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A SHRINKING PROJECTION APPROACH FOR SPLIT EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS IN HILBERT SPACES

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In this work, we employ the iterative shrinking projection algorithm to find an approximate common solution to an equilibrium problem and a fixed point problem in the setting of Hilbert spaces. In particular, we establish strong convergence of the proposed iterative algorithm towards a common element in the set of solutions of a finite family of split equilibrium problems and the set of common fixed points of a finite family of total asymptotically nonexpansive mappings in such setting. Our results can be viewed as a generalization and improvement of various existing results in the current literature.

Keywords: Split equilibrium problem; fixed point problem; total asymptotically nonexpansive mapping; inverse strongly monotone mapping; shrinking projection algorithm; Hilbert space.

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1. Introduction

Recent developments in nonlinear functional analysis reflect that fixed point theory, in particular metric fixed point theory, has emerged as a powerful tool to solve various practical problems arising in different branches of pure and applied sciences. The branch metric fixed point theory has its roots in the celebrated Banach Contraction Principle (BCP) which asserts that every contraction in a complete metric space has a unique fixed point. It is worth mentioning that the existence of fixed points of certain nonlinear mappings has valuable applications in nonlinear analysis and topology. However, there are certain situations where it is hard to derive the conditions for the existence of fixed point. In such a situation, the approximation of fixed points is much more desirable. Note that the BCP is not only theoretical in nature but also provides an iterative algorithm for the approximation of such unique fixed points. As a consequence, various problems such as Fredholm and Volterra integral equations, ordinary differential equations, partial differential equations and image processing problems are addressed via the equivalent fixed point problems.

The class of nonexpansive mappings, the limit case of a contraction mapping, has a diverse range of applications to solve problems such as variational inequality problem, convex minimization, zeros of a monotone operator, initial value problems of differential equations, game-theoretic model and image recovery. It is therefore, natural to extend such powerful results of the class of nonexpansive mappings to more general class of mappings such as total asymptotically nonexpansive mapping. It is worth mentioning that the iterative algorithms are the only main tool for the approximation of fixed points of various generalizations of nonexpansive mappings (see, for example [12, 16, 19, 20, 23, 28] and the references cited therein). Such iterative algorithms can be compared w.r.t. their efficiency and convergence characteristics (weak or strong). In many situations, the strong convergence of an iterative

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algorithm involving a nonlinear mapping is much more desirable than the weak convergence. We remark that the shrinking projection algorithm [22] converges strongly even in the setting of Hilbert spaces. We, therefore, employ the shrinking projection algorithm for an approximate solution in Hilbert spaces.

Equilibrium problem theory provides a unified approach to address a variety of mathematical problems arising in disciplines such as physics, optimization, variational inequalities, transportation, economics, network and noncooperative games, see, for example [2, 3, 11] and the references cited therein. This theory flourishes significantly, due to an excellent paper of Combettes and Hirstoaga [10], with the use of iterative algorithms to solve an equilibrium problem assuming the set of solutions of the equilibrium problem is nonempty. The classical equilibrium problem theory [3, 10] has been generalized in several interesting ways to solve real world problems. In 2012, Censor et al. [8] proposed a theory regarding split variational inequality problem (SVIP) which aims to solve a pair of variational inequality problems in such a way that the solution of a variational inequality problem, under a given bounded linear operator, solves another variational inequality.

Motivated by the work of Censor et al. [8], Moudafi [17] generalized the concept of SVIP to that of split monotone variational inclusions which includes, as a special case, the split variational inequality problem, the split common fixed point problem, the split zeroes problem, the split equilibrium problem and the split feasibility problem. These problems have already been studied and successfully employed as a model in intensity-modulated radiation therapy treatment planning, see [6, 7]. This formalism is also at the core of modeling of many inverse problems arising for phase retrieval and other real-world problems; for instance, in sensor networks in computerized tomography and data compression; see, for example, [5, 9]. Some methods have been proposed and analyzed to solve the above mentioned problems in Hilbert spaces, see, for example [13, 15, 21, 24, 25, 26, 27] and the references cited therein.

Inspired and motivated by the above mentioned results and the ongoing research in the direction of split equilibrium problem, we aim to employ a hybrid shrinking projection algorithm to find a common element in the set of solutions of a finite family of split equilibrium problems and the set of common fixed points of a finite family of total asymptotically nonexpansive mappings in Hilbert spaces. Our results can be viewed as a generalization and improvement of various existing results in the current literature.

The rest of the paper is organized as follows: Section 2 is devoted to the development of necessary concepts, mathematical tools and lemmas required in the sequel. In Section 3, we propose a hybrid shrinking projection algorithm and establish strong convergence results under certain assumptions.

2. **Preliminaries**

Throughout this paper, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{N}$ denotes the set of natural numbers, respectively. We write $x_n \to x$ (resp. $x_n \rightharpoonup x$) to indicate the strong convergence (resp. the weak convergence) of a sequence $\{x_n\}_{n=1}^{\infty}$. Let C be a nonempty subset of a real Hilbert space H and let $T : C \to C$ be a mapping. The set of fixed points of the mapping T is defined and denoted as: $F(T) = \{x \in C : T(x) = x\}$. A self-mapping T is said to be total asymptotically nonexpansive [1] if there exist nonnegative real sequences $\{\lambda_n\}_{n=1}^{\infty}$, $\{\mu_n\}_{n=1}^{\infty}$ with $\lambda_n, \mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\xi(0) = 0$ then for all $x, y \in C$, we have

$$\|T^n x - T^n y\| \leq \|x - y\| + \lambda_n \xi(\|x - y\|) + \mu_n. \tag{2.1}$$

The class of total asymptotically nonexpansive mappings is the most general class of nonlinear mappings and contains properly various classes of nonlinear mappings such as generalized asymptotically nonexpansive mappings, asymptotically nonexpansive mappings, asymptotically nonexpansive type mappings, asymptotically nonexpansive in the intermediate sense and nonexpansive mappings.

Remark 2.1. We now elaborate how total asymptotically nonexpansive mapping defined in (2.1) unifies various definitions of classes of mappings associated with the class of asymptotically nonexpansive mappings. Observe that:

(i) if $\xi(x) = x$, (2.1) becomes generalized asymptotically nonexpansive mapping [18];

(ii) if $\xi(x) = x$ and $\mu_n = 0$ for all $n \geq 1$, then (2.1) becomes asymptotically nonexpansive mapping [14];

(iii) if $\xi(x) = 0$, then (2.1) becomes asymptotically nonexpansive type mapping provided C is bounded and T^n is continuous for some integer $n \geq 1$;

(iv) if $\xi(x) = x$, $\lambda_n = 0$ and $\mu_n = \max\{0, a_n\}$ where

$$a_n = \max\left\{0, \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|)\right\},$$

then (2.1) becomes asymptotically nonexpansive mapping in the intermediate sense [4] (i.e., T satisfies the inequality $\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\| \leq 0)$);

(v) if $\lambda_n = \mu_n = 0$, then (2.1) becomes nonexpansive mapping.

