_i\|^2.$$

From the hypothesis

$$\frac{1}{B}\sup_{i\in\mathbb{I}}\sum_{j\in\mathbb{I},\ j\neq i}|\langle f_j, f_i\rangle| < \frac{1}{B}\inf_{i\in\mathbb{I}}\|f_i\|^2 \leqslant \frac{1}{B}\sup_{i\in\mathbb{I}}\|f_i\|^2 \leqslant 1$$

We obtain the inequality

$$1-\frac{1}{B}\inf_{i\in\mathbb{I}}\|f_i\|^2 < 1-\frac{1}{B}\sup_{i\in\mathbb{I}}\sum_{j\in\mathbb{I},\ j\neq i}|\langle f_j, f_i\rangle| < 1,$$

and therefore

$$1-\frac{1}{B}\inf_{i\in\mathbb{I}}\|f_i\|^2+\frac{1}{B}\sup_{i\in\mathbb{I}}\sum_{j\in\mathbb{I},\ j\neq i}|\langle f_j, f_i\rangle| < 1.$$

Hence

$$\begin{aligned}K(\{f_i\}_{i\in\mathbb{I}})&=\sup_{i\in\mathbb{I}}\left(1-\frac{1}{B}\|f_i\|^2+\frac{1}{B}\sum_{i\neq j}|\langle f_j, f_i\rangle|\right)\\ &\leq 1-\frac{1}{B}\inf_{i\in\mathbb{I}}\|f_i\|^2+\frac{1}{B}\sup_{i\in\mathbb{I}}\sum_{j\in\mathbb{I},\ j\neq i}|\langle f_j, f_i\rangle| < 1.\end{aligned}$$

For the latter assertion, because $\sup_{i\in\mathbb{I}}\sum_{j\in\mathbb{I},\ j\neq i}|\langle f_j, f_i\rangle| = 0$, by (7), $0 < \inf_{i\in\mathbb{I}}\|f_i\|^2$. Every Riesz sequence is a Bessel sequence and so $\sup_{i\in\mathbb{I}}\|f_i\|^2 < \infty$. □

3. Some Extensions of Orthogonality

In the next assertion a characterization for Riesz sequences which are linear combinations of an orthonormal basis is given.

Proposition 3.1. *Let $\{e_i\}_{i\in\mathbb{Z}}$ be an orthonormal basis for $\mathcal{H}$ and $\{\alpha_k\}_{k\in\mathbb{Z}}$ be a sequence of complex numbers such that $\sum_{k\in\mathbb{Z}}|\alpha_k|^2 < \infty$ and*

$$|\alpha_0| \geqslant |\alpha_1| = |\alpha_{-1}| \geqslant |\alpha_2| = |\alpha_{-2}| \geqslant \ \ldots.$$

For every $i\in\mathbb{Z}$ define

$$f_i = \sum_{k\in\mathbb{Z}} \alpha_{k-i} e_k. \tag{8}$$

The following condition implies that $\{f_i\}_{i\in\mathbb{Z}}$ is a Riesz sequence:

$$\sum_{j\in\mathbb{Z}, j\neq 0}\sum_{k\in\mathbb{Z}} \left|\alpha_k\overline{\alpha_{k-j}}\right| < \sum_{k\in\mathbb{Z}}|\alpha_k|^2, \tag{9}$$

Proof. Clearly,

$$B := \sup_{i\in\mathbb{Z}}\|f_i\|^2 = \inf_{i\in\mathbb{Z}}\|f_i\|^2 = \sum_{k\in\mathbb{Z}}|\alpha_k|^2 < \infty.$$

For $j\neq 0$, we compute

$$\langle f_0, f_j\rangle = \sum_{k\in\mathbb{Z}} \alpha_k\overline{\alpha_{k-j}}.$$

Hence, for $i\in\mathbb{Z}$,

$$\sum_{j\in\mathbb{Z}, j\neq i}|\langle f_i, f_j\rangle| = \sum_{j\in\mathbb{Z}, j\neq i}|\langle f_0, f_{j-i}\rangle| = \sum_{j\in\mathbb{Z}, j\neq 0}|\langle f_0, f_j\rangle|.$$

