QUANTIZATION ERRORS IN SAMPLED-DATA SYSTEMS WITH BACKSTEPPING CONTROLLERS

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The present article deals with the problem of evaluating the effects of quantization on a nonlinear sampled-data control system. The control strategy is based on a suitable digital design of "backstepping" type controllers and it was proposed in a previous article. The advantage of this design is that it preserves, under sampling, the stabilizing performances imposed in the continuous-time design. The control solution comprises heavily nonlinear expressions and to study the effects of quantization is strongly requested. This work captures the contributions of the sampling period, of the degree of controllers' approximations and of the numerical precision in the performance of a sampled-data controller.

Keywords: digital backstepping control, quantization, nonlinear and sampled-data systems.

1. Introduction

Nowadays, most of the modern solutions adopted in the process control are based on digital control schemes. Many solutions exist for designing suitable digital controllers for linear systems [1] which are not the case for nonlinear systems. In a sampled-data nonlinear context[4] there are common three approaches (see the references therein [2]). Because exact discretization is not always available, a first approach is based on the approximated discrete-time models under which digital controllers are derived. The results obtained in this case are rather local and assure practical stabilization [3]. A second approach, which is the most used tool nowadays, is represented by the emulation of continuous-time controllers. In this case the control design is carried out in continuous-time domain and the controller is implemented digitally by means of sampling and holding devices. Emulation is not the best solution, since it does not take into account the sampling period, and it recovers the continuous-time properties in the case of fast sampling. A third approach, which is here used, proposes a digital

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4 A sampled data control system is a control scheme where a continuous-time plant is driven by a digital controller

controller exploiting the sampled-data dynamics of the plant. In this approach better stabilizing performances can be achieved not only under the condition of fast sampling.

The attention is restricted in this paper to a particular stabilizing procedure backstepping for systems admitting strict-feedback forms. The backstepping control strategy introduced in [4] is widely developed in various contexts and it is an acknowledged powerful stabilizing procedure for nonlinear continuous-time dynamics.

In a recent work [5], a digital version was proposed for improving – in particular with respect to the admissible sampling period- the performance of its usual implementation through emulation. The admissible sampling period denotes the time over which the control can be kept constant without damaging the stabilizing objective. Such an approach is inspired from [6],[7], where digital redesign methods are developed for finding piecewise constant controllers which match - at the sampling instants - a suitably chosen target behavior associated with the continuous-time closed loop dynamics. In this way, the digital solutions are described by their series expansions in powers of the sampling period. In practice, approximate solutions referring to the degree of the polynomial truncation, are implemented. This approach has been developed on various academic examples and tested on experimental platforms as in [8].

When considering the implementation of these digital controllers others issues arise due to the finite number representation of the controller. The quantization error affects the performances of the control law and a close attention should be given.

The object of this work is to study the effects of the quantization on a sampled-data nonlinear system, with a digital controller issued from a backstepping procedure. This study captions the contributions of the sampling period, of the degree of the controller's approximations and of the numerical precision in the performance of the sampled-data controller.

The paper is organized as follows: section 2 recalls the underlying theory of the continuous-time and sampled-data backstepping designs; an analysis of quantization errors based on the literature survey for linear and nonlinear systems is given in section 3. An analysis of the e_ects of the quantization errors when employing the digital solution proposed is given in 3.3.

2. Controller design

In this section the results on the backstepping controller design are recalled from [5, 8].

2.1. Mathematical notations and main assumptions. Through the paper, maps and vector fields are assumed smooth (i.e. infinitely differentiable of class

C^{∞}). This condition can be relaxed when one considers practical situations, where approximated solutions are given. L_f denotes the Lie derivative operator as $L_f = \sum_{i=1}^{n} f_i(\cdot)\frac{\partial}{\partial x_i}$, associated with a vector field f, and e^{L_f} or simply denoted e^f is the Lie series exponential operator associated to f, i.e. $e^f := 1 + \sum_{i\geq 1}\frac{L_f^i}{i!}$; "$(x)$" or "$|_x$" denotes the evaluation at a point x of a generic map. Given two vector fields f, g on R^n, then the Lie bracket is defined as $ad_{fg} = [f, g] = [L_f, L_g] = L_f \circ L_g - L_g \circ L_f$. By abuse of language, we will drop the composition sign as $L_f L_g$ is the same with $L_f \circ L_g$. For any smooth real valued function h, the following result holds $e^f h(x) = h(e^f(x))$. The evaluation of a function at time $t = k\delta$ indicated by "$|_{t=k\delta}$"or is it omitted, when it is obvious from the context.