In [17], the following split equilibrium problem (SEP) is introduced: Let C be a nonempty subset of a real Hilbert space H_1, Q be a nonempty subset of a real Hilbert space H_2 and let $A\colon H_1 \to H_2$ be a bounded linear operator. Let $F\colon C \times C \to \mathbb{R}$ and $G\colon Q \times Q \to \mathbb{R}$ be two bifunctions. A SEP is to find:

$$x^* \in C \quad \text{such that } F(x^*, x) \geq 0 \text{ for all } x \in C, \tag{2.2}$$

and

$$y^* = Ax^* \in Q \quad \text{such that } G(y^*, y) \geq 0 \text{ for all } y \in Q. \tag{2.3}$$

It is remarked that inequality (2.2) represents the classical equilibrium problem and its solution set is denoted as $EP(F)$. Moreover, inequalities (2.2) and (2.3) constitute a pair of equilibrium problems which aim to find a solution x^* of an equilibrium problem (2.2) such that its image $y^* = Ax^*$ under a given bounded linear operator A also solves another equilibrium problem (2.3). The set of solutions of SEP (2.2) and (2.3) is denoted $\Omega = \{z \in EP(F) : Az \in EP(G)\}$.

Let C be a nonempty closed convex subset of a Hilbert space H_1. For each $x \in H_1$, there exists a unique nearest point of C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \text{ for all } y \in C.$$

Such a mapping $P_C : H_1 \to C$ is known as a metric projection or a nearest point projection of H_1 onto C. Moreover, P_C satisfies nonexpansiveness in a Hilbert space and $\langle x - P_C x, P_C x - y\rangle \geq 0$ for all $x, y \in C$. It is remarked that P_C is firmly nonexpansive mapping from H_1 onto C, that is,

$$\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y\rangle, \text{ for all } x, y \in C.$$

Recall that a nonlinear mapping $A\colon C \to H_1$ is λ-inverse strongly monotone if it satisfies

$$\langle x - y , Ax - Ay\rangle \geq \lambda \|Ax - Ay\|^2 .$$

Note that, if $A := I - T$ is a λ-inverse strongly monotone mapping, then:

(i): A is a $\left(\frac{1}{\lambda}\right)$-Lipschitz continuous mapping;

(ii): if T is a nonexpansive mapping, then A is a $\left(\frac{1}{2}\right)$-inverse strongly monotone mapping;

(iii): if $\eta \in (0, 2\lambda]$, then $I - \eta A$ is a nonexpansive mapping.

The following lemma collects some well-known equations in the context of a real Hilbert space.

Lemma 2.2. Let H_1 be a real Hilbert space, then:

(i): $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle$, for all $x, y \in H_1$;

(ii): $\|x+y\|^2 \le \|x\|^2 + 2\langle x-y, y\rangle$, for all $x, y \in H_1$;

(iii): $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$ for all $x, y \in H_1$ and $\alpha \in [0,1]$.

It is well-known that H_1 satisfies Opial's condition, that is, for any sequence $\{x_n\}$ in H_1 with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| \le \liminf_{n\to\infty} \|x_n - x\|.$$

Recall that a mapping $T : H_1 \to H_1$ is said to be demiclosed at origin if for any sequence $\{x_n\}$ in H_1 with $x_n \rightharpoonup x$ and $\|x_n - Tx_n\| \to 0$, we have $x = Tx$.

In order to solve an equilibrium problem, the bifunction F must satisfy certain conditions as summarized in the following lemma (c.f. [3] and [10]):

Lemma 2.3. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying the following conditions:

1. $F(x,x) = 0$ for all $x \in C$;
2. F is monotone, that is, $F(x,y) + F(y,x) \le 0 \; for \; all \; x, y \in C$;
3. F is upper hemicontinuous, that is, for each $x, y, z \in C$,

$$\lim_{t\to 0} F(tz + (1-t)x, y) \le F(x,y);$$

4. for each $x \in C$, the function $y \mapsto F(x,y)$ is convex and lower semi-continuous.

Lemma 2.4 [10]. Let C be a closed convex subset of a real Hilbert space H_1 and let $F\colon C \times C \to \mathbb{R}$ be a bifunction satisfying Lemma 2.3. For $r > 0$ and $x \in H_1$, there exists $z \in C$ such that

$$F(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \ge 0, \text{ for all } y \in C.$$

Moreover, define a mapping $T_r^F\colon H_1 \to C$ by

$$T_r^F(x) = \left\{ z \in C : F(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \ge 0, \quad \text{for all } y \in C \right\},$$

for all $x \in H_1$. Then, the following hold:

(i) T_r^F is single-valued;

(ii) T_r^F is firmly nonexpansive, i.e., for every $x, y \in H$,

$$\left\|T_r^F x - T_r^F y\right\|^2 \le \left\langle T_r^F x - T_r^F y, x-y \right\rangle$$

(ii) $F(T_r^F) = EP(F)$;

(iv) $EP(F)$ is closed and convex.

It is remarked that if $G\colon Q \times Q \to \mathbb{R}$ is a bifunction satisfying Lemma 2.3, then for $s > 0$ and $w \in H_2$ we can define a mapping:

$$T_s^G(w) = \left\{ d \in C : G(d,e) + \frac{1}{s}\langle e-d, d-w\rangle \ge 0, \quad \text{for all } e \in Q \right\},$$

which is, nonempty, single-valued and firmly nonexpansive. Moreover, $EP(G)$ is closed and convex, and $F(T_s^G) = EP(G)$.

3. **Main results**

We first set some of the notions required in the sequel for our main result of this section. For a nonempty subset C of a real Hilbert space H_1, we assume that:

(i) $S_i(\text{mod}N) : C \to C$ is a finite family of total asymptotically nonexpansive mappings where $i \in \{1, 2, 3, \dots, N\}$;

(ii) $F_i(\text{mod}N)\colon C \times C \to \mathbb{R}$ and $G_i(\text{mod}N)\colon Q \times Q \to \mathbb{R}$ are two finite families of bifunctions satisfying Lemma 2.3;

(iii) $A_i(\text{mod}N) : H_1 \to H_2$ is a finite family of bounded linear operators.

We are now in a position to prove our main result of this section.

