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η-PSEUDOLINEARITY IN MULTIOBJECTIVE PROGRAMMING USING THE BIFUNCTIONS CLASS

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This paper is concerning about the nonlinear programming problems where the functions involved are η-pseudolinear with respect to a bifunction. Starting from differentiable η-pseudolinear case described by Giorgi and Rueda we first design a framework for case of η-pseudolinearity with respect to a bifunction then we present our main results in two theorems where we prove the necessary and sufficient conditions of efficiency for a feasible point and we investigate the conditions when an efficient solution is properly efficient solution.

Keywords: multiobjective nonlinear programming, efficiency.

1. Introduction

The convexity concept is one of the most significant concepts in mathematical optimization theory. Since Hanson (see [1]) introduced the class of η-convex functions as a generalization of convex function, a lot of authors (see [2],[3],[4],[5],[6],[7],[8]) developed this research domain. An important contribution in study of the multiobjective programming problems was the pseudolinear functions class gave by Chew and Choo (see [9]). Their approach was later extended to the semilocal pseudoliniarity and η-pseudolinearity concepts (see [10],[11],[12]).

On the other hand, there are in the specialized literature (see [13], [14], [15]) results regarding the multiobjective programming problem when the objective functions and the constrain functions are not differentiable. In the following we establish our framework, then we give in Section 2, two results concerning the efficiency and finally, the last Section contains our conclusion.

Definition 1 (Hanson see [1]) A differentiable function $f: I \subseteq R^n \longrightarrow R$ is said to be η-pseudo-convex or pseudo-invex with respect to a vector function η(y,x) if:

$$\nabla f(x)^t \eta(y,x) \geq 0 \Rightarrow f(y) \geq f(x) \; \forall \, x, y \in I$$

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Ansari et al. (see [10]) defined for differentiable functions, the notion of η-pseudolinear functions.

Definition 2 Given a function $f: I \longrightarrow R$. The function is said to be η-pseudolinear if f and $-f$ are pseudo-invex with respect to the same η.

In our paper we work with η-pseudolinearity with respect to a bifunction instead η-pseudolinearity for differentiable functions. Let a function $f: I \longrightarrow R$, $\eta(y,x)$ a vector function and $\hat{f}(x,\eta(y,x))$ a bifunction. We suppose that there exists a function $p: I \times I \to R$ with $p(x,y) > 0$ called the proportional functional, such that:

$$f(y) = f(x) + p(x,y)\hat{f}(x,\eta(y,x)) \text{ for all } x,y \in I \quad (1)$$

Definition 3 Let η: $I \times I \to R^n$ a vector function. A function $f: I \subseteq R^n \longrightarrow R$ is called η-pseudo-convex with respect to the bifunction $\hat{f}$ if:

$$\hat{f}(x,\eta(y,x)) \geq 0 \ \Rightarrow \ f(y) \geq f(x) \ \forall\, x,y \in I$$

Definition 4 Given a function $f: I \longrightarrow R$. The function f is said to be η-pseudolinear with respect to bifunction $\hat{f}$ if f and $-f$ verify the following conditions for all $\forall\, x,y \in I$:

$\hat{f}(x,\eta(y,x)) \geq 0 \Rightarrow f(y) \geq f(x)$ and $\widehat{-f}(x,\eta(y,x)) \geq 0 \Rightarrow f(y) \leq f(x)$.

2. Efficiency for η-pseudolinear with respect to a bifunction case

Let the multiobjective programming problem (MP): $max f(x)$, $g(x) \leq b$ where b is a m-dimensional vector. Giorgi and Rueda (see [16]) state the efficiency conditions in the case of differentiable η-linear functions. The following theorems extends their results to the case of η-pseudolinearity with respect to the bifunction $\hat{f}$.

Theorem 1 Let a programming problem (MP) where $f_i: I \longrightarrow R$ are η_i-pseudolinear with respect to $\hat{f}_i$, $i = \overline{1,p}$ and $g_j: I \longrightarrow R$ are $\tilde{\eta}_j$-pseudolinear with respect to $\hat{g}_j, j = \overline{1,m}$, on the open set I with p_i and q_j their proportional functionals. If $\hat{f}_i(x^0,\eta_i(.,x^0))$ and $\hat{g}_j\left(x^0,\tilde{\eta}_j(.,x^0)\right)$ are likeconvexe for all i and j, then a feasible point x^0 is an efficient solution of (MP) if and only if there exist multipliers $\lambda_i > 0$ and $\mu_j \geq 0$ with $i \in \{1,..,p\}$ and $j \in J(x^0)$ such that:

$$\sum_{i=1}^{p} \lambda_i \hat{f}_i\left(x^0,\eta_i(y,x^0)\right) \leq \sum_{j\in J(x^0)} \mu_j\, \hat{g}_j\left(x^0,\tilde{\eta}_j(y,x^0)\right) \quad (2)$$