So,

$$\begin{aligned}\sup_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}, j\neq i}|\langle f_i, f_j\rangle| &= \sup_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}, j\neq 0}|\langle f_0, f_j\rangle| \\ &= \sum_{j\in\mathbb{Z}, j\neq 0}|\langle f_0, f_j\rangle| \\ &= \sum_{j\in\mathbb{Z}, j\neq 0}\sum_{k\in\mathbb{Z}}\left|\alpha_k\overline{\alpha_{k-j}}\right| \end{aligned} \tag{10}$$

Since $\inf_{i\in\mathbb{I}}\|f_i\|^2 = \sum_{i\in\mathbb{Z}}|\alpha_i|^2 < \infty$, the above relation and (9) yield that

$$\sup_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}, j\neq i}|\langle f_j, f_i\rangle| < \inf_{i\in\mathbb{I}}\|f_i\|^2.$$

Now, use Corollary 2.2.

□

The following example is a special case of the above proposition.

Example 3.1. *For every $i \in \mathbb{Z}$ define*

$$f_i = \alpha e_{i-1} + e_i + \alpha e_{i+1},$$

where $0 < \alpha < \infty$. In the previous proposition put $\alpha_0 = 1$, $\alpha_{-1} = \alpha_1 = \alpha$ and $\alpha_k = 0$ otherwise. We see that for $|j| > 2$, $\alpha_k \overline{\alpha_{k-j}} = 0$. We consider the cases $j \in \{\pm 1, \pm 2\}$. When $j = \pm 1$, $\left\{\alpha_k \overline{\alpha_{k-j}} | k \in \mathbb{Z}\right\} = \{0, \alpha, \alpha\}$. and when $j = \pm 2$, $\left\{\alpha_k \overline{\alpha_{k-j}} | k \in \mathbb{Z}\right\} = \left\{0, \alpha^2, \alpha^2\right\}$. Using these computations,

$$\sum_{j \in \mathbb{Z}, j \neq 0} \sum_{k \in \mathbb{Z}} \left|\alpha_k \overline{\alpha_{k-j}}\right| = \sum_{0 < |j| \leq 2} \sum_{k \in \mathbb{Z}} \left|\alpha_k \overline{\alpha_{k-j}}\right| = 2\alpha^2 + 4\alpha.$$

Since $\sum_{k \in \mathbb{Z}} |\alpha_k|^2 = 2\alpha^2 + 1$, when $\alpha < \frac{1}{4}$ the condition of Proposition 3.1 is satisfied and hence $\{f_i\}_{i \in \mathbb{Z}}$ is a Riesz sequence.

As well as the above example, we introduce some extensions of orthogonality as follows: Let $B = \sup_{i \in \mathbb{I}} \|f_i\|^2 < \infty$ and $1 \leqslant m \in \mathbb{N}$, $\eta > 0$. Suppose that for $i \in \mathbb{I}$,

$$|\langle f_j, f_i \rangle| \leq \eta, \qquad i - m \leqslant j \leqslant i + m, \ j \neq i,$$

and $\langle f_j, f_i \rangle = 0$, otherwise. With $m = 0$, $\{f_i\}_{i \in \mathbb{I}}$ becomes an orthogonal sequence.

Proposition 3.2. *With the orthogonality described above, if $\{f_i\}_{i \in \mathbb{I}}$ satisfies the condition*

$$2m\eta < \inf_{i \in \mathbb{I}} \|f_i\|^2, \tag{11}$$

then $\{f_i\}_{i \in \mathbb{I}}$ is a Riesz sequence.

Proof. We compute that

$$\begin{aligned} K(\{f_i\}_{i \in \mathbb{I}}) &= \sup_{i \in \mathbb{I}} \left(1 - \frac{1}{B}\|f_i\|^2 + \sum_{j=i-m, j \neq i}^{i+m} \left|\frac{\langle f_j, f_i \rangle}{B}\right|\right) \\ &\leqslant 1 - \frac{1}{B} \inf_{i \in \mathbb{I}} \|f_i\|^2 + \frac{2m\eta}{B}. \end{aligned} \tag{12}$$