Theorem 3.1. Let H_1 and H_2 be two real Hilbert spaces and let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $F_i : C \times C \to \mathbb{R}$ and $G_i : Q \times Q \to \mathbb{R}$ be two finite families of bifunctions satisfying Lemma 2.3 such that G_i be upper semicontinuous for each $i \in \{1, 2, 3, \ldots, N\}$. Let $S_i\colon C \to C$ be a finite family of uniformly continuous total asymptotically nonexpansive mappings and let $A_i : H_1 \to H_2$ be a finite family of bounded linear operators. Suppose that $\mathbb{F} := \left[\bigcap_{i=1}^N F(S_i)\right] \cap \Theta \neq \emptyset$, where $\Theta = \left\{z \in C : z \in \bigcap_{i=1}^N EP(F_i) \text{ and } A_i z \in EP(G_i) \text{ for } 1 \le i \le N\right\}$. Let $\{x_n\}$ be a sequence generated by:

$$\begin{aligned} & x_1 \in C_1 = C, \\ & u_{n,i} = T^{F_i}_{r_{n,i}} \left(I - \gamma A_i^* \left(I - T^{G_i}_{s_{n,i}}\right) A_i\right) x_n, \\ & y_{n,i} = \alpha_{n,i} x_n + (1 - \alpha_{n,i}) S_i^n u_{n,i}, \\ & C_{n+1} = \left\{z \in C_n : \|y_{n,i} - z\|^2 \le \|x_n - z\|^2 + \theta_{n,i}\right\}, \\ & x_{n+1} = P_{C_{n+1}} x_1, \quad n \ge 1, \end{aligned} \tag{3.1}$$

where $\theta_{n,i} = (1 - \alpha_{n,i}) \{\lambda_n \xi_n(M_n) + \lambda_n M_n^* D_n + \mu_n\}$ with $D_n = \sup\{\|x_n - p\| : p \in \mathbb{F}\}$. Let $\{r_{n,i}\}, \{s_{n,i}\}$ be two positive real sequences and let $\{\alpha_{n,i}\}$ be in $(0, 1)$. Let $\gamma \in \left(0, \frac{1}{L}\right)$, where $L = \max\{L_1, L_2, \cdots, L_N\}$ and L_i is the spectral radius of the operator $A_i^* A_i$ and A_i^* is the adjoint of A_i for each $i \in \{1, 2, 3, \cdots, N\}$. Assume that if following set of conditions holds:

(C1): $0 < a \le \alpha_{n,i} \le b < 1$;

(C2): $\liminf\limits_{n\to\infty} r_{n,i} > 0$ and $\liminf\limits_{n\to\infty} s_{n,i} > 0$;

(C3): $\sum\limits_{n=1}^{\infty} \lambda_n < \infty$ and $\sum\limits_{n=1}^{\infty} \mu_n < \infty$;

(C4): there exist constants $M_i,\ M_i^* > 0$ such that $\xi_i(\phi_i) \le M_i^* \phi_i$ for all $\phi_i \ge M_i$, $i = 1, 2, 3, \ldots, N$, then the sequence $\{x_n\}$ generated by (3.1) converges strongly to $x = P_{\mathbb{F}} x_1$.

Proof. We first show that each $A_i^* \left(I - T^{G_i}_{s_{n,i}}\right) A_i$ is a $\frac{1}{L}$-inverse strongly monotone mapping. For this, we utilize the firm nonexpansiveness of $T^{G_i}_{s_{n,i}}$ which implies that $\left(I - T^{G_i}_{s_{n,i}}\right)$ is a 1-inverse strongly monotone mapping. Now, observe that

$$\begin{aligned} & \left\|A_i^* \left(I - T^{G_i}_{s_{n,i}}\right) A_i x - A_i^* \left(I - T^{G_i}_{s_{n,i}}\right) A_i y\right\|^2 \\ = & \left\langle A_i^* \left(I - T^{G_i}_{s_{n,i}}\right) (A_i x - A_i y), A_i^* \left(I - T^{G_i}_{s_{n,i}}\right) (A_i x - A_i y)\right\rangle \\ = & \left\langle \left(I - T^{G_i}_{s_{n,i}}\right) (A_i x - A_i y), A_i^* A_i \left(I - T^{G_i}_{s_{n,i}}\right) (A_i x - A_i y)\right\rangle \\ \le & L \left\langle \left(I - T^{G_i}_{s_{n,i}}\right) (A_i x - A_i y), \left(I - T^{G_i}_{s_{n,i}}\right) (A_i x - A_i y)\right\rangle \\ = & L \left\|\left(I - T^{G_i}_{s_{n,i}}\right) (A_i x - A_i y)\right\|^2 \\ \le & L \left\langle x - y, A_i^* \left(I - T^{G_i}_{s_{n,i}}\right) (A_i x - A_i y)\right\rangle, \text{ for all } x, y \in H_1. \end{aligned}$$

Hence, it follows from the above estimate that $A_i^* \left(I - T^{G_i}_{s_{n,i}}\right) A_i$ is a $\frac{1}{L}$-inverse strongly monotone. Moreover, $I - \gamma A_i^* \left(I - T^{G_i}_{s_{n,i}}\right) A_i$ is nonexpansive provided $\gamma \in \left(0, \frac{1}{L}\right)$.

Next, we show by mathematical induction that $\mathbb{F} \subset C_{n+1}$ for all $n \geq 1$. Obviously, $\mathbb{F} \subset C_1$ as if $p \in \mathbb{F}$ implies that $T^{F_i}_{r_{n,i}} p = p$ and $(I - \gamma A^*(I - T^{G_i}_{s_{n,i}})A)p = p$, then $p \in C = C_1$. Now, assume that $\mathbb{F} \subset C_k$ for some $k \geq 1$. Observe that

$$\begin{aligned}
\|u_{k,i} - p\| &= \left\| T^{F_i}_{r_{k,i}} \left(I - \gamma A_i^* \left(I - T^{G_i}_{s_{k,i}} \right) A_i \right) x_k - T^{F_i}_{r_{k,i}} \left(I - \gamma A_i^* \left(I - T^{G_i}_{s_{k,i}} \right) A_i \right) p \right\| \\
&\leq \left\| \left(I - \gamma A_i^* \left(I - T^{G_i}_{s_{k,i}} \right) A_i \right) x_k - \left(I - \gamma A_i^* \left(I - T^{G_i}_{s_{k,i}} \right) A_i \right) p \right\| \\
&\leq \|x_k - p\| . \qquad (3.2)
\end{aligned}$$

Utilizing (3.2), we have

$$\begin{aligned}
\|y_{k,i} - p\|^2 &= \|\alpha_{k,i} x_k + (1 - \alpha_{k,i}) S_k^n u_{k,i} - p\|^2 \\
&\leq \alpha_{k,i} \|x_k - p\|^2 + (1 - \alpha_{k,i}) \|S_k^n u_{k,i} - S_k^n p\|^2 - \alpha_{k,i} (1 - \alpha_{k,i}) \|x_k - S_k^n u_{k,i}\|^2 \\
&\leq \alpha_{k,i} \|x_k - p\|^2 + (1 - \alpha_{k,i}) \left\{ \|u_{k,i} - p\|^2 + \lambda_k \xi_k (\|u_{k,i} - p\|) + \mu_k \right\} . \qquad (3.3)
\end{aligned}$$

Note that $\xi_k (\|u_{k,i} - p\|) \leq \xi_k (M_k)$ for $\|u_{k,i} - p\| \leq M_k$. Moreover, $\xi_k (\|u_{k,i} - p\|) \leq \|u_{k,i} - p\| M_k^*$ for $\|u_{k,i} - p\| \geq M_k$ (by C4). In either case, we have

$$\begin{aligned}
\xi_k (\|u_{k,i} - p\|) &\leq \xi_k (M_k) + \|u_{k,i} - p\| M_k^* \\
&\leq \xi_k (M_k) + \|x_k - p\| M_k^*,
\end{aligned}$$

where $M_k,\ M_k^* > 0$.
Utilizing the above estimate and simplifying (3.3), we have

$$\|y_{k,i} - p\|^2 \leq \|x_k - p\|^2 + \theta_{k,i}, \qquad (3.4)$$

where $\theta_{k,i} = (1 - \alpha_{k,i}) \{\lambda_k \xi_k(M_k) + \lambda_k M_k^* D_k + \mu_k\}$ and $D_k = \sup \{\|x_k - p\| : p \in \mathbb{F}\}$. Hence $p \in C_{k+1}$ implies $\mathbb{F} \subset C_{k+1}$ and consequently $\mathbb{F} \subset C_{n+1}$ for all $n \geq 1$.

Next, we show that C_{n+1} is closed and convex for all $n \geq 1$. For $n = 1$, it is obvious that $C_1 = C$ is closed and convex. Assume that C_k is closed and convex for some $k \geq 1$. Let $z_m \in C_{k+1} \subset C_k$ with $z_m \to z$. Since, C_k is closed, it follows that $z \in C_k$ and $\|y_{k,i} - z_m\|^2 \leq \|x_k - z_m\|^2 + \theta_{k,i}$. Then, observe that

$$\begin{aligned}
\|y_{k,i} - z\|^2 &= \|y_{k,i} - z_m + z_m - z\|^2 \\
&= \|y_{k,i} - z_m\|^2 + \|z_m - z\|^2 + 2 \langle y_{k,i} - z_m, z_m - z \rangle \\
&\leq \|x_k - z_m\|^2 + \theta_{k,i} + \|z_m - z\|^2 + 2 \|y_{k,i} - z_m\| \|z_m - z\| .
\end{aligned}$$

Letting $m \to \infty$, we have

$$\|y_{k,i} - z\|^2 \leq \|x_k - z\|^2 + \theta_{k,i}.$$

This implies that $z \in C_{k+1}$. Let $z = \alpha x + (1 - \alpha) y$ for some $x, y \in C_{k+1} \subset C_k$ and $\alpha \in (0, 1)$. Since $z \in C_k$ and C_k is convex, we have $\|y_{k,i} - x\|^2 \leq \|x_k - x\|^2 + \theta_{k,i}$ and

$\|y_{k,i} - y\|^2 \leq \|x_k - y\|^2 + \theta_{k,i}$. The following estimate:

$$\begin{aligned}
\|y_{k,i} - z\|^2 &= \|y_{k,i} - (\alpha x + (1-\alpha) y)\|^2 \\
&= \|\alpha (y_{k,i} - x) + (1-\alpha)(y_{k,i} - x)\|^2 \\
&= \alpha \|y_{k,i} - x\|^2 + (1-\alpha) \|(y_k - y)\|^2 - \alpha (1-\alpha) \|y_{k,i} - x - (y_{k,i} - y)\|^2 \\
&\leq \alpha \left(\|x_k - x\|^2 + \theta_{k,i} \right) + (1-\alpha) \left(\|x_k - y\|^2 + \theta_{k,i} \right) - \alpha (1-\alpha) \|y - x\|^2 \\
&\leq \alpha \|x_k - x\|^2 + (1-\alpha) \|x_k - y\|^2 + \theta_{k,i} - \alpha(1-\alpha) \|(x_k - x) - (x_k - y)\|^2 \\
&= \|\alpha (x_k - x) + (1-\alpha)(x_k - y)\|^2 + \theta_{k,i} \\
&= \|x_k - z\|^2 + \theta_{k,i},
\end{aligned}$$

implies that C_{k+1} is closed and convex. Consequently, C_n is closed and convex for all $n \geq 1$. Hence the sequence $\{x_n\}$ is well defined.

Note that $x_{n+1} = P_{C_{n+1}} x_1$, therefore $\|x_{n+1} - x_1\| \leq \|x^* - x_1\|$ for all $x^* \in C_{n+1}$. In particular, we have $\|x_{n+1} - x_1\| \leq \|P_{\mathbb{F}} x_1 - x_1\|$. This implies that $\{x_n\}$ is bounded, so are $\{u_{n,i}\}$ and $\{y_{n,i}\}$. On the other hand, $x_n = P_{C_n} x_1$ and $x_{n+1} = P_{C_{n+1}} x_1 \in C_n$, we have

$$\begin{aligned}
0 &\leq \langle x_1 - x_n, x_n - x_{n+1} \rangle \\
&= \langle x_1 - x_n, x_n - x_1 + x_1 - x_{n+1} \rangle \\
&\leq -\|x_1 - x_{n1}\|^2 + \|x_{n+1} - x_1\| \|x_n - x_1\|.
\end{aligned}$$

The above estimate implies that $\|x_n - x_1\| \leq \|x_{n+1} - x_1\|$. That is, the sequence $\{\|x_n - x_1\|\}$ is nondecreasing. This implies that

$$\lim_{n\to\infty} \|x_n - x_1\| \text{ exists.} \tag{3.5}$$

Further, observe that

$$\begin{aligned}
\|x_{n+1} - x_n\|^2 &= \|x_{n+1} - x_1 + x_1 - x_n\|^2 \\
&= \|x_{n+1} - x_1\|^2 + \|x_n - x_1\|^2 - 2 \langle x_n - x_1, x_{n+1} - x_1 \rangle \\
&= \|x_{n+1} - x_1\|^2 + \|x_n - x_1\|^2 - 2 \langle x_n - x_1, x_{n+1} - x_n + x_n - x_1 \rangle \\
&= \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2 - 2 \langle x_n - x_1, x_{n+1} - x_n \rangle \\
&\leq \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2.
\end{aligned}$$

Taking lim sup on both sides of the above estimate and utilizing (3.5), we have $\limsup_{n\to\infty} \|x_{n+1} - x_n\|^2 = 0$. That is

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.6}$$

Since $x_{n+1} \in C_{n+1}$, we have $\|y_{n,i} - x_{n+1}\| \leq \|x_n - x_{n+1}\| + \theta_{n,i}$. This implies that

$$\lim_{n\to\infty} \|y_{n,i} - x_{n+1}\| = 0 \text{ for } 1 \leq i \leq N. \tag{3.7}$$

Utilizing (3.6), (3.7) and the following triangular inequality:

$$\|y_{n,i} - x_n\| \leq \|y_{n,i} - x_{n+1}\| + \|x_{n+1} - x_n\|,$$

we get

$$\lim_{n\to\infty} \|y_{n,i} - x_n\| = 0 \text{ for } 1 \leq i \leq N. \tag{3.8}$$

Now, consider the following estimate

$$
\begin{aligned}
\|u_{n,i}-p\|^2 &= \left\|T^{F_i}_{r_{n,i}}\left(I-\gamma A_i^*\left(I-T^{G_i}_{s_{n,i}}\right)A_i\right)x_n - T^{F_i}_{n,i}p\right\|^2 \\
&\leq \left\|x_n-\gamma A_i^*\left(I-T^{G_i}_{s_{n,i}}\right)A_ix_n-p\right\|^2 \\
&\leq \|x_n-p\|^2+\gamma^2\left\|A_i^*\left(I-T^{G_i}_{s_{n,i}}\right)A_ix_n\right\|^2+2\gamma\left\langle p-x_n, A_i^*\left(I-T^{G_i}_{s_{n,i}}\right)A_ix_n\right\rangle \\
&\leq \|x_n-p\|^2+\gamma^2\left\langle A_ix_n-T^{G_i}_{s_{n,i}}A_ix_n, A_iA_i^*\left(I-T^{G_i}_{s_{n,i}}\right)A_ix_n\right\rangle \\
&\quad +2\gamma\left\langle p-x_n, A_i^*\left(I-T^{G_i}_{s_{n,i}}\right)A_ix_n\right\rangle \\
&\leq \|x_n-p\|^2+L\gamma^2\left\langle A_ix_n-T^{G_i}_{s_{n,i}}A_ix_n, A_ix_n-T^{G_i}_{s_{n,i}}A_ix_n\right\rangle \\
&\quad +2\gamma\left\langle p-x_n, A_i^*\left(I-T^{G_i}_{s_{n,i}}\right)A_ix_n\right\rangle \qquad (3.9) \\
&= \|x_n-p\|^2+L\gamma^2\left\|A_ix_n-T^{G_i}_{s_{n,i}}A_ix_n\right\|^2+2\gamma\left\langle p-x_n, A_i^*\left(I-T^{G_i}_{s_{n,i}}\right)A_ix_n\right\rangle.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&2\gamma\left\langle p-x_n, A_i^*\left(I-T^{G_i}_{s_{n,i}}\right)A_ix_n\right\rangle \\
=\ &2\gamma\left\langle A_i(p-x_n), A_ix_n-T^{G_i}_{s_{n,i}}A_ix_n\right\rangle \\
=\ &2\gamma\left\langle A_i(p-x_n)+\left(A_ix_n-T^{G_i}_{s_{n,i}}A_ix_n\right)-\left(A_ix_n-T^{G_i}_{s_{n,i}}A_ix_n\right), A_ix_n-T^{G_i}_{s_{n,i}}A_ix_n\right\rangle \\
=\ &2\gamma\left\{\left\langle A_ip-T^{G_i}_{s_{n,i}}A_ix_n, A_ix_n-T^{G_i}_{s_{n,i}}A_ix_n\right\rangle-\left\|A_ix_n-T^{G_i}_{s_{n,i}}A_ix_n\right\|^2\right\} \\
\leq\ &2\gamma\left\{\frac{1}{2}\left\|A_ix_n-T^{G_i}_{s_{n,i}}A_ix_n\right\|^2-\left\|A_ix_n-T^{G_i}_{s_{n,i}}A_ix_n\right\|^2\right\} \\
=\ &-\gamma\left\|A_ix_n-T^{G_i}_{s_{n,i}}A_ix_n\right\|^2.
\end{aligned}
$$

Substituting the above estimate in (3.9), we get

$$
\|u_{n,i}-p\|^2 \leq \|x_n-p\|^2+\gamma(L\gamma-1)\left\|A_ix_n-T^{G_i}_{s_{n,i}}A_ix_n\right\|^2. \qquad (3.10)
$$

Note that, the estimates (3.3) and (3.10) imply that

$$
\begin{aligned}
\|y_{n,i}-p\|^2 &\leq \alpha_{n,i}\|x_n-p\|^2+(1-\alpha_{n,i})\left\{\|u_{n,i}-p\|^2+\lambda_n\xi_n(\|u_{n,i}-p\|)+\mu_n\right\} \\
&\quad -\alpha_{n,i}(1-\alpha_{n,i})\|x_n-S_i^nu_{n,i}\|^2 \\
&\leq \alpha_{n,i}\|x_n-p\|^2+(1-\alpha_{n,i})\|u_{n,i}-p\|^2+(1-\alpha_{n,i})\{\lambda_n\xi_n(\|u_{n,i}-p\|)+\mu_n\} \\
&\leq \|x_n-p\|^2+(1-\alpha_{n,i})\left\{\gamma(\gamma L-1)\left\|A_ix_n-T^{G_i}_{s_{n,i}}A_ix_n\right\|^2\right\}+\theta_{n,i}.
\end{aligned}
$$

Re-arranging the terms of the above estimate, we have

$$
\begin{aligned}
\gamma(1-\gamma L)\left\|A_ix_n-T^{G_i}_{s_{n,i}}A_ix_n\right\|^2 &\leq \|x_n-p\|^2-\|y_{n,i}-p\|^2+\theta_{n,i} \\
&\leq (\|x_n-p\|+\|y_{n,i}-p\|)\|x_n-y_{n,i}\|+\theta_{n,i}.
\end{aligned}
$$

Since $\gamma(1-\gamma L)>0$, therefore letting $n\to\infty$ and utilizing (3.8), we have

$$
\lim_{n\to\infty}\left\|A_ix_n-T^{G_i}_{s_{n,i}}A_ix_n\right\|^2=0 \text{ for } 1\leq i\leq N. \qquad (3.11)
$$

Since $T^{F_i}_{r_{n,i}}$ is firmly nonexpansive, then

$$\begin{aligned}
\|u_{n,i}-p\|^2 &= \left\|T^{F_i}_{r_{n,i}}\left(I-\gamma A_i^*\left(I-T^{G_i}_{s_{n,i}}\right)A_i\right)x_n - T^{F_i}_{r_{n,i}}p\right\| \\
&\leq \left\langle u_{n,i}-p, x_n-\gamma A_i^*\left(I-T^{G_i}_{s_{n,i}}\right)A_ix_n-p\right\rangle \\
&= \frac{1}{2}\{\|u_{n,i}-p\|^2+\left\|x_n-\gamma A_i^*\left(I-T^{G_i}_{s_{n,i}}\right)A_ix_n-p\right\|^2 \\
&\quad -\left\|u_{n,i}-x_n-\gamma A_i^*\left(I-T^{G_i}_{s_{n,i}}\right)A_ix_n\right\|^2\} \\
&\leq \frac{1}{2}\left\{\|u_{n,i}-p\|^2+\|x_n-p\|^2-\left\|u_{n,i}-x_n-\gamma A_i^*\left(I-T^{G_i}_{s_{n,i}}\right)A_ix_n\right\|^2\right\} \\
&= \frac{1}{2}\{\|u_{n,i}-p\|^2+\|x_n-p\|^2-(\|u_{n,i}-x_n\|^2+\gamma^2\left\|A_i^*\left(I-T^{G_i}_{s_{n,i}}\right)A_ix_n\right\|^2 \\
&\quad -2\gamma\left\langle u_{n,i}-x_n, A_i^*\left(I-T^{G_i}_{s_{n,i}}\right)A_ix_n\right\rangle)\}.
\end{aligned}$$

This implies that

$$\begin{aligned}
\|u_{n,i}-p\|^2 &\leq \|x_n-p\|^2-\|u_{n,i}-x_n\|^2+2\gamma\left\langle u_{n,i}-x_n, A_i^*\left(I-T^{G_i}_{s_{n,i}}\right)A_ix_n\right\rangle \\
&\leq \|x_n-p\|^2-\|u_{n,i}-x_n\|^2 \\
&\quad +2\gamma\|u_{n,i}-x_n\|\left\|A_i^*\left(I-T^{G_i}_{s_{n,i}}\right)A_ix_n\right\| \qquad (3.12)
\end{aligned}$$

Note that

$$\|y_{n,i}-p\|^2 \leq \alpha_{n,i}\|x_n-p\|^2+(1-\alpha_{n,i})\|u_{n,i}-p\|^2+\theta_{n,i}. \qquad (3.13)$$

Substituting (3.12) in (3.13) and re-arranging the terms, we get

$$\begin{aligned}
(1-\alpha_{n,i})\|u_{n,i}-x_n\|^2 &\leq (\|x_n-p\|+\|y_{n,i}-p\|)\|x_n-y_{n,i}\| \\
&\quad +2\gamma\|u_{n,i}-x_n\|\left\|A_i^*\left(I-T^{G_i}_{s_{n,i}}\right)A_ix_n\right\|+\theta_{n,i}.
\end{aligned}$$

Now, letting $n\to\infty$ and utilizing (3.8) and (3.11), we have

$$\lim_{n\to\infty}\|u_{n,i}-x_n\|=0 \text{ for } 1\leq i\leq N. \qquad (3.14)$$

Utilizing (3.8) and (3.14) and the following triangular inequality:

$$\|y_{n,i}-u_{n,i}\|\leq\|y_{n,i}-x_n\|+\|x_n-u_{n,i}\|,$$

we get

$$\lim_{n\to\infty}\|y_{n,i}-u_{n,i}\|=0 \text{ for } 1\leq i\leq N. \qquad (3.15)$$

Consider the following variant of the estimate (3.3):

$$\|y_{n,i}-p\|^2\leq\|x_n-p\|^2-\alpha_{n,i}(1-\alpha_{n,i})\|x_n-S_i^nu_{n,i}\|^2+\theta_{n,i}.$$

Re-arranging the terms of the above estimate and using condition (C1), we have

$$a(1-b)\|x_n-S_i^nu_{n,i}\|^2\leq(\|x_n-p\|+\|y_{n,i}-p\|)\|x_n-y_{n,i}\|+\theta_{n,i}.$$

Again, letting $n\to\infty$ and utilizing (3.8), we have

$$\lim_{n\to\infty}\|x_n-S_i^nu_{n,i}\|=0 \text{ for } 1\leq i\leq N. \qquad (3.16)$$

Similarly, utilizing (3.14) and (3.16) and the following triangular inequality:

$$\|u_{n,i}-S_i^nu_{n,i}\|\leq\|u_{n,i}-x_n\|+\|x_n-S_i^nu_{n,i}\|,$$

we get

$$\lim_{n\to\infty}\|u_{n,i}-S_i^nu_{n,i}\|=0 \text{ for } 1\leq i\leq N. \qquad (3.17)$$

Observe that $\|y_{n,i}-x_n\| = (1-\alpha_{n,i})\|S_i^n u_{n,i}-x_n\|$. Then it follows from condition (C1) and (3.8) that

$$\lim_{n\to\infty}\|S_i^n u_{n,i}-x_n\| = 0 \text{ for } 1\le i\le N. \tag{3.18}$$

Note that

$$\begin{aligned}\|S_i^n x_n - x_n\| &\le \|S_i^n x_n - S_i^n u_{n,i}\| + \|S_i^n u_{n,i}-x_n\| \\ &\le \|x_n-u_{n,i}\| + \lambda_n\xi_n(\|x_n-u_{n,i}\|) + \mu_n + \|S_i^n u_{n,i}-x_n\| \\ &\le \|x_n-u_{n,i}\| + \lambda_n(\xi_n(M_n) + \|x_n-u_{n,i}\| M_n^*) + \mu_n + \|S_i^n u_{n,i}-x_n\|.\end{aligned}$$

Using (3.14) and (3.18), the above estimate implies that

$$\lim_{n\to\infty}\|S_i^n x_n-x_n\| = 0 \text{ for } 1\le i\le N. \tag{3.19}$$

Moreover, utilizing the uniform continuity of S_i, the estimate:

$$\|x_n - S_i x_n\| \le \|x_n - S_i^n x_n\| + \|S_i^n x_n - S_i x_n\|$$

implies that

$$\lim_{n\to\infty}\|S_i x_n-x_n\| = 0 \text{ for } 1\le i\le N. \tag{3.20}$$

Now, we show that $\omega(x_n)\subset\mathbb{F}$, where $\omega(x_n)$ is the set of all weak ω-limits of $\{x_n\}$. Since $\{x_n\}$ is bounded, therefore $\omega(x_n)\neq\emptyset$. Let $q\in\omega(x_n)$, then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j}\rightharpoonup q$. It follows from the estimate of (3.14) that $u_{n_j,i}\rightharpoonup q$. Using demiclosed principle for S_i (it is evident that $x_{n_j}\rightharpoonup q$ and $\lim_{n\to\infty}\|S_i x_{n_j}-x_{n_j}\|=0$), we have $q\in\bigcap_{i=1}^N F(S_i)$. Next, we show that $q\in\Theta$, i.e., $q\in\bigcap_{i=1}^N EP(F_i)$ and $A_i q\in EP(G_i)$ for each $1\le i\le N$. We first show that $q\in EP(F_1)$, where $F_1=F_{n_j}$ for some $j\ge 1$. Note that, for a finite family of equilibrium problems, the indexing $F_1=F_{n_j}$ results from the modulo function $j\equiv 1(\text{mod } N)$ whereas the corresponding term of the infinite sequence $\{x_n\}$ would then be $\{x_{n_j}\}$. Similarly, we can have $F_{n_k}=F_2$ for some $k\ge 1$. From $u_{n_j,i} = T_{r_{n_j,i}}^{F_i}\left(I-\gamma A_i^*\left(I-T_{s_{n_j,i}}^{G_i}\right)A_i\right)x_{n_j}$ for all $n\ge 1$, we have

$$F_1(u_{n_j,i},y) + \frac{1}{r_{n_j,i}}\left\langle y-u_{n_j,i}, u_{n_j,i}-x_{n_j}-\gamma A_i^*\left(I-T_{s_{n_j,i}}^{G_i}\right)A_i x_{n_j}\right\rangle \ge 0, \quad \text{for all } y\in C.$$

This implies that

$$F_1(u_{n_j,i},y) + \frac{1}{r_{n_j,i}}\left\langle y-u_{n_j,i}, u_{n_j,i}-x_{n_j}\right\rangle - \frac{1}{r_{n_j,i}}\left\langle y-u_{n_j,i}, \gamma A_i^*\left(I-T_{s_{n_j,i}}^{G_i}\right)A_i x_{n_j}\right\rangle \ge 0$$

From (A2), we have

$$\frac{1}{r_{n_j,i}}\left\langle y-u_{n_j,i}, u_{n_j,i}-x_{n_j}\right\rangle - \frac{1}{r_{n_j,i}}\left\langle y-u_{n_j,i}, \gamma A_i^*\left(I-T_{s_{n_j,i}}^{G_i}\right)A_i x_{n_j}\right\rangle \ge F_1(y,u_{n_j,i}), \tag{3.21}$$

for all $y\in C$. Since $\liminf_{j\to\infty} r_{n_j,i}>0$ (by (C2)), therefore it follows from (3.11) and (3.14) that

$$F_1(y,q)\le 0, \quad \text{for all } y\in C.$$

Let $y_t = ty+(1-t)q$ for some $0<t<1$ and $y\in C$. Since $q\in C$, this implies that $y_t\in C$. Using (A1) and (A4), the following estimate:

$$0 = F_1(y_t,y_t) \le tF_1(y_t,y) + (1-t)F_1(y_t,q) \le tF_1(y_t,y),$$

implies that

$$F_1(y_t,y)\ge 0.$$

Letting $t\to 0$, we have $F_1(q,y)\ge 0$ for all $y\in C$. Thus, $q\in EP(F_1)$. Similarly, we can show that $q\in EP(F_2)$ where $F_2=F_{n_k}$ for some $k\ge 1$. Therefore, $q\in\bigcap_{i=1}^N EP(F_i)$. Next, we show that $A_i q\in EP(G_i)$ for each $1\le i\le N$. Reasoning as above, we first show that

$A_iq \in EP(G_1)$, where $G_1 = G_{n_l}$ for some $l \geq 1$. Since $x_{n_j} \rightharpoonup q$ and A_i is a bounded linear operator, therefore $A_ix_{n_j} \rightharpoonup A_iq$ for $1 \leq i \leq N$. Hence, it follows from (3.11) that

$$T^{G_i}_{s_{n_j},i} A_ix_{n_j} \rightharpoonup A_iq \quad \text{as} \quad j \to \infty. \tag{3.22}$$

Now, from Lemma 2.4 we have

$$G_1\left(T^{G_i}_{s_{n_l},i}A_ix_{n_l}, z\right) + \frac{1}{s_{n_l,i}}\left\langle z - T^{G_i}_{s_{n_l},i}A_ix_{n_l}, T^{G_i}_{s_{n_l},i}A_ix_{n_l} - A_ix_{n_l}\right\rangle \geq 0, \quad \text{for all } z \in Q.$$

Since G_1 is upper semicontinuous in the first argument, therefore taking lim sup on both sides of the above estimate as $l \to \infty$ and utilizing (C2) and (3.22), we get

$$G_1\left(A_iq, z\right) \geq 0, \quad \text{for all } z \in Q.$$

Hence $A_iq \in EP(G_1)$. Similarly, we can show that $A_iq \in EP(G_i)$ for each $1 \leq i \leq N$ and consequently $q \in \mathbb{F}$. Let $x = P_{\mathbb{F}}x_1$ and since $\|x_{n+1} - x_1\| \leq \|x - x_1\|$, therefore we have

$$\begin{aligned}
\|x - x_1\| &\leq \|q - x_1\| \\
&\leq \liminf_{j\to\infty} \left\|x_{n_j} - x_1\right\| \\
&\leq \limsup_{j\to\infty} \left\|x_{n_j} - x_1\right\| \\
&\leq \|x - x_1\|.
\end{aligned}$$

This implies that

$$\lim_{j\to\infty} \left\|x_{n_j} - x_1\right\| = \|q - x_1\|.$$

Hence $x_{n_j} \to q = P_{\mathbb{F}}x_1$. From the arbitrariness of the subsequence $\{x_{n_j}\}$ of $\{x_n\}$, we conclude that $x_n \to x$ as $n \to \infty$. It is easy to see that $y_{n,i} \to x$ and $u_{n,i} \to x$. This completes the proof. □

If T_i - in iteration (3.1) - is a finite family of nonexpansive mappings, then we have the following result:

Corollary 3.2. Let H_1 and H_2 be two real Hilbert spaces and let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $F_i : C \times C \to \mathbb{R}$ and $G_i : Q \times Q \to \mathbb{R}$ be two finite families of bifunctions satisfying Lemma 2.3 such that G_i be upper semicontinuous for each $i \in \{1, 2, 3, \cdots, N\}$. Let $S_i : C \to C$ be a finite family of nonexpansive mappings and let $A_i : H_1 \to H_2$ be a finite family of bounded linear operators. Suppose that $\mathbb{F} := \left[\bigcap_{i=1}^N F(S_i)\right] \cap \Theta \neq \emptyset$, where $\Theta = \left\{z \in C : z \in \bigcap_{i=1}^N EP(F_i) \text{ and } A_iz \in EP(G_i) \text{ for } 1 \leq i \leq N\right\}$. Let $\{x_n\}$ be a sequence generated by:

$$\begin{aligned}
&x_1 \in C_1 = C, \\
&u_{n,i} = T^{F_i}_{r_{n,i}}\left(I - \gamma A_i^*\left(I - T^{G_i}_{s_{n,i}}\right)A_i\right)x_n, \\
&y_{n,i} = \alpha_{n,i}x_n + (1 - \alpha_{n,i})S_iu_{n,i}, \\
&C_{n+1} = \left\{z \in C_n : \|y_{n,i} - z\|^2 \leq \|x_n - z\|^2 + \theta_{n,i}\right\}, \\
&x_{n+1} = P_{C_{n+1}}x_1, \quad n \geq 1,
\end{aligned} \tag{3.23}$$

where $\theta_{n,i} = (1 - \alpha_{n,i})\{\lambda_n\xi_n(M_n) + \lambda_nM_n^*D_n + \mu_n\}$ with $D_n = \sup\{\|x_n - p\| : p \in \mathbb{F}\}$. Let $\{r_{n,i}\}, \{s_{n,i}\}$ be two positive real sequences and let $\{\alpha_{n,i}\}$ be in $(0, 1)$. Let $\gamma \in \left(0, \frac{1}{L}\right)$, where $L = \max\{L_1, L_2, \cdots, L_N\}$ and L_i is the spectral radius of the operator $A_i^*A_i$ and A_i^* is the adjoint of A_i for each $i \in \{1, 2, 3, \cdots, N\}$. Assume that if following set of conditions holds:

(C1): $0 < a \leq \alpha_{n,i} \leq b < 1$;

(C2): $\liminf_{n\to\infty} r_{n,i} > 0$ and $\liminf_{n\to\infty} s_{n,i} > 0$;
then the sequence $\{x_n\}$ generated by (3.23) converges strongly to $x = P_{\mathbb{F}}x_1$.

In order to solve the classical equilibrium problem together with the fixed point problem, we prove the following result:

Theorem 3.3. Let H_1 and H_2 be two real Hilbert spaces and let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $F_i : C \times C \to \mathbb{R}$ and $G_i : Q \times Q \to \mathbb{R}$ be two finite families of bifunctions satisfying Lemma 2.3 such that G_i be upper semicontinuous for each $i \in \{1, 2, 3, \cdots, N\}$. Let $S_i : C \to C$ be a finite family of uniformly continuous total asymptotically nonexpansive mappings and let $A_i : H_1 \to H_2$ be a finite family of bounded linear operators. Suppose that $\mathbb{F} := \left[\bigcap_{i=1}^{N} F(S_i)\right] \cap \Theta \neq \emptyset$, where $\Theta = \left\{z \in C : z \in \bigcap_{i=1}^{N} EP(F_i) \text{ and } A_i z \in EP(G_i) \text{ for } 1 \leq i \leq N\right\}$. Let $\{x_n\}$ be a sequence generated by:

$$\begin{aligned} &x_1 \in C_1 = C,\\ &u_{n,i} = T_{r_{n,i}}^{F_i}\left(I - \gamma A_i^*\left(I - T_{s_{n,i}}^{G_i}\right)A_i\right)x_n,\\ &y_{n,i} = \alpha_{n,i}x_n + (1-\alpha_{n,i})\,S_i^n u_{n,i},\\ &C_{n+1} = \left\{z \in C_n : \|y_{n,i} - z\|^2 \leq \|x_n - z\|^2 + \theta_{n,i}\right\},\\ &x_{n+1} = P_{C_{n+1}}x_1, \quad n \geq 1, \end{aligned} \tag{3.24}$$

where $\theta_{n,i} = (1-\alpha_{n,i})\{\lambda_n\xi_n(M_n) + \lambda_n M_n^* D_n + \mu_n\}$ with $D_n = \sup\{\|x_n - p\| : p \in \mathbb{F}\}$. Let $\{r_{n,i}\}, \{s_{n,i}\}$ be two positive real sequences and let $\{\alpha_{n,i}\}$ be in $(0,1)$. Let $\gamma \in \left(0, \frac{1}{L}\right)$, where $L = \max\{L_1, L_2, \cdots, L_N\}$ and L_i is the spectral radius of the operator $A_i^* A_i$ and A_i^* is the adjoint of A_i for each $i \in \{1, 2, 3, \cdots, N\}$. Assume that the following set of conditions holds:

(C1): $0 < a \leq \alpha_{n,i} \leq b < 1$;
(C2): $\liminf_{n\to\infty} r_{n,i} > 0$ and $\liminf_{n\to\infty} s_{n,i} > 0$;
(C3): $\sum_{n=1}^{\infty} k_{in} < \infty$ and $\sum_{n=1}^{\infty} \varphi_{in} < \infty$;
(C4): there exist constants $M_i,\ M_i^* > 0$ such that $\xi_i(\lambda_i) \leq M_i^*\lambda_i$for all $\lambda_i \geq M_i, i = 1, 2, 3, \ldots, N$, then the sequence $\{x_n\}$ generated by (3.24) converges strongly to $x = P_{\mathbb{F}}x_1$.

Proof. Set $H_1 = H_2$, $C = Q$ and $A = I$(the identity mapping) then the desired result then follows from Theorem 3.1 immediately. □

Remark 3.4. It is worth mentioning that the proof of Theorem 3.1 can also be adopted to solve:

(i): split variational inequality problem (SVIP) introduced by Censor et al.[8] by setting $F_i(x,y) = \langle f_i(x), y - x\rangle$ for all $x, y \in C$ and $G_i(u,v) = \langle g_i(u), v - u\rangle$ for all $u, v \in Q$ where f_i and g_i are φ- and ψ-inverse-strongly monotone mappings, respectively,

(ii): split optimization problem by setting $F_i(x,y) = f_i(x) - f_i(y)$ for all $x, y \in C$ and $G_i(u,v) = g_i(u) - g_i(v)$ for all $u, v \in Q$ where $f_i : C \to \mathbb{R}$ and $g_i : Q \to \mathbb{R}$ are two functions satisfying:

(a): $f_i(tx + (1-t)y) \leq f_i(y)$ for all $x, y \in C$ and $g_i(tu + (1-t)v) \leq f_i(v)$ for all $u, v \in Q$,

(b): $f_i(x)$ is concave and upper semicontinuous for all $x \in C$ and $g_i(u)$ is concave and upper semicontinuous for all $u \in Q$.

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