A positive function α is called class K if it is continuous, strictly increasing and zero at zero. If it is unbounded then it is of class K_∞.

A positive function β is called class KL if it is continuous and if for each s fixed the map $\beta(r,s)$ belongs to class K with respect to r and, for each r fixed, the map $\beta(r,s)$ is decreasing with respect to s and $\beta(r,s) \to 0$ as $r \to 0$.

A function $Q(x,\delta)$, with $\delta < 1$ is of P order in δ, e.g. $O(\delta^P)$, if whenever Q is defined then it can be written as $Q(x,\delta) = \delta^P \widetilde{Q}(x,\delta)$ and there is a function $\psi \in K_\infty$ such that for each $\Delta > 0$ there exists $\delta^* > 0$ such that $|x| \leq \Delta$ and $\delta < \delta^*$ implies $\left|\widetilde{Q}(x,\delta)\right| \leq \psi(|x|)$, where $|\cdot|$ represents any norm in R^n.

2.2. Continuous-time backstepping design. Given a continuous-time dynamics in strict-feedback form

$$\dot{\eta}(t) = f(\eta(t)) + g(\eta(t))\xi \tag{1}$$
$$\dot{\xi}(t) = f_a(\eta(t),\xi(t)) + g_a(\eta(t),\xi(t))u_c(t) \tag{2}$$

where the states η and ξ are in R^n and R^m respectively and the control vector $u_c \in R^n$; f, g, f_a, g_a are vector fields of appropriate dimensions assumed to be smooth and complete. Analogously, all the functions are assumed to be smooth. The following result recalls the backstepping approach.

Proposition 2.1. [9] *Continuous-time backstepping - Consider the system (1)-(2), and assume the existence of* $\phi(\eta)$ *with* $\phi(0)=0$ *and* $W(\eta)$ *a Lyapunov function such that*

$$\frac{\partial W}{\partial \eta}\left(f(\eta)+g(\eta)\phi(\eta)\right)<0, \forall \eta \in R^n \setminus \{0\} \tag{3}$$

Then, if $g_a(\eta,\xi)$ *is invertible for all* (η,ξ)*, the state-feedback control law*

$$u_c(t)=g_a(\eta,\xi)^{-1}\left(\dot{\phi}(\eta)-\frac{\partial W}{\partial \eta}g(\eta)-f_a(\eta,\xi)+v\right) \tag{4}$$

with $\dot{\phi}(\eta)=\frac{\partial \phi}{\partial \eta}\left(f(\eta)+g(\eta)\xi\right)$*, and* v *an external input, asymptotically stabilizes the origin of (1)-(2), with Lyapunov function*

$$V(\eta,\xi)=W(\eta,\xi)+\frac{1}{2}(\xi-\phi(\eta))^2 \tag{5}$$

In fact, setting the output error $y=\xi-\varphi(\eta)$*, (1)-(2) can be rewritten as an* y *-error dynamics:*

$$\dot{\eta}(t)=f(\eta(t))+g(\eta(t))(y(t)+\phi(\eta)) \tag{6}$$

$$\dot{y}(t)=-\frac{\partial W}{\partial \eta}g(\eta)+v \tag{7}$$

The damping is improved by setting $v=-K_y y$ with $K_y>0$ and also u_c provides passivity of the link y/v.

Following [10, Th. 4.1], the asymptotic stabilization of the origin allow us to characterize the Lyapunov evolution V for the continuous-time dynamics as follows.

Proposition 2.2. *Given the dynamics (6)-(7), with* $v=-K_y y$ *which asymptotically (globally) stabilizes the origin with a Lyapunov function* V *candidate (5) then, there exist* K (K_∞) *functions* α_1, α_2 *and* α_3 *such that the following conditions hold:*

$$\alpha_1(\|x\|)\leq V(x)\leq \alpha_2(\|x\|) \tag{8}$$

$$\dot{V}(x)\leq -\alpha_3(\|x\|) \tag{9}$$

for all $t>0$, $\forall x\in D$ *and* $x^T=\left[\eta^T, y^T\right]^T$. *If* $D=R^{n+m}$ *then the conditions hold globally.*

It is well known that the passivity and consequently the stabilizing performances are lost under sampling when setting the controller u_c constant over time intervals of length $\delta>0$ since the negativity of $\dot{V}$ is no longer assured. Let us recall the steps of the digital design strategy proposed in [5].

