POLYNOMIAL APPROXIMATION ON UNBOUNDED SETS
AND THE MULTIDIMENSIONAL MOMENT PROBLEM

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We solve a two dimensional moment problem on a space of absolutely integrable functions in a strip. To this end, we approximate nonnegative continuous compactly supported functions by sums of tensor products of positive polynomials on the corresponding intervals. Thus, one characterizes the solutions in terms of “computable” quadratic mappings. Next, we prove an application of an extension theorem for linear operators defined on a subspace that is “distanced” with respect to a bounded convex subset. Finally, one considers an application of the abstract Markov moment problem to a space of analytic functions. A common characteristic of all these results is the Hahn-Banach principle and its generalizations.

Keywords: approximation; extension of linear operators; constraints; unbounded sets; the moment problem

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1. Introduction

Using polynomial decomposition or approximation in existence, uniqueness and construction of the solution of classical moment problems is a well-known and natural technique [1] - [22]. For the study of the uniqueness and the construction of the solutions, one uses $L^2$ - approximation [20], [17]. The approximation in $L^1$ norm is sometimes sufficient for the characterization of the existence of the solution. This is one of the ideas of the present work. In the real multidimensional moment problem, one of the main difficulties is the fact that the positive polynomials are not writable in terms of sums of squares. We solve this difficulty by means of appropriate polynomial $L^1$ approximation. For various results concerning decomposition of polynomials of several variables see [1], [3], [6] - [9]. For approximation results applied to the complex moment problem, similar to our results in the real case, see [11] and the reference there. Uniqueness of the solution is considered in [20]-[22], and in many other works of the References and of outside. Probabilistic approach of the uniqueness problem appears in [22]. The background is partially contained in [1], [23], [24]. For earlier basic related results, see the Introduction of [17] and the references therein.

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The paper is organized as follows. In Section 2, we prove an application of polynomial approximation results to a Markov moment problem in an unbounded strip. Section 3 is devoted to an application of an earlier result concerning extension of linear operators defined on a subspace distanced with respect to a convex set. Section 4 contains an application of the solution of an abstract moment problem to a space of analytic functions. Section 5 concludes the paper.

2. Approximation and the Markov moment problem

The following lemma appeared firstly in [15]. A complete proof has been published in [16].

**Theorem 1.** Let $A \subseteq \mathbb{R}^n$ be an arbitrary closed subset and $\nu$ a positive regular determinate Borel measure on $A$, with finite moments of all orders. Then for any nonnegative continuous vanishing at infinity function $\psi \in (C_0(A))_+$, there exists a sequence $(p_m)_m$ of polynomials on $A$, $p_m \geq \psi$, $p_m \rightarrow \psi$ in $L^1(\nu)$. We have

$$\lim_{A} \int p_m d\nu = \int \psi d\nu,$$

the cone $P_+$ of positive polynomials is dense in $(L^1(\nu))_+$ and $P$ is dense in $L^1(\nu)$.

Recall that a determinate measure is, by definition, uniquely determinate by its moments [22]. Here the novelty is that approximation holds by “dominating” polynomials. Let $\nu = \nu_1 \times \nu_2$ be the product of two measures on the strip $S = \mathbb{R} \times [0,1]$, $\nu_1$ verifying the conditions from Lemma 1 for $n = 1$, $A = \mathbb{R}$, and $\nu_2$ being a positive Borel regular measure on $[0,1]$. Let $Y$ be an order complete Banach lattice with solid norm:

$$|y_1| \leq |y_2| \Rightarrow \|y_1\| \leq \|y_2\|$$

**Theorem 2.** Let

$$X = L^1(\nu(S), \phi_{j,k}(t_1, t_2) = t_1^{i_1} t_2^{i_2}, (j, k) \in \mathbb{N}^2, (t_1, t_2) \in S).$$

Let $(y_{j,k})_{(j,k) \in \mathbb{N}^2}$ be a sequence in $Y$ and $G \in B_+(X, Y)$ a linear positive bounded operator. The following statements are equivalent:

(a) there exists a unique operator $F$ satisfying the conditions

$$F \in B(X, Y), F(\phi_{j,k}) = y_{j,k}, \forall (j, k) \in \mathbb{N}^2,$$

$$0 \leq F(\psi) \leq G(\psi), \forall \psi \in X_+, \|F\| \leq \|G\|.$$
(b) for any finite subsets \((m, n, p) \in \mathbb{N}^3\) and any \(\{\alpha_j\}_{j=0}^m\), we have:

\[
0 \leq \sum_{i, j=0}^{m} \alpha_i \alpha_j \left( \sum_{k=0}^{p} (-1)^k \binom{k}{p} y_{i+j+n+k} \right) \leq \sum_{i, j=0}^{m} \alpha_i \alpha_j \left( \sum_{k=0}^{p} (-1)^k \binom{k}{p} y_{i+j+n+k} \right)
\]

**Proof.** Let \(\psi\) be a continuous nonnegative compactly supported function, and \(K\) its support contained in \(S\). One chooses a rectangle \(R_2\) containing \(K = [0,1] \times\).

One approximates the extension of \(\psi\) to \(R_2\) vanishing outside its support by means of Luzin’s theorem. Next, we approximate this continuous function on the rectangle by the corresponding Bernstein polynomials in two variables. Each term of such Bernstein polynomial is the tensor product of two positive polynomials on \(K_1, j = 1,2, K_2 = [0,1]\).

Extend the polynomial in the first variable \(t_1\) such that it vanishes outside \(pn_1(R_2)\). Then we use Luzin’s theorem, followed by Theorem 1, applied to \(n = 1, A = R\). Hence one obtains a positive approximating polynomial on the whole real axes in the first variable, which is a sum of two squares. On the other hand, the polynomial in the second variable \(t_2\) is a linear combination with positive coefficients of special polynomials \(t_2 = (1-t_2)^p, (n, p) \in \mathbb{N}^2\) [6]. Using these conclusions, one obtains approximation of \(\psi\) in \(L_1(S)\) by sums of tensor products

\[
p_1 \otimes p_2, p_1(t_1) = q^2(t_1) \forall t_1 \in R, p_2(t) = t_2^n(1-t_2)^p.
\]

From the preceding arguments, we infer that the assertion (b) says that

\[
0 \leq F_0(\tilde{p}_1 \otimes \tilde{p}_2) \leq G(\tilde{p}_1 \otimes \tilde{p}_2), \tilde{p}_j(t_j) > 0, j = 1,2, (t_1, t_2) \in S,\]

where \(F_0\) is defined on the subspace of polynomials, such that the moment conditions are accomplished. Application of Theorem from Section 5.1.2 [23] p. 160, leads to the existence of a positive linear extension \(F \in L_+(X_1,Y)\) of \(F_0\), where \(X_1 \subset X\) is the subspace of all functions from \(X\) having their modulus dominated by a polynomial. This subspace contains the subspace of continuous compactly supported functions. Hence \(h \circ F\) has a representing positive measure for all linear positive functional \(h\) on \(Y\). Using these conclusions, one obtains approximation of \(\psi\) in \(L_1(S)\) by sums of tensor products.
Now, using the density of sums of tensor products of positive polynomials in $X_+$, we prove that

$$0 \leq F(\psi) \leq G(\psi), \psi \in C_c(S), \psi \geq 0.$$ 

To this end, we proceed as follows. Applying Fatou’s lemma, one obtains:

$$0 \leq h(F(\psi)) \leq \liminf_m \left(h \circ F \left( \sum_{j=0}^{k(m)} p_{m,1,j} \otimes p_{m,2,j} \right) \right) \leq \lim_m \left(h \circ F_2 \left( \sum_{j=0}^{k(m)} p_{m,1,j} \otimes p_{m,2,j} \right) \right) = h(F_2(\psi)), \psi \in (C_c(S))_+, h \in Y^*.$$ 

Assume that

$$F_2(\psi) - F(\psi) \not\in Y_+$$

Using a separation theorem, it should exist a positive linear continuous functional $h \in Y_+^*$ such that

$$h(F_2(\psi) - F(\psi)) < 0,$$

that is $h(F_2(\psi)) < h(F(\psi))$. This relation contradicts (1). The conclusion is that we must have

$$F(\psi) \leq F_2(\psi), \psi \in (C_c(S))_+.$$ 

Then for arbitrary $g \in C_c(S)$ one writes

$$|F(g)| \leq F_2(g^+) + F_2(g^-) = F_2(\|g\|) \Rightarrow \|F(g)\| \leq \|F_2\| \cdot \|g\|.$$ 

The conclusion is that the operator $F$ is positive and continuous, of norm dominated by $\|F_2\|$, on a dense subspace of $L^1_v(S)$. It has a unique linear extension preserving these properties. This concludes the proof. □
3. Distanced subspace with respect to a convex subset and the moment problem

The following extension result for linear operators has a nice geometric meaning and leads to interesting results in the moment problem.

If $V$ is a convex neighborhood of the origin in a locally convex space, we denote by $p_V$ the gauge attached to $V$. See also [17] and the references there.

**Theorem 3.** Let $X$ be a locally convex space, $Y$ an order complete vector lattice with strong order unit $u_0$ and $S \subset X$ a vector subspace. Let $A \subset X$ be a convex subset with the following qualities:

(a) there exists a (convex) neighborhood $V$ of the origin such that $(S + V) \cap A = \Phi$

(A and $S$ are distanced);

(b) $A$ is bounded.

Then for any equicontinuous family of linear operators $\{f_j\}_{j \in J} \subset L(S, Y)$ and for any $\bar{y} \in Y_+ \setminus \{0\}$, there exists an equicontinuous family $\{F_j\}_{j \in J} \subset L(X, Y)$ such that

$$F_j(s) = f_j(s), s \in S \text{ and } F_j(a) \geq \bar{y}, a \in A, j \in J.$$

Moreover, if $V$ is a neighborhood of the origin such that $f_j(V \cap S) \subset [-u_0, u_0]$ and $A$ is bounded, then the following relations hold

$$F_j(x) \leq (1 + \alpha + \alpha_1) p_V(x) \cdot u_0, \quad x \in X, j \in J.$$

We denote by $X$ the space of all continuous functions in the polydisc $\overline{D} = \prod_{j=1}^{n} \{ z_j \mid |z_j| \leq 1 \}$, which can be written as a power series with real coefficients, centered at $(0, \ldots, 0)$ in the open polydisc $D$. Let

$$\varphi_j(z_1, \ldots, z_n) = z_1^{j_1} \cdots z_n^{j_n}, \quad j = (j_k)_{k=1}^{n}, \quad |j| = \sum_{k=1}^{n} j_k \geq 1.$$

On the other hand, consider a complex Hilbert space $H$, $U_0 \in A(H)$ a selfadjoint operator acting on $H$. Denote

$$Y_1 = \{ U \in A(H); UU_0 = U_0 U \}, \quad Y = \{ U \in Y_1; U V = V U \ \forall V \in Y_1 \},$$

$$Y_+ = \{ U \in Y; \langle U(h), h \rangle \geq 0 \ \forall h \in H \}$$
Here $A(H)$ stands for the real vector space of all selfadjoint operators acting on $H$. Obviously, $Y$ is a commutative algebra of selfadjoint operators. Moreover, $Y$ is an order complete vector lattice [23], [10], and the operatorial norm is solid on $Y$:

$$\|U\| \leq \|V\| \Rightarrow \|U\| \leq \|V\|, U, V \in Y.$$

**Theorem 4.** Let $(B_j)_{j \in \mathbb{N}^n}, \sum_{k=1}^{n} j_k \geq 1$ be a sequence in $Y$, $0 < \varepsilon < 1$, such that

$$\|B_j\| \leq M \cdot \varepsilon^{j_1 + \cdots + j_n}, \quad \forall j = (j_1, \ldots, j_n) \in \mathbb{N}^n, |j| \geq 1.$$

Let $\{\psi_k\}_{k \in \mathbb{N}^n}$ be a sequence in $X$, such that $\psi_k(0,\ldots,0) = 1, \|\psi_k\| \leq 1, \forall k \in \mathbb{N}^n$.

Let $\tilde{B} \in Y_+$. Then there is a linear operator applying $X$ into $Y$ such that:

$$F(\phi_j) = B_j, \quad F(\psi_k) = \tilde{B},$$

$$F(\phi) \leq (2 + \|\tilde{B}\| \left(M^{-1}(1-\varepsilon)^n\right)) \|\phi\| \cdot u_0, \quad u_0 := \left(M(1-\varepsilon)^{-n}\right) \cdot I.$$

**Proof.** Due to the behavior at $(0,\ldots,0)$ of the functions $\phi_j$, $|j| := \sum_{k=1}^{n} j_k \geq 1$ and $\psi_k, k \in \mathbb{N}^n$, we have

$$\|s - a\|_{\infty} \geq |s(0) - a(0)| \geq 1, \quad \forall s \in S := \text{Span} \{\phi_j; |j| \geq 1\},$$

$$\forall a \in A := \text{conv} \{\psi_k; k \in \mathbb{N}^n\} \Rightarrow (S + B(0,1)) \cap A = \Phi.$$

Thus using the hypothesis on the norms of the functions $\psi_k, k \in \mathbb{N}^n$, (where the unit ball $B(0,1)$ stands for $V$, and $\|\|\|$ stands for $p_T$), the above relations hold.

Now let $s = \sum_{j \in J_0} \lambda_j \phi_j \in S \cap B(0,1)$ and define the linear operator $F_0$ on the subspace $S$, such that the moment conditions $F_0(\phi_j) = B_j, |j| \geq 1$ be accomplished. Cauchy’s inequalities yield

$$|\lambda_j| \leq \|s\|_{\infty} \leq 1, \quad j \in J_0 \Rightarrow f(s) = \sum_{j \in J_0} \lambda_j B_j \leq \sum_{j \in J_0} |\lambda_j| \cdot |B_j| \leq$$

$$\left(\sum_{j \in J_0} \|B_j\| \cdot I \leq M \cdot \left(\sum_{j_1 \in \mathbb{N}} \varepsilon^{j_1}\right) \cdots \cdot \left(\sum_{j_n \in \mathbb{N}} \varepsilon^{j_n}\right) \cdot I = M \cdot (1-\varepsilon)^{-n} \cdot I = u_0.$$
On the other hand, we have:

\[ \bar{B} \leq \left\| \bar{B} \right\| \cdot I = \left\| \bar{B} \right\| \cdot \left( M^{-1} (1 - \varepsilon)^n \right) \cdot u_0. \]

Application of theorem 3 leads to the conclusion.

\[ \square \]

4. An application to a space of analytic functions

We recall the following result [19] on the abstract Markov moment problem, as an extension with two constraints theorem for linear operators.

**Theorem 5.** Let \( X \) be an ordered vector space, \( Y \) an order complete vector lattice, \( \{x_j\}_{j \in J} \subset X \), \( \{y_j\}_{j \in J} \subset Y \) given families and \( F_1, F_2 \in L(X, Y) \) two linear operators. The following statements are equivalent:

(a) there is a linear operator \( F \in L(X, Y) \) such that

\[ F_1(x) \leq F(x) \leq F_2(x) \quad \forall x \in X_+, \quad F(x_j) = y_j \quad \forall j \in J; \]

(b) for any finite subset \( J_0 \subset J \) and any \( \{\lambda_j\}_{j \in J_0} \subset R \), we have:

\[ \sum_{j \in J_0} \lambda_j x_j = \psi_2 - \psi_1, \quad \psi_1, \psi_2 \in X_+ \Rightarrow \sum_{j \in J_0} \lambda_j y_j \leq F_2(\psi_2) - F_1(\psi_1). \]

From Theorem 5 we deduce the following result. Let \( Y \) be an commutative real Banach algebra, which is also an order complete Banach lattice, with solid norm. Let

\[ a_k, b_k \in Y_+, \quad \left\| a_k \right\| < 1, \quad \left\| b_k \right\| < 1, \quad k = 1, \ldots, n. \]

Let \( \{y_j\}_{j \in N^n} \) be a sequence in \( Y_+ \). Consider the space \( X \) of all continuous functions in the unit closed polydisc, which can be represented by sums of absolutely convergent power series with real coefficients in the open polydisc. The order relation on \( X \) is given by the coefficients of the power series. Namely,

\[ X_+ = \left\{ \sum_{j \in N^n} c_j z^j ; c_j \geq 0 \quad \forall j \in N^n \right\}. \]

Let
Theorem 6. (see also [18]) With the above notations, the following statements are equivalent:

(a) there exists $F \in B(X, Y)$ such that
$$F(\varphi_j) = y_j, \quad j \in \mathbb{N}^n, \quad 0 \leq F(\psi) \leq \psi(a_1, \ldots, a_n) + \varepsilon \cdot \psi(b_1, \ldots, b_n),$$
$$\psi \in X_+, \quad \|F\| \leq 1 + \varepsilon;$$

(b) we have: $0 \leq y_j \leq a_1^{j_1} \ldots a_n^{j_n} + \varepsilon \cdot b_1^{j_1} \ldots b_n^{j_n}, \quad j = (j_1, \ldots, j_n) \in \mathbb{N}^n$.

Proof. The implication $(a) \Rightarrow (b)$ is obvious, because of the relations
$$\varphi_j \in X_+ \Rightarrow y_j = F(\varphi_j) \in [0, \varphi_j(a_1, \ldots, a_n) + \varepsilon \cdot \varphi_j(b_1, \ldots, b_n)] = \left[0, a_1^{j_1} \ldots a_n^{j_n} + \varepsilon \cdot b_1^{j_1} \ldots b_n^{j_n}\right], \quad j \in \mathbb{N}^n.$$

Conversely, assume that $(b)$ holds. We verify the implication in $(b)$, Theorem 5. Namely, we have:
$$\sum_{j \in J_0} \lambda_j \varphi_j = \psi_2 - \psi_1 = \sum_{m \in \mathbb{N}^n} \alpha_m \varphi_m - \sum_{m \in \mathbb{N}^n} \beta_m \varphi_m, \quad \alpha_m, \beta_m \geq 0, \quad m \in \mathbb{N}^n \Rightarrow$$
$$\sum_{j \in J_0} \lambda_j y_j \leq \sum_{j \in J_0^+} \lambda_j y_j \leq \sum_{j \in \mathbb{N}^n} \alpha_j y_j \leq \sum_{j \in \mathbb{N}^n} \alpha_j \left(a_1^{j_1} \ldots a_n^{j_n} + \varepsilon \cdot b_1^{j_1} \ldots b_n^{j_n}\right) = \psi_2(a_1, \ldots, a_n) + \varepsilon \cdot \psi_2(b_1, \ldots, b_n) = F_2(\psi_2) - F_1(\psi_1),$$
$$F_1 := 0, \quad J_0^+ = \left\{j \in J_0; \lambda_j \geq 0\right\}.$$

A direct application of Theorem 5 leads to the existence of a linear operator $F \in L(X, Y)$, such that
$$0 \leq F(\psi) \leq \psi(a_1, \ldots, a_n) + \varepsilon \cdot \psi(b_1, \ldots, b_n), \quad \forall \psi \in X_+.$$

For an arbitrary $\varphi \in X$, one obtains:
$$|F(\varphi)| \leq F(\varphi^+) + F(\varphi^-) \leq |\varphi(a_1, \ldots, a_n) + \varepsilon \cdot \varphi(b_1, \ldots, b_n)| \Rightarrow$$
$$\|F(\varphi)\| \leq (1 + \varepsilon) \cdot \|\varphi\|_{\infty}, \quad \forall \varphi \in X \Rightarrow \|F\| \leq 1 + \varepsilon.$$

This concludes the proof. □

5. Conclusions

In the first part of this work, we apply polynomial approximation results on unbounded subsets to the real multidimensional Markov moment problem on a strip. One approximates nonnegative compactly supported continuous functions,
having their support contained in the strip, by sums of tensor products of positive polynomials in each separate variable, on the corresponding intervals. For such polynomials, the analytic expression is well known. Thus, one characterizes the existence of the solution in terms of quadratic mappings. Secondly, one proves applications of a theorem of extension of linear operators with two constraints to the Markov moment problem.

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