By the hypothesis of the proposition

$$\frac{2m\eta}{B} < \frac{1}{B} \inf_{i \in \mathbb{I}} \|f_i\|^2 \leq \frac{1}{B} \sup_{i \in \mathbb{I}} \|f_i\|^2 = 1.$$

And so,

$$1 - \frac{1}{B} \inf_{i \in \mathbb{I}} \|f_i\|^2 < 1 - \frac{2m\eta}{B}.$$

Therefore

$$1 - \frac{1}{B} \inf_{i \in \mathbb{I}} \|f_i\|^2 + \frac{2m\eta}{B} < 1.$$

This relation and (12) yield that $K(\{f_i\}_{i \in \mathbb{I}}) < 1$. Theorem 2.1(1) ensures us that $\{f_i\}_{i \in \mathbb{I}}$ is a Riesz basis. □

Relation (11) says that with the orthogonality described above, for keeping Riesz property, m and η should be restricted by relations $\eta \leq \frac{\inf_{i \in \mathbb{I}} \|f_i\|^2}{2m}$ or $m = \frac{\inf_{i \in \mathbb{I}} \|f_i\|^2}{2\eta}$.

In the sequel, two special types of Gram matrices will be introduced. A Matrix $A = \{a_{i,j}\}_{i,j\in\mathbb{I}}$ is said to be a sub-polynomial matrix (see [6]), if there are two constants $\varepsilon, r > 0$ such that

$$|a_{i,j}| \le \varepsilon\,(1+|i-j|)^{-r}, \quad i,j\in\mathbb{I}.$$

A matrix $A = \{a_{i,j}\}_{i,j\in\mathbb{I}}$ is called a sub-exponential matrix if there exist constants $\varepsilon, \alpha > 0$ such that

$$|a_{i,j}| \le \varepsilon e^{-\alpha|i-j|}, \quad i,j\in\mathbb{I}.$$

We mention ε, α or (ε, r) as the parameters of sub-exponential (sub-polynomial) matrix $A = \{a_{i,j}\}_{i,j\in\mathbb{I}}$. The following theorem shows the importance of such Gram matrices.

Theorem 3.1. [6, Theorem 5.6] *Assume that A is a sub-exponential (sub-polynomial) matrix which is invertible. Then its inverse is also a sub-exponential (sub-polynomial) matrix.*

By a little computation we see that the following conditions are equivalent:

(1) $\sup_{i\in\mathbb{I}} \|f_i\|^2 \le B$, $i\in\mathbb{I}$
(2) $0 \leqslant \frac{1}{B}\|f_i\|^2 \le 1$.
(3) $0 \leqslant 1-\frac{1}{B}\|f_i\|^2 \le 1$.
(4) $|1-\frac{1}{B}\|f_i\|^2| \le 1$.

For the proof the reader can follow these implications $(1)\to(2)\to(3)\to(4)\to(1)$. The equivalence above is used in the following Lemma.

Lemma 3.1. *Let $\{f_i\}_{i\in\mathbb{I}} \subseteq \mathcal{H}$ with $\sup_{i\in\mathbb{I}}\|f_i\|^2 \le B < \infty$ and $0<\alpha<\infty$. The following statements are equivalent for $\varepsilon > 1$:*

(1) $\left|\delta_{i,j} - \frac{1}{B}\langle f_j, f_i\rangle\right| \le \varepsilon e^{-\alpha|i-j|}, \ i,j\in\mathbb{I},$

(2) $\frac{1}{B}|\langle f_j, f_i\rangle| \le \varepsilon e^{-\alpha|i-j|}, \ i,j\in\mathbb{I}.$

If the above conditions are satisfied we have $\|f_i\| \le \sqrt{B}$. For the case $0<\varepsilon<1$,

$$\left|\delta_{i,j} - \frac{1}{B}\langle f_j, f_i\rangle\right| \le \varepsilon e^{-\alpha|i-j|}, \quad i,j\in\mathbb{I}, \tag{13}$$

if and only if

$$\sqrt{B(1-\varepsilon)} \le \|f_i\| \le \sqrt{B(1+\varepsilon)},\ i\in\mathbb{I} \ \textit{and} \ \ |\langle f_j, f_i\rangle| \le B\varepsilon e^{-\alpha|i-j|}, \ \ i,j\in\mathbb{I}\ (i\ne j). \tag{14}$$