Proof First we are going to prove the converse. If we suppose that x^0 is not an efficient solution for (MP) then there exists a feasible point y such that $f_i(x^0) \leq f_i(y)$ for all i and for some j, $f_j(x^0) < f_j(y)$. Since f_i are η_i-pseudolinear with respect to $\hat{f}_i$, $i = 1,p$ and g_j are $\tilde{\eta}_j$-pseudolinear and using the relation (1), we have:

$$\hat{f}_i\left(x^0, \eta_i(y, x^0)\right) = \frac{f_i(y) - f_i(x^0)}{p_i(x^0, y)} \Rightarrow$$

$$\sum_{i=1}^{p} \lambda_i \hat{f}_i\left(x^0, \eta_i(y, x^0)\right) = \sum_{i=1}^{p} \frac{\lambda_i (f_i(y) - f_i(x^0))}{p_i(x^0, y)} \quad (3)$$

$$\hat{g}_j\left(x^0, \tilde{\eta}_j(y, x^0)\right) = \frac{g_j(y) - g_j(x^0)}{q_j(x^0, y)} \Rightarrow$$

$$\sum_{j \in J(x^0)} \mu_j \hat{g}_j\left(x^0, \tilde{\eta}_j(y, x^0)\right) = \sum_{j \in J(x^0)} \frac{\mu_j (g_j(y) - g_j(x^0))}{q_j(x^0, y)} \quad (4)$$

From (2), (3) and (4) $\Rightarrow \sum_{i=1}^{p} \frac{\lambda_i (f_i(y) - f_i(x^0))}{p_i(x^0, y)} \leq \sum_{j \in J(x^0)} \frac{\mu_j (g_j(y) - g_j(x^0))}{q_j(x^0, y)}$ (5)

In relation (5) the right member is negative or zero because $g_j(y) \leq b_j = g_j(x^0)$ and the left one is strictly positive because x^0 is not efficient solution. That situation is impossible so our presumption is false.

For the direct theorem implication, we consider that x^0 is an efficient solution. We consider the following system:

$$\begin{aligned} &\hat{g}_j\left(x^0, \tilde{\eta}_j(y, x^0)\right) \leq 0 \text{ for } j \in J(x^0) \\ &\hat{f}_i\left(x^0, \eta_i(y, x^0)\right) \geq 0 \text{ for } i \in \{1, .., p\} - \{s\} \\ &\hat{f}_s\left(x^0, \eta_i(y, x^0)\right) > 0 \end{aligned} \quad (6)$$

Since f_i, g_j are η_i-pseudolinear, then using the relation (1) the previous system can be written:

$$\begin{aligned} &\frac{g_j(x) - g_j(x^0)}{q_j(x^0, x)} \leq 0 \text{ for } j \in J(x^0) \\ &\frac{f_i(x) - f_i(x^0)}{p_i(x^0, x)} \geq 0 \text{ for } i \in \{1, .., p\} - \{s\} \\ &\frac{f_s(x) - f_s(x^0)}{p_s(x^0, x)} > 0 \end{aligned} \quad (7)$$

If there exists a solution y for the system (7) then:

$$\frac{g_j(y) - g_j(x^0)}{q_j(x^0, y)} \leq 0 \text{ for } j \in J(x^0) \quad (8)$$

$$\frac{f_i(y) - f_i(x^0)}{p_i(x^0, y)} \geq 0 \text{ for } i \in \{1, .., p\} - \{s\} \quad (9)$$

$$\frac{f_s(y) - f_s(x^0)}{p_s(x^0, y)} > 0 \quad (10)$$

But $g_j(x^0, y) > 0$, so the relation (8) implies: $g_j(y) \leq g_j(x^0) = b_j$ for $j \in J(x^0)$. Hence y is a feasible point. On the other hand, $p_i(x^0, y) > 0$,then the relations (9) and (10) implies: $f_i(y) \geq f_i(x^0)$ for $i \in \{1, .., p\} - \{s\}$ and $f_s(y) > f_s(x^0)$. Therefor $x^0$is not an efficient solution that contradicts our hypothesis. So, if x^0 is an efficient solution then the system (6) has no solution. That implies the following system has no solution:

$$\hat{g}_j\left(x^0, \tilde{\eta}_j(x, x^0)\right) + \varepsilon < 0 \quad \text{for } j \in J(x^0) \tag{11}$$
$$\hat{f}_i\left(x^0, \eta_i(x, x^0)\right) - \varepsilon > 0 \quad \text{for } i \in \{1, .., p\} - \{s\}$$
$$\hat{f}_s\left(x^0, \eta_i(x, x^0)\right) > 0$$