2.3. Digital backstepping redesign. Setting $x^T = \left[\eta^T, y^T\right]^T$, the transformed system (6)-(7) can be rewritten in a general form called input-affine representation:

$$\dot{x}(t) = f_c(x) + g_c(x)u_c \tag{10}$$

with

$$f_c(x) = \begin{bmatrix} f(\eta) + g(\eta)(\phi(\eta) + y) \\ f_a(\eta, y + \phi) - \dot{\phi} \end{bmatrix},\ g_c(x) = \begin{bmatrix} 0 \\ g_a(\eta, y + \phi) \end{bmatrix}$$

Given a finite time interval length T^*, let $\delta \in (0, T^*]$ be the sampling period and assume $u_c(t) = u_k$ constant over each interval of length δ. Arguing so, the sampled-data dynamics equivalent to (6)-(7) is described by

$$x_{k+1} = F^{\delta}(x_k, u_k) = e^{\delta(f_c + u_k g_c)} x_k \tag{11}$$

where $e^{\delta(f_c + u_k g_c)} = 1 + \sum_{i \geq 1} \frac{L^i_{c + u_k g_c}}{i!}$. Under u_c as in (4), integrating the Lyapunov function $V(\eta, \xi)$ over intervals of length δ, one gets

$$V(x_c(t = (k+1)\delta)) - V(x_c(t = k\delta)) = \int_{k\delta}^{(k+1)\delta} \dot{V}(x_c(\tau)) d\tau \tag{12}$$

which is referred to as the target difference; x_c indicates the closed loop continuous-time x-dynamics under u_c.

On the other hand, the first order discrete-time difference $V(x_{k+1}) - V(x_k)$ associated with the sampled dynamics (11) for a given δ and under u_k constant, can be computed as

$$V(x_{k+1}) - V(x_k) = V\left(e^{\delta(f_c + g_c u_k)} x_k\right) - V(x_k) \tag{13}$$

The sampled-data redesign strategy consists in designing the constant control law u_k to match, at the sampled-instants, the target difference (12); i.e. to satisfy the equality of series

$$V\left(e^{\delta(f_c + g_c u_k)} x_k\right) - V(x_k) = \int_{k\delta}^{(k+1)\delta} \dot{V}(x_c(\tau)) d\tau \tag{14}$$

when setting $x_k = x_c(t = k\delta)$. Approximated solutions correspond to satisfy the equality up to any desired order in δ. This is resumed in the next Theorem.

Theorem 2.1. [5] *Consider the system (1)-(2), and suppose the existence of a continuous-time controller $u_c(t)$ designed as in (4), such that condition (3) is satisfied. Then, there exists $T^* > 0$ and a piecewise constant controller $u_k = u_d^{\delta}$*

which matches the Lyapunov function evolution at the sampling instants and guarantees asymptotic stability of the sampled-data equivalent system (11) for any sampling period $\delta < T^*$.

The proof is given in the referred papers. The solution is described by a series expansion in δ:

$$u_k = u_d^{\delta} = u_{d0} + \sum_{i\geq 1} \frac{\delta^i}{(i+1)!} u_{di} \tag{15}$$

with the first terms

$$u_{d0} = u_c\big|_{t=k\delta} \tag{16}$$

$$u_{d1} = \dot{u}_c\big|_{t=k\delta} = \frac{\partial u_c}{\partial \eta}\left(f + g\xi\right)\big|_{t=k\delta} + \frac{\partial u_c}{\partial \xi}\left(f_a + g_a u_c\right)\big|_{t=k\delta} \tag{17}$$

$$u_{d2} = \ddot{u}_c\big|_{t=k\delta} + \frac{u_{d1}}{2}\left(L_{f_c} L_{g_c} - L_{g_c} L_{f_c}\right)V(x_k) / L_{g_c} V(x_k) \tag{18}$$

Remark. We note that $L_{g_c} V(x_k) \neq 0$ by construction. A complete solution exists in the form (15); the higher order terms can be iteratively computed, details are given in [5].

For practical applications approximated controllers are computed. Let us consider that a P order controller means, in the subsequents sections, a truncation of the in_nite series (15) in the power p of δ:

$$u_k = u_d^{\delta P} = u_{d0} + \sum_{i=1}^{P} \frac{\delta^i}{(i+1)!} u_{di} \tag{19}$$

The stability performance of the sampled-data dynamics with a P order controller is captured by the next proposition by means of Lyapunov functions.