Proof. Let $i=j$. By the comment before the lemma, since $\sup_{i\in\mathbb{I}}\|f_i\|^2 \le B$, we always have

$$\left|\delta_{i,i} - \frac{1}{B}\langle f_i, f_i\rangle\right| = \left|1-\frac{1}{B}\ \|f_i\|^2\right| \le 1 \le \varepsilon = \varepsilon e^{-\alpha|i-i|}.$$

and

$$\frac{1}{B}|\langle f_i, f_i\rangle| = \frac{1}{B}\|f_i\|^2 \le 1 \le \varepsilon = \varepsilon e^{-\alpha|i-i|}.$$

Thus, for $i=j$, (1) and (2) are the same. But, for $i\ne j$, since $\delta_{i,j}=0$.

$$\left|\delta_{i,j} - \frac{1}{B}\langle f_j, f_i\rangle\right| = \frac{1}{B}|\langle f_j, f_i\rangle|.$$

Therefore, for $i \neq j$, (1) and (2) are identical.
Now, for the case $0 < \varepsilon < 1$ we prove the second assertion. We prove the inequality (13) in two cases $i \neq j$ and $i = j$. Assuming $i = j$, (13) becomes $|1 - \frac{1}{B}\|f_i\|^2| \leq \varepsilon$. This is equivalent to

$$1 - \varepsilon \leqslant \frac{1}{B}\|f_i\|^2 \leq 1 + \varepsilon,$$

and since $0 < \varepsilon < 1$ we take the root $\sqrt{B(1-\varepsilon)}$ and then

$$\sqrt{B(1-\varepsilon)} \leq \|f_i\| \leq \sqrt{B(1+\varepsilon)}.$$

For the case $i \neq j$, obviously, (13) is equivalent to the second part of (14). □

Proposition 3.3. *Let* $\{f_i\}_{i\in\mathbb{I}} \subseteq \mathcal{H}$ *with* $\sup_{i\in\mathbb{I}} \|f_i\|^2 \leq B < \infty$. *Then* $I - \frac{1}{B}G$ *is a sub-exponential matrix with parameter* $0 < \varepsilon < 1$, $0 < \alpha$ *if and only if*

$$\sqrt{B(1-\varepsilon)} \leq \|f_i\| \leq \sqrt{B(1+\varepsilon)},\ i \in \mathbb{I}\ \ \textit{and}\ \ |\langle f_j, f_i\rangle| \leq B\varepsilon e^{-\alpha|i-j|},\ \ i,j \in \mathbb{I}\ (i \neq j).$$

The above conditions yield that $\{f_i\}_{i\in\mathbb{I}}$ *is a Riesz sequence and the inverse of the Gram matrix is sub-exponential.*

Note that the first part of Lemma 3.1 has the following consequence that: for $\varepsilon \geq 1$, $I - \frac{1}{B}G$ is a sub-exponential matrix if and only if G is a sub-exponential matrix. Also, similar results for a sub-polynomial Gram matrix can be derived. A special type of a sub-exponential orthogonality is described below.

Proposition 3.4. *Let* $\{f_i\}_{i\in\mathbb{I}} \subseteq \mathcal{H}$ *with* $\sup_{i\in\mathbb{I}} \|f_i\|^2 \leq B < \infty$ *and* $k \in \mathbb{I}$ *such that* $2^{-k+2} < \inf_{i\in\mathbb{I}} \|f_i\|^2$. *Assume that*

$$|\langle f_j, f_i\rangle| \leq 2^{-|i-j|-k},\ \ i,j \in \mathbb{I}, (i \neq j).$$

Then $\{f_i\}_{i\in\mathbb{I}}$ *is a Riesz sequence and*

$$|\langle f_j, f_i\rangle| \leq 2^{-|i-j|-2} \inf_{i\in\mathbb{I}} \|f_i\|^2,\ \ i,j \in \mathbb{I}.$$