The relation (11) and the alternative theorem H-K imply:

$$\sum_{i=1}^{p} \lambda_i \hat{f}_i\left(x^0, \eta_i(y, x^0)\right) \leq \sum_{j \in J(x^0)} \mu_j\, \hat{g}_j\left(x^0, \tilde{\eta}_j(y, x^0)\right) + \varepsilon\left(\sum_{i=1}^{p} \lambda_i + \sum_{j \in J(x^0)} \mu_j\right)$$

If $\varepsilon \to 0$ we get:

$$\sum_{i=1}^{p} \lambda_i \hat{f}_i\left(x^0, \eta_i(y, x^0)\right) \leq \sum_{j \in J(x^0)} \mu_j\, \hat{g}_j\left(x^0, \tilde{\eta}_j(y, x^0)\right).$$

Theorem 2 Let a programming problem (MP) with $f_i: I \longrightarrow R$ and are η_i-pseudolinear with respect to $\hat{f}_i$ and $g_j: I \longrightarrow R$ are $\tilde{\eta}_j$-pseudolinear with respect to $\hat{g}_j$ on the open set I and p_i, q_j their proportional functionals. If $x^0$is an efficient solution that satisfy the boundness condition, then $x^0$is properly efficient solution of the programing problem.

Proof From the hypothesis x^0 is an efficient solution and using theorem 1, it follows that there exist λ_i and μ_j such that:

$$\sum_{i=1}^{p} \lambda_i \hat{f}_i\left(x^0, \eta_i(x, x^0)\right) \leq \sum_{j \in J(x^0)} \mu_j\, \hat{g}_j\left(x^0, \tilde{\eta}_j(x, x^0)\right) \Rightarrow$$
$$\sum_{i=1}^{p} \frac{\lambda_i(f_i(x) - f_i(x^0))}{p_i(x^0, x)} \leq \sum_{j \in J(x^0)} \frac{\mu_j(g_j(x) - g_j(x^0))}{q_j(x^0, x)} \tag{12}$$

For x a feasible point, we have: $g_j(x) \leq g_j(x^0) = b_j$, then:

$$\sum_{j \in J(x^0)} \frac{\mu_j(g_j(x) - g_j(x^0))}{q_j(x^0, x)} \leq 0$$

and relation (12) becomes:

$$\sum_{i=1}^{p} \frac{\lambda_i(f_i(x) - f_i(x^0))}{p_i(x^0, x)} \leq 0 \ \text{ or } \ \sum_{i=1}^{p} \lambda_i \hat{f}_i\left(x^0, \eta_i(x, x^0)\right) \leq 0 \tag{13}$$

On the other hand, the hypothesis that x^0 is an efficient solution that satisfies the boundness condition implies that the following set is bounded from above also:

$$\left\{(p-1)\frac{\lambda_j p_i(x^0, x)}{\lambda_i p_j(x^0, x)} \middle| x \in I, f_i(x^0) < f_i(x), f_j(x^0) > f_j(x), i, j = 1, p\right\} \tag{14}$$

We denote with a the positive real number that is upper bound of the set previously defined.

If $r \in \{1, .., p\}$ is the index with:

$$-\lambda_r \hat{f}_r\left(x^0, \eta_i(x, x^0)\right) = max_{\hat{f}_i\left(x^0, \eta_i(x, x^0)\right)}\left\{-\lambda_i \hat{f}_i\left(x^0, \eta_i(x, x^0)\right)\right\}$$

And if there exist $t \in \{1, .., p\}$ and $x \in I$ such that $f_t(x) > f_t(x^0)$, which means $\hat{f}_t\left(x^0, \eta_i(x, x^0)\right) > 0$, then the relation (13) implies:

$$\lambda_t \hat{f}_t\big(x^0, \eta_i(x, x^0)\big) \leq (p-1)\left(-\lambda_r \hat{f}_r\big(x^0, \eta_i(x, x^0)\big)\right) \Rightarrow$$

$$\lambda_t \frac{f_t(x) - f_t(x^0)}{p_t(x^0, x)} \leq (p-1)\lambda_r \frac{f_r(x^0) - f_r(x)}{p_r(x^0, x)}$$

Since a is a upper bound for set defined in (14), then:

$$f_s(x) - f_s(x^0) \leq a\big(f_r(x^0) - f_r(x)\big)$$

Thus x^0 is properly efficient solution of (MP).

3. Conclusions

Our paper deal with the nonlinear programming problems where the functions involved are η-pseudolinear with respect to a bifunction. In this context we prove the necessary and sufficient conditions of efficiency for a feasible point and we described the conditions when an efficient solution is properly efficient solution. Our theorems supplies results that can be applied even in the study of case when a multiobjective programming problem has the objective functions and the constrains functions nondifferentiable.

R E F E R E N C E S

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