Proposition 2.3. [9] *Given the dynamics (10), there exists* $T_1^* > 0$ *such that the* P*-order approximated controller computed according to* ***Theorem 2.1*** *satisfies the following conditions for any* $\delta < T_1^*$ *and* $x_k \in D$

$$\alpha_1(\|x_k\|) \leq V(x_k) \leq \alpha_2(\|x_k\|) \tag{20}$$

$$V(x_{k+1}) - V(x_k) \leq -\delta\alpha_3(\|x\|) + O(\delta^{P+2}) \tag{21}$$

under the assumption $x_k = x_c(t = k\delta)$.

The complete proof of this proposition is given in [9]. The proof underlines on the idea that there exists a discrete-time equivalent of the continuous-time dynamics (6)-(7) which preserves the same evolutions at each sampling instant. The discrete-time dynamics is described by infinite series in δ and its convergence to the continuous-time solution is assured only for $\delta < T_1^*$. Under the assumption that there exist no finite escape time of the solutions of the sampled-data dynamics then the stability analysis can be carried out at discrete-time instants only.

Then the conditions (8)-(9) are evaluated in a discrete-time context so getting the new conditions for a p approximated controller (19)-(20) under the assumption of the same initial conditions of the continuous and sampled-data dynamics. Studding these conditions it is clear that the exact solution provided by **Theorem 2.1** provides the same Lyapunov evolution at the sampling instants as the continuous-time solution, and hence the same asymptotic stabilization property. If in the continuous-time case the stabilization is globally, the digital solution holds globally for any $x_k \in R^{n+m}$ and any $\delta < T_1^*$.

In the presence of controller approximations, the Lyapunov stability depends on terms parameterized by higher powers of the sampling period δ. In this case it cannot be guaranteed that the Lyapunov difference (20) remains negative for all $\delta < T_1^*$ and all $x_k \in R^{n+m}$. The stabilization properties of the sampled-data scheme can be ensured only if the terms in $O(\delta^{P+2})$ do not influence the negativity of the right hand term. To reduce the influence of the terms $O(\delta^{P+2})$ one can either reduce the sampling period or to increase the order P. It is clear that the sampling period length and the initial condition will directly influence the performances. The Lyapunov difference may also becomes equal to zero under digital feedback. In such a case, one refers to practical stabilization [3], as the trajectories remain bounded near origin.

3. Analysis of quantization errors in digital control systems

A significant number of papers deals with the evaluation of the quantization error contributions in digital linear systems and fewer dedicate this problem to the nonlinear case. In the next lines some conclusions about quantizations of linear controllers are drawn, based on the literature survey.

3.1. Linear system case. Earlier studies on quantizations of linear sampled-data systems are given in [11], [12] or [13]. Based on these works, in [14], there are summarized 3 models that can be used to analyze the effects of the roundoff error. The first one, which is due to Bertram [11] is known as the worst-case error bound. In this analysis, the most pessimistic case is considered when the roundoff errors occur in a way to cause maximum harm. The Bertram's worst-case bound can be used to state that the output error will not grow beyond this bound. Another conclusion given is that if the linear system is stable then the system with quantization is also stable.

Another model used is the steady-state worst case. In this approach the analysis is carried out when the system is in steady state. This approach evaluates how large the errors of the steady-states are as a result of the roundoff. In this case, the output error bound is depending on *q/2* multiplied by the static gain of the linear system (where *q* is the roundoff error of the quantizer).

However, the previous models give simpler forms but are often excessively pessimistic. A third solution is to employ a stochastic analysis. The basic idea is that the quantization error is a signal that can be modeled as a white random signal with a probability function uniformly distributed over the range of quantization. Then by applying the stochastic procedure some estimates of the errors on a linear system can be expressed. Another aspect studied, is the contribution of the quantization of the controller parameters in the digital devices. The usual approach that can be used for analyzing the effects of coefficient quantization is referred to as the coefficient sensitivity analysis. The main idea is to compare the response differences (called variations) of the ideal system with the one with quantized coefficients. The evaluations of these responses become difficult for higher order linear controllers and the only solution is the use of simulation tools.

In the last decade the interest for sampled-data systems with quantizers has been renewed due to the success of networked control systems (see [15]). But all these works still consider the linear time characterization of the plants. In this context, new solutions have been proposed by employing quantizers with variable precision which are adapted accordingly to the quantized measurements. In this way better stabilization properties can be achieved.

In the end of this paragraph let us summarize some conclusions about quantization effects on linear digital controllers.

- The amount of the error introduced by quantization may depend on the choice of the sampling rate (especially when discretizing continuous-time controllers), [14], on the type of the system, and on the form of the controller [16];
- Due to finite word length in the controller there are met limit cycles (sustained oscillations) even in the absence of any applied input; these limit cycles exist in fixed-point digital controllers but can be ignored in floating-point architectures. To alleviate this effect a solution is to add to the input a low amplitude oscillation known as *dither*.

3.2. The nonlinear case. When considering a nonlinear input-affine system with a nonlinear state-feedback controller it is clear that most of the methods recalled for the linear case are not suitable to be used since the superposition principle do not apply.

A qualitative and singular work concerning the stability of nonlinear sampled-data systems with quantization is conducted in [17]. Instead of the fact that the nonlinear system provided is a simpler version of the standard input-affine case and also the fact that a linear controller is considered, this represents a first attempt concerning the stabilization properties of such systems.

The results obtained there state that if the linear version of the sampled-data system, without quantization, is asymptotically stable then the nonlinear sampled-data system with quantization is uniformly ultimately bounded. The bound of the solutions can be made as small as desired by making the quantization size sufficiently small. There have not yet been formulated the bounds of a quantized systems with nonlinear controllers. An idea is to consider that the quantizations act as a perturbation on the measurements used in the controller computations. If one considers the case of a state-feedback controller, constant on time-intervals of length δ, then the perturbed controller can be also expressed as a series expansion around the real state measurements as:

$$u_k(x_k+\varepsilon_k)=u(x_k)+\varepsilon_k D_u(x_k)+\frac{1}{2}\varepsilon_k H_u(x_k)\varepsilon_k^T \tag{22}$$

where $D_u(x_k)=\left.\left[\frac{\partial u}{\partial x_1}\cdots\frac{\partial u}{\partial x_n}\right]\right|_{t=k\delta}$ is the gradient of u along the state directions, and $H_u(x_k)$ is the Hessian matrix of u. It is clear that when dealing with linear state feedback controllers the $D_u(x_k)$ is a scalar which amplifies the quantization error ε_k. In the case of nonlinear controllers the series expression order (21) is equal to the highest power of the states and a discussion about the quantization becomes intractable. It is clear that if one considers the quantization error as a perturbation, the difficult part is to estimate its bound since it depends on the state variables. In the case of perturbed nonlinear systems, many results do exist for continuous or discrete-time systems which underlie on the level sets of suitable Lyapunov functions or in the case of sampling and hold devices, the robustness of the perturbed controller is often analyzed with the help of the notion of input-to-state stability. When the perturbation acts in the inputs measurements a more general result is more tedious to be stated. A complete work that handles this problem is [18]. Anyway, these results have not been linked with the quantization errors due to the fact that in this case the perturbations bounds also depends on the state variable.

Other studies give some qualitative results on the robustness of nonlinear controls with adaptive quantizers. In this case the quantizers are adapted accordingly in order to assure the stabilizing properties [19]. Related to the coefficient sensitivity a study has been performed for the backstepping type controllers in [20]. In this article the digital controller is designed by using a different methodology based on the adaptive approach. The important conclusion that is drawn there is that the design parameter should be chosen according to the level of precision desired.

3.3. The case of the proposed solution. In the context of this work, the heavily nonlinear nature of the digital controller and the lack of specific results in the literature make difficult any theoretical analysis of the effects of the quantization. As the digital controllers are build to satisfy specific stability properties the aim is to estimate the effects of the quantization on the stabilization.

We are interested next to evaluate the Lyapunov difference mismatches, at sampling instants, between the digital solution and the solution affected by the quantization error. Let us define the next Lyapunov difference:

$$\Delta V_q(x_k, u_k) = V(x_{k+1}, u_k^q) - V(x_{k+1}, u_k) \tag{23}$$

Some computations give:

$$\begin{aligned} V(x_{k+1}, u_k^q) - V(x_{k+1}, u_k) &= V(x_k) + \delta\big(L_{f_c}V(x_k) + u_k^q L_{g_c}V(x_k)\big) + O_1(\delta^2) \\ &- V(x_k) + \delta\big(L_{f_c}V(x_k) + u_k L_{g_c}V(x_k)\big) + O_2(\delta^2) \\ &= \delta(u_k^q - u_k)L_{g_c}V(x_k) + O_q(\delta^2) \end{aligned} \tag{24}$$

with $O_q(\delta^2)$ including all the mismatches terms provided by the presence of the quantization error. After some simple calculus we can state the following proposition.

Proposition 3.1. *Given the dynamics (10), there exists* $T_1^* > 0$ *such that the* P *-order approximated controller computed according to* ***Theorem 2.1*** *satisfies the following conditions for any* $\delta < T_1^*$ *and* $x_k \in D$ *in the presence of the measurement quantization errors:*

$$\alpha_1(|x_k^q|) \le V(x_k^q) \le \alpha_2(|x_k^q|) \tag{25}$$

$$V(x_{k+1}^q) - V(x_k^q) \le -\delta\alpha_3(|x_k^q|) + \delta\varepsilon_{u_k^q} L_{g_c}V(x_k) + O_q(\delta^2) + O(\delta^{P+2}) \tag{26}$$

under the assumption $x_k^q = x_c(t = k\delta)$ *and* $\varepsilon_{u_k^q} = u(x_k + \varepsilon_k) - u(x_k)$.

Proof. The proof is based on the results stated in Proposition 2.3. The Lyapunov difference of the quantized scheme can be computed from the 'ideal' one as follows:

$$\begin{aligned} V(x_{k+1}, u_k^q) - V(x_{k+1}, u_k^q) &= \Delta V_q(x_k, u_k) + V(x_{k+1}) - V(x_k) \\ &\le \Delta V_q(x_k, u_k) - \delta\alpha_3(|x_k|) + O(\delta^{P+2}) \end{aligned} \tag{27}$$

Then it follows the condition (25) from the last inequality and (23). □

In the case of the exact controller ($P = 1$) it is clear that the stabilization property of the sampled data dynamics cannot be guaranteed to hold asymptotically for any $\delta < T_1^*$ and $x_k \in R^{n+m}$. The presence of the quantized terms $O_q(\delta)$ makes difficult any general statement of the type of the stability. Depending on the nonlinearities and for certain values of δ the term $\alpha_3(|x_k|)$ is sufficiently large to dominate the terms in $O_q(\delta)$. But when x_k converge to the origin, the size of $\alpha_3(|x_k|)$ is decreasing and the negativity of (25) can be affected. The previous observation suggests that in the presence of quantizations, the expected stability property will refer to practical stability [3]. This means that the states ultimately enter a ball with a specific radius that could not be estimated. By increasing the precision this ball can be reduced to lower radius. The sign of the terms included in $O_q(\delta)$ can vary for each x_k and for each δ.

The same conclusions can be drawn for the approximated controller where also the contributions of the terms included in $O(\delta^{P+2})$ have to be taken into account. What is interesting is that the quantization terms can either improve the negativity of the Lyapunov difference (20) or either destroy; this fact is depending on the sign of the terms in $O_q(\delta)$ and on the size of the terms in $O(\delta^{P+2})$.

Another interesting fact, is that in the Lyapunov difference (25) the quantization error in δ is present if $L_{g_c}V(x_k) \neq 0$ for all $x_k \neq 0$. This condition is assumed from the beginning, when the strict-feedback form is considered. This condition reveals that the relative degree index in respect to V is equal to 1. We can think for other type of systems, when this index is greater than 1. In this case the Lyapunov difference is less depending on the quantization error (in fact is depending on higher powers of δ).

4. Conclusions

In this article the quantization effects have been discussed on a digital backstepping design that was proposed in a previous article. A survey on the actual realizations on this topic has been done. An analysis was conducted on the evaluation of the Lyapunov difference that can reveal some of the stabilizing properties of the control solutions. The conclusions that can be draw are:

- stabilization properties, in terms of Lyapunov evolutions, are affected by the presence of quantization errors; It is impossible, for a general case, to establish if these effects ameliorate or by contrary destroy the stabilizing properties imposed by the ideal digital solution (without quantizations).
- There can be given cases when the performances of an approximated P controller are improved by the presence of quantization errors. For an exact controller, the asymptotic stabilization in the presence of quantization cannot be guaranteed.
- By increasing the static gain of the controllers, it does not generally imply that the quantizations errors increase also as it is the case for linear controllers;
- By increasing the order of the approximated controller, and with this the complexity and the number of computations, this not imply that the quantization error is increasing accordingly.
- It is clear that the means of simulations tools are of great importance in establishing the proper values of the sampling period and of the suitable parameters of the control laws. In conclusion, the proposed digital design is robust in the presence of quantization that occurs in the state variable measurements.

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