Proof. For $i \in \mathbb{I}$,

$$\begin{aligned} \sum_{j\in\mathbb{I},\ j\neq i} \left|\frac{\langle f_j, f_i\rangle}{B}\right| &\leq \frac{1}{B} \sum_{j\in\mathbb{I},\ j\neq i} 2^{-|i-j|-k} \\ &= \frac{2^{-k}}{B} \sum_{j\in\mathbb{I},\ j\neq i} 2^{-|i-j|} \\ &\leq \frac{2^{-k}}{B} 2 \sum_{l\in\mathbb{I},} 2^{-l} \qquad (15) \\ &= \frac{2^{-k+2}}{B}. \end{aligned}$$

Using the above relation and the conditions of the theorem we compute

$$K(\{f_i\}) = \sup_{i\in\mathbb{I}} \left(1 - \frac{1}{B}\|f_i\|^2 + \sum_{j\in\mathbb{I},\ j\neq i} \left|\frac{\langle f_j, f_i\rangle}{B}\right|\right)$$
$$\leq 1 - \inf_{i\in\mathbb{I}} \frac{1}{B}\|f_i\|^2 + \frac{2^{-k+2}}{B} < 1 - \frac{2^{-k+2}}{B} + \frac{2^{-k+2}}{B} = 1.$$

The reason for the inequality (15) is stated below:

$$\sum_{j\in\mathbb{I},\ j\neq i} 2^{-|i-j|} = \sum \left\{2^{-|i-j|} \mid \ j\in\mathbb{I},\ j\neq i\right\}$$
$$= \sum \left\{2^{-l} \middle| \ l \in \{|i-1|, |i-2|, ..., |i-(i-1)|, |i-(i+1)|, ... \right.$$
$$\left. ..., |i-(2i+1)|, |i-(2i+2)|, ...\} \right\}$$
$$= \sum \left\{2^{-l} \middle| \ l \in \{|i-1|, |i-2|, ..., 2, 1, 1, 2, ... \right.$$
$$\left. ..., |i-(2i+1)|, |i-(2i+2)|, ...\} \right\}$$
$$= \sum \left\{2^{-l} \middle| \ l \in \{|i-1|, |i-2|, ..., 2, 1, 1, 2, ..., |i-1|, |i-2|, ...\} \right\}$$
$$= \sum \left\{2^{-l} \middle| \ l \in \{1, 2, ..., |i-2|, |i-1|\} \right\} + \sum \left\{2^{-l} \middle| \ l \in \mathbb{I} \right\}$$
$$\leqslant 2\sum \left\{2^{-l} \mid \ l \in \mathbb{I}\right\} = 2 \sum_{l\in\mathbb{I},} 2^{-l} = 2^2.$$

□

REFERENCES

[1] *O. Christensen,* An Introduction to Frames and Riesz Bases, 2ed Birkhauser, Boston, MA. 2016.

[2] *O. Christensen, H. O. Kim, R. Y. Kim,* Gabor windows supported on $[1,1]$ and compactly supported dual windows, Appl. Comput. Harmon. Anal., **28** (2010), 89-103.

[3] *J.B. Conway,* A course in functional analysis. Springer, Science 2013.

[4] *I. Daubechies,* Ten Lectures on Wavelets, SIAM, Philadelphia, PA, 1992.

[5] *I. Daubechies,* The wavelet transform, time-frequency localization and signal analysis, IEEE Trans. Inform. Theory, **39** (1990), 961-1005.

[6] *K. Grochenig* Localization of frames, Banach frames, and the invertibility of the frame operator. J. Fourier Anal. Appl., **10** (2004), No. 2, 105–132.

[7] *A. Fereydooni, A. Safapour*, Pair frames, Results Math. **66** (2014), 247-263.

[8] *C. Heil, D. Walnut,* Continuous and discrete wavelet transforms, SIAMRev., **31** (1989), 628-666.

[9] *A.J.E.M. Janssen,* The duality condition for Weyl-Heisenberg frames, in: H.G. Feichtinger, T. Strohmer (Eds.), Gabor Analysis: Theory and Application, Birkhauser, Boston, MA, 1998.

[10] *F. Sergio, J. R.A. Zalik,* On the stability of frames and Riesz bases, Appl. Comput. Harmon. Anal., **2** (1995), No. 2 160-173.

[11] *A. Ron, Z. Shen,* Frames and stable for shift-invariant subspaces of $L^2(\mathbb{R})$, Canad. J. Math., **47** (1995), 1051-1094.