DYNAMIC OF EVOLUTIVE OPTIMIZATION PROBLEMS

Constantin Udriste[1], Ionel Ţevy[2], Ali Sapeeh Rasheed[3]

This is a paper about continuous time deterministic evolutionary dynamics for optimization problems. The topics include: (i) evolution of objective function; (ii) evolution of constraints; (iii) evolution of minimum value; (iv) evolution of catastrophe manifold; (v) evolution of an optimal control problem. For the context of optimal control problems, we also described: (1) a general theory about evolution of a curve described by an ODE; (2) a general theory about evolution of a surface described by a PDE; (3) evolution of a curve following a first order ODE.

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1. **Evolution of an optimization problem**

In general terms, "optimization problems that change over time" are called dynamic problems/time-dependent problems. The first idea about the evolution of a free optimization problem appeared in [5]. Now this subject can be included in the geometric evolution of ODEs and PDEs [1],[3], [4]. Related topics are found in [2], [6], [7].

In *trade-off analysis of an optimum problem* we vary the constraints, and see the effect on the optimal value of the problem. *Sensitivity analysis of an optimum problem* is closely related to trade-off analysis. In sensitivity analysis, we consider how small changes in the constraints affect the optimal objective value. Both problems reflect the idea that in many practical problems, the constraints are not really set in stone, and can be changed, especially if there is a compelling reason to do so (such as a drastic improvement in the objective obtained). We extend trade-off analysis and sensitivity analysis to evolution of an optimization problem.

Let $x = (x^1, ..., x^n) \in \mathbb{R}^n$ be the vector of decision variables, $f : \mathbb{R}^n \to \mathbb{R}$ be a C^2 objective function and $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, ..., k$, be C^2 functions that describe the constraints. A general optimization problem is of the form:

$$\min_x f(x)$$

subject to

$$g_1(x) \leq 0, ..., g_k(x) \leq 0.$$

[1]Professor, Department of Mathematics and Informatics, Faculty of Applied Sciences, University Politehnica of Bucharest, Splaiul Independentei 313, Bucharest 060042, Romania, E-mail: udriste@mathem.pub.ro

[2]Professor, Department of Mathematics and Informatics, Faculty of Applied Sciences, University Politehnica of Bucharest, Splaiul Independentei 313, Bucharest 060042, Romania, E-mail: vascatevy@yahoo.fr

[3]Ph.D. Student at Department of Mathematics and Informatics, Faculty of Applied Sciences, University Politehnica of Bucharest, Splaiul Independentei 313, Bucharest 060042, Romania; University of Wasit, Faculty of Science, Iraq, E-mail: ali.math2018@yahoo.com

The t-evolution of the graph of f and the t-evolution of the "manifold" described by constraints have a geometrical sense and are very clear. Here we solve the following problem: what is the significance of the t-evolution of an optimization problem? Of course, though the transition from the evolution single time parameter t to the evolution multi-time parameter $t = (t^1, ..., t^m)$ is not so difficult, the multitime case has own specifical problems.

1.1. Evolution of objective function

Without loss of generality, we consider a simplified problem of optimization:

$$\min_x f(x) \text{ subject to } g(x) = 0.$$

Suppose the objective function $f\colon R^n \to R$ in this problem is smooth and regular. Its graph $G(f)\colon y = f(x)$ is a submanifold in R^{n+1}, characterized by the metric g_{ij}, g^{ij},

$$g_{ij} = \delta_{ij} + f_i f_j,\ g^{ij} = \delta^{ij} - \frac{f_i f_j}{1 + |\nabla f|^2},$$

the normal versor n, the second fundamental form h_{ij},

$$n = \frac{((f_i), -1)}{\sqrt{1 + |\nabla f|^2}},\ h_{ij} = \frac{f_{ij}}{\sqrt{1 + |\nabla f|^2}},$$

the mean curvature H, and the Gauss curvature K,

$$H = div\left(\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}}\right),\ K = \det \frac{\nabla^2 f}{(1 + |\nabla f|^2)^{\frac{n+2}{2}}}.$$

The graph submanifold is described by the embedding vector field $X(x) = (x, f(x))$. Therefore, its evolution in time, after the normal vector, is introduced by an extension

$$F = F(x,t), F(x,0) = f(x),\ t \in [0,T]$$

and the associated vector field $Y(x,t) = (x, t, F(x,t))$, which satisfies the evolution PDE

$$\partial_t Y(x,t) = -H(x,t) n(x,t),\ Y(x,0) = X(x).$$

The last PDE is equivalent to a flow PDE. It follows

Proposition 1.1. *The normal evolution of objective function is described by the flow*

$$\dot{F} = \sqrt{1 + |\nabla F|^2}\ H,\ F(x,0) = f(x).$$

1.2. Evolution of constraint manifold

Let $S : g(x) = 0$ be a regular implicit hypersurface representing the constraint in an optimization problem

$$\min_x f(x) \text{ subject to } g(x) = 0.$$

Suppose $x = x(u), u \in R^{n-1}$ is a parametrization of S, i.e., $g(x(u)) = 0, \forall u \in R^{n-1}$. The time evolution $S(t)$ is described by the implicit equation

$$S(t)\colon G(x(u,t),t) = 0,\ u \in R^{n-1}, t \in [0,T], S(0) = S,$$

i.e., $G(x,0) = g(x)$. Using $X = x(u,t)$, and derivation with respect to t, we find that the function G must verify the relation

$$\nabla G \cdot X_t + G_t = 0$$

or, explicitly, the PDE

$$\frac{\partial G}{\partial x^i}\frac{\partial x^i}{\partial t} + \frac{\partial G}{\partial t} = 0.$$

If we impose $\frac{\partial}{\partial t}x = \alpha Y$, then we find the PDE $G_t = -\alpha(\nabla G, Y)$, with unknown G, fixed by the initial condition.

Let $n = \frac{\nabla G}{||\nabla G||}$ be the versor normal to the hypersurface $S(t)$. If we accept the evolution after the normal versor, i.e., $\frac{\partial}{\partial t}x = \beta n$, then we get $G_t = -\beta||\nabla G||$. Some authors introduce $\beta = K$, attending a nonlinear second order PDE in the unknown G.

Summing up, we find

Proposition 1.2. *The normal evolution of constraint function is described by the flow* $G_t = -\beta||\nabla G||,\ G(x,0) = g(x)$.

1.3. Evolution of minimum value

Suppose that x^* and $f^* = f(x^*)$ is solution of the problem

$$\min_x f(x) \text{ subject to } g(x) = 0.$$

Let us analyse what happen with these data during the evolution.

Theorem 1.1. *The minimum value* F^* *satisfies the relation*

$$\frac{dF^*}{dt} - \frac{\partial F^*}{\partial t} = \lambda(t)\ \frac{\partial G^*}{\partial t}.$$

Proof. By t-evolution, the Lagrange function $l(x) = f(x) + \lambda g(x)$ becomes

$$L(x, t, \lambda) = F(x, t) + \lambda G(x, t).$$

The critical point condition $\frac{\partial L}{\partial x} = 0$ gives $x = x(\lambda, t)$. From the constraint condition $G(x, t) = 0$, i.e., $G(x(\lambda, t), t) = 0$, we find $\lambda = \lambda(t)$ and finally $x(t) = x(\lambda(t), t)$. Consequently the minimum value satisfies the relation in Theorem. □

Remark 1.1. *For example, if* $F(x, t) = f(x)$ *and* $G(x, t) = g(x) - t$*, then the previous PDE reduces to the well-known relation*

$$\frac{df^*}{dt} = -\lambda(t).$$

In other words, a classical optimization problem "$f(x)$*=extremum, subject to* $g(x) = c$*" is a particular case of evolution of the optimization problem "*$f(x)$*=extremum, subject to* $g(x) = 0$*".*

If $F(x, t)$ *is combined to* $G(x, t) = g(x)$*, then we get*

$$\frac{df^*}{dt} = \frac{\partial f^*}{\partial t}$$

(see also the properties of "minimum functions").

If G *is a vectorial function, then in the relation of Theorem, the right hand term becomes a scalar product. The optimization problem "*$f(x)$*=extremum, subject to* $g_1(x) = t,\ g_2(x) = t$*" leads to*

$$\frac{df^*}{dt} = -(\lambda_1(t) + \lambda_2(t)).$$

2. Evolution of catastrophe manifold associated to a Lagrange potential

We start with a Lagrange function

$$L : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R},\ L(x, \lambda) = f(x) + < \lambda, g(x) >,$$

where f and g are of class C^2. The point $x \in \mathbb{R}^n$ is called *state*. The point $\lambda \in \mathbb{R}^k$ is called *control*. The partial function $x \to L(x, \lambda)$ is called *potential*.

The set of all critical points $M \subset \mathbb{R}^n \times \mathbb{R}^k$ of the potentials $x \to L(x, \lambda)$, characterized by

$$\frac{\partial f}{\partial x^i} + < \lambda, \frac{\partial g}{\partial x^i} > = 0, \; i = 1, ..., n,$$

is called the *catastrophe manifold*.

Let $\pi : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^k, \; \pi(x, \lambda) = \lambda$ be the natural projection. The restriction of π to M is denoted by χ and is called *catastrophe application*. The subset $S \subset M$ consisting of the singular points of the catastrophe application $\chi : M \to \mathbb{R}^k$, i.e., the points at which the rank of the Jacobian matrix $J(\chi)$ is smaller than m, is called the *singularity set*. The set S is described by the equations

$$\frac{\partial f}{\partial x^i} + < \lambda, \frac{\partial g}{\partial x^i} > = 0, \; \det \left(\frac{\partial^2 L}{\partial x^i \partial x^j} \right) = 0.$$

The image $B = \chi(S) \subset \mathbb{R}^k$ is called *bifurcation set*. The bifurcation set B is the set on which the number and nature of critical points change; the change takes place only when passing through a degenerate critical point.

2.1. Evolution preserving the catastrophe manifold

Now we introduce a variation parameter $\epsilon \in [0, T)$ independent of x. Then it appears the perturbed functions $f(x, \epsilon), \lambda(\epsilon), g(x, \epsilon)$. We impose the initial conditions

$$f(x, 0) = f(x), \lambda(0) = \lambda, g(x, 0) = g(x).$$

Also, we need a new Lagrange function

$$L(x, \lambda(\epsilon), \epsilon) = f(x; \epsilon) + < \lambda(\epsilon), g(x; \epsilon) >$$

and the associated critical point condition

$$\left(\frac{\partial f}{\partial x^i} + < \lambda, \frac{\partial g}{\partial x^i} > \right) (x, \lambda(\epsilon), \epsilon) \; = 0.$$

To continue, we must declare the variational objects

$$\frac{\partial f}{\partial \epsilon}|_{\epsilon=0} = \xi(x), \; \frac{\partial g}{\partial \epsilon}|_{\epsilon=0} = \eta(x), \; \frac{\partial \lambda}{\partial \epsilon}|_{\epsilon=0} = \varsigma.$$

Theorem 2.1. *Let $x(\lambda)$ be a critical point. The vector $\nabla \xi(x(\lambda))$ belongs to the space generated by $\nabla \eta(x(\lambda))$ and $\nabla g(x(\lambda))$, i.e.,*

$$\frac{\partial \xi}{\partial x^i} + < \lambda, \frac{\partial \eta}{\partial x^i} > + < \varsigma, \frac{\partial g}{\partial x^i} > = 0.$$

Proof. **Variant 1** Taking the derivative with respect to ϵ, in critical point condition, setting $\epsilon = 0$, we find the equation on variations in the Theorem.

Variant 2 We set

$$f(x, \epsilon) = f(x) + \epsilon \xi(x) + o(\epsilon),$$

$$g(x, \epsilon) = g(x) + \epsilon \eta(x) + o(\epsilon),$$

$$\lambda(\epsilon) = \lambda + \epsilon \varsigma + o(\epsilon).$$

Then

$$L(x, \lambda, \epsilon) = L(x, \lambda) + \epsilon(\xi(x) + < \lambda, \eta(x) > + < \varsigma, g(x) >) + o(\epsilon).$$

Taking the partial derivative with respect to x^i and using the initial critical point condition, we find the equation in Theorem. □

2.2. All ingredients are evolutive

Initially, to an optimization problem, we attach the Lagrange function

$$L(x,\lambda) = f(x) + < \lambda, g(x) >$$

and the critical point conditions

$$\frac{\partial f}{\partial x^i} + < \lambda, \frac{\partial g}{\partial x^i} > = 0,\ i = 1, ..., n.$$

We consider a differentiable variation $x(\epsilon)$ and we introduce the general perturbation

$$L(x(\epsilon), \lambda(\epsilon), \epsilon) = f(x(\epsilon), \epsilon) + < \lambda(\epsilon), g(x(\epsilon), \epsilon) >$$

together critical point condition

$$\left(\frac{\partial f}{\partial x^i} + < \lambda, \frac{\partial g}{\partial x^i} >\right)(x(\epsilon), \lambda(\epsilon), \epsilon)\ = 0.$$

By derivation with respect to ϵ, setting $\epsilon = 0$, and denoting $\frac{dx^i}{d\epsilon}|_{\epsilon=0} = y$, we produce a variational linear system associated to the system describing catastrophe manifold. Of course, we use the differential operator

$$\frac{d}{d\epsilon} = \frac{\partial}{\partial x^j}\frac{dx^j}{d\epsilon} + \frac{\partial}{\partial \epsilon},$$

together variational objects (hypothesis)

$$\frac{dx^j}{d\epsilon}|_{\epsilon=0} = y^j,\ \frac{\partial f}{\partial \epsilon}|_{\epsilon=0} = \xi(x),\ \frac{\partial g}{\partial \epsilon}|_{\epsilon=0} = \eta(x),\ \frac{\partial \lambda}{\partial \epsilon}|_{\epsilon=0} = \varsigma,$$

The differential operators $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial \epsilon}$ commute.

Theorem 2.2. *At a critical point $x(\lambda)$, the tangent space to catastrophe manifold, if it exists, is described by the system*

$$\left(\frac{\partial^2 f}{\partial x^i \partial x^j} + < \lambda, \frac{\partial^2 g}{\partial x^i \partial x^j} >\right) y^j + \frac{\partial \xi}{\partial x^i} + < \varsigma, \frac{\partial g}{\partial x^i} > + < \lambda, \frac{\partial \eta}{\partial x^i} >= 0.$$

Proof. Applying $\frac{\partial}{\partial \epsilon}$ to (1), we find

$$\frac{\partial}{\partial x^i}\left(\frac{df}{d\epsilon}\right) + < \frac{\partial \lambda}{\partial \epsilon}, \frac{\partial g}{\partial x^i} > + < \lambda, \frac{\partial}{\partial x^i}\frac{dg}{d\epsilon} >= 0,$$

where

$$\frac{df}{d\epsilon} = \frac{\partial f}{\partial x^j}\frac{dx^j}{d\epsilon} + \frac{\partial f}{\partial \epsilon},\ \frac{dg}{d\epsilon} = \frac{\partial g}{\partial x^j}\frac{dx^j}{d\epsilon} + \frac{\partial g}{\partial \epsilon}.$$

Setting $\epsilon = 0$, we get the linear system in Theorem. More precisely, at a nonsingular critical point $x(\lambda)$ of catastrophe manifold, we have a unique solution $y^j(x(\lambda))$, i.e., an n-dimensional vector of parameter λ; at a singular point $x(\lambda)$ of catastrophe manifold, we have a linear variety of solutions, i.e., n-dimensional vectors depending on λ and p parameters, $y^j(x(\lambda); \alpha_1, ..., \alpha_p)$. □

3. Evolution of an optimal control problem

The constituents of an optimal control problem are: the independent variable (generally speaking, time) t, the initial time t_0, the terminal time t_f, the state vector $x(t)$, the control vector $u(t)$, a cost functional J, and a dynamic constraint (controlled ODE system) or an isoperimetric constraint.

Let us consider an abstract *optimal control problem* of the form: *minimize the continuous-time cost functional*

$$J = \int_{t_0}^{t_f} L(t, x(t), u(t))dt,$$

subject to the first-order dynamic constraints (the state controlled ODE)

$$\dot{x}(t) = X(t, x(t), u(t)).$$

The function L is called *Lagrangian*.

We consider an evolution after $\tau \in [0, \epsilon)$, where $w = w(t, \tau)$, $\tau \in [0, \epsilon)$, is fixed by the partial flow

$$w_\tau = \varphi(t, \tau, w(t, \tau), v(t, \tau)),\ w(t, 0) = x(t),\ v(t, 0) = u(t).$$

The objective functional is changed into

$$\mathcal{J} = \int_{t_0}^{t_f} \mathcal{L}(t, \tau, w(t, \tau), v(t, \tau))dt,$$

with the initial condition

$$\mathcal{L}(t, 0, w(t, 0), v(t, 0)) = L(t, x(t), u(t)).$$

The ODE constraint is changed into a dynamic PDE

$$w_t(t, \tau) = \mathcal{X}(t, \tau, w(t, \tau), v(t, \tau)),$$

with the properties

$$\mathcal{X}(t, \tau, w(t, \tau), v(t, \tau))|_{\tau=0} = X(t, x(t), u(t)),\ \varphi_t = \mathcal{X}_\tau + \mathcal{X}_w \varphi + \mathcal{X}_v v_\tau.$$

For the new problem we can use the Hamiltonian

$$\mathcal{H} = \mathcal{L}(t, \tau, w(t, \tau), v(t, \tau)) + q_i(t)\mathcal{X}^i(t, \tau, w(t, \tau), v(t, \tau)).$$

Theorem 3.1. *The evolution of an optimal control problem is characterized by Euler-Lagrange PDE for*

$$\mathcal{L}_\tau = \frac{\partial \mathcal{L}}{\partial \tau} + \frac{\partial \mathcal{L}}{\partial w}\frac{\partial w}{\partial \tau} + \frac{\partial \mathcal{L}}{\partial v}\frac{\partial v}{\partial \tau} = \frac{\partial \mathcal{L}}{\partial \tau} + \frac{\partial \mathcal{L}}{\partial w}\ \varphi + \frac{\partial \mathcal{L}}{\partial v}\frac{\partial v}{\partial \tau}.$$

3.1. Evolution of a curve described by a second order ODE

Let us consider a C^2 curve $x = x(t)$, $t \in I$, solution of the second order ODE

$$F(t, x(t), \dot{x}(t), \ddot{x}(t)) = 0,\ t \in J \supset I.$$

Its evolution $w = w(t, \tau)$, $\tau \in [0, \epsilon)$, given by $w_\tau = \varphi(t, \tau, w(t, \tau))$, $w_{\tau=0} = x$, is a surface, solution of the PDE

$$\mathcal{F}(t, \tau, w(t, \tau), w_t, w_{tt}) = 0,\ \mathcal{F}_{\tau=0} = F.$$

The new function $\mathcal{F}$ is solution of a linear PDE,

$$\frac{\partial \mathcal{F}}{\partial \tau} + \frac{\partial \mathcal{F}}{\partial w} w_\tau + \frac{\partial \mathcal{F}}{\partial w_t} w_{\tau t} + \frac{\partial \mathcal{F}}{\partial w_{tt}} w_{\tau tt} = 0,$$

since the derivatives $w_{\tau t}, w_{\tau tt}$ are obtained from evolution condition.

Remark 3.1. *This theory can be applied to auto-parallel curves,*

$$\ddot{x}^i(t) + \Gamma^i_{jk}(x(t))\dot{x}^j(t)\dot{x}^k(t) = 0,\ x(t_0) = x_0,\ \dot{x}^i(t_0) = \xi^i_{x_0}.$$

3.2. Evolution of a surface described by a second order PDE

We start with a C^2 surface $u = u(x, y)$, $(x, y) \in D$, solution of second order PDE

$$F(x, y, u(x, y), u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0,\ (x, y) \in E \supset D.$$

Its evolution $w = w(x, y, t)$, $t \in [0, \epsilon)$ is given by the condition

$$w_t = \varphi(t, x, y, w(x, y, t)),\ w_{t=0} = u,$$

satisfying a new PDE

$$\mathcal{F}(t, x, y, w(x, y, t), w_x, w_y, w_{xx}, w_{xy}, w_{yy}) = 0,\ \mathcal{F}_{t=0} = F.$$

The function $\mathcal{F}$ is solution of a linear PDE,

$$\frac{\partial \mathcal{F}}{\partial t} + \frac{\partial \mathcal{F}}{\partial w} w_t + \frac{\partial \mathcal{F}}{\partial w_x} w_{xt} + \frac{\partial \mathcal{F}}{\partial w_y} w_{yt} + \frac{\partial \mathcal{F}}{\partial w_{xx}} w_{xxt} + \frac{\partial \mathcal{F}}{\partial w_{xy}} w_{xyt} + \frac{\partial \mathcal{F}}{\partial w_{yy}} w_{yyt} = 0,$$

since the derivatives $w_{xt}, ..., w_{yyt}$ are obtained from evolution condition.

Example 3.1. Let us take the Laplace PDE $u_{xx} + u_{yy} = 0$, with a solution (harmonic surface) $u(x, y) = x^2 - y^2$. We impose the evolution $w_t = w + x^2$. It follows the Poisson PDE $w_{xx} + w_{yy} = 2e^t - 2$, as equation $\mathcal{F} = 0$. The function $w(x, y, t) = (2e^t - 1)x^2 - e^t y^2$ verifies both the evolution condition and the Poisson PDE. The uniqueness is connected to problems on PDEs with unique solutions.

Example 3.2. Now we start with wave PDE $u_{xx} - u_{yy} = 0$, and the solution $u(x, y) = x^2 + y^2$. We impose the evolution $w_t = w + x^2$. It follows the nonhomogeneous wave PDE $w_{xx} - w_{yy} = 2e^t - 2$, as equation $\mathcal{F} = 0$. The general solution of the last PDE is

$$w(x, y, t) = f(x + y, t) + g(x - y, t) + e^t x^2 - y^2$$

and the function that verifies both the initial condition and the evolution relation is

$$w(x, y, t) = (2x^2 + y^2)e^t - x^2.$$

Example 3.3. Let us consider the Tzitzeica PDE

$$z_{xx}z_{yy} - z_{xy}^2 = c(xz_x + yz_y - z)^4,$$

and a solution $z = \frac{a}{xy},\ xy \neq 0,\ 9a^2 = \frac{1}{c}$. We impose the evolution $w_t = w + x^2,\ w|_{t=0} = z$. It follows the evolution PDE

$$((w_{xx} - 2)e^{-t} + 2)w_{yy}e^{-t} - w_{xy}^2 e^{-2t} = c((xw_x + yw_y - w)e^{-t} + x^2(1 - e^{-t}))^4,$$

as equation $\mathcal{F} = 0$. The function $w(x, y, t) = \left(\frac{a}{xy} + x^2\right) e^t - x^2$ verifies both the evolution condition and the equation $\mathcal{F} = 0$.

Example 3.4. Let us take into discussion the Monge-Ampere PDE

$$z_{xx}z_{yy} - z_{xy}^2 = H(x, y, z, z_x, z_y),$$

and a solution $z = \varsigma(x, y)$. We impose the evolution $w_t = w + x^2,\ w|_{t=0} = \varsigma$. It follows the evolution PDE

$$((w_{xx} - 2)e^{-t} + 2)w_{yy}e^{-t} - w_{xy}^2 e^{-2t} = \mathcal{H}(t, x, y, w, w_x, w_y),$$

as equation $\mathcal{F} = 0$. The function $w(x, y, t) = \left(\varsigma(x, y) + x^2\right) e^t - x^2$ verifies both the evolution condition and the equation $\mathcal{F} = 0$.

Remark 3.2. *Evolution PDE - based methods are widely used in image processing and computer vision. For many of these evolution PDEs, we can formulate the following questions: (i) how are they created? (ii) what we can say about the existence and regularity of solutions? (iii) how to implement them effectively to produce the desired effects? In this work, we study the generation of an arbitrary evolution PDE starting from a stationary ODE or PDE.*

On the other hand, PDEs can be regarded as evolution equations on an infinite dimensional state space. The solution $u(x, t)$ belongs to a function space in x at each instant of time t. For example, second order PDEs are evolution equations for second order ODEs.

The monographs [1] *offers the reader a treatment of the theory of evolution PDEs with nonstandard growth conditions. This class includes parabolic and hyperbolic equations with variable or anisotropic nonlinear structure. Similar problems are found in* [4].

Problem Given a second order PDE regarded as evolutionary equation,

$$\mathcal{F}(t, \tau, w(t, \tau), w_t, w_\tau, w_{tt}, w_{t\tau}, w_{\tau\tau}) = 0,\ (t, \tau) \in E \supset D$$

with respect to the parameter τ, find an originating ODE

$$F(t, u(t), \dot{u}(t), \ddot{u}(t)) = 0,\ t \in J \supset I,$$

where $F = \mathcal{F}_{\tau=0}$, $u(t) = w(t, 0)$.

Example 3.5. Let us take the Laplace PDE

$$u_{xx} + u_{yy} = 0$$

as evolution equation with respect to the parameter y. Since the general solution of the Laplace PDE is of the form $u(x, y) = f(x + iy) + g(x - iy)$, the attached generating ODE must be of the form

$$u_{xx}(x, y) = \varphi(x + iy) + \psi(x - iy).$$

Let us consider the wave PDE

$$a^2 u_{xx} - u_{yy} = 0$$

as evolution equation with respect to the parameter y. Since the general solution of the wave PDE is of the form $u(x, y) = f(x+ay)+g(x-ay)$, the attached generating ODE must be of the form

$$u_{xx}(x, y) = \varphi(x + ay) + \psi(x - ay).$$

Example 3.6. A conditioning system of two diffusion PDEs

$$w(x, y, \tau = (\tau^1, \tau^2));\ w_{\tau^1} = w_{xx}, w_{\tau^2} = w_{yy},\ w_{\tau^1} + w_{\tau^2} = 0,$$

on $Oxy\tau^1\tau^2$, has as trace on the plane Oxy, the Laplace PDE

$$w_{xx} + w_{yy} = 0.$$

Example 3.7. Let us consider two Newton Laws

$$u_{xx}(x, y) = F(x, y, u),\ u_{yy}(x, y) = G(x, y, u).$$

The equilibrium condition

$$F(x, y, u) + G(x, y, u) = 0$$

leads to Laplace PDE $u_{xx} + u_{yy} = 0$. Another condition, as for example,

$$F(x, y, u) = c^2 G(x, y, u)$$

gives the wave PDE $u_{xx} - c^2 u_{yy} = 0$.

3.3. Evolution of a curve satisfying a first order PDE

Problem *Solve the first order pseudo-linear PDE*

$$a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = f(x,y,u),$$

with the condition that, at $y = 0$, the function $u(x,0)$ to be solution of the Cauchy problem

$$\dot{u} = g(x,u),\ u(x_0,0) = u_0.$$

Let us show that this problem lies with a problem of evolution. We look for the evolution of the curve $u = u(x)$ satisfying the Cauchy problem

$$\dot{u} = g(x,u),\ u(x_0) = u_0,$$

with respect to the parameter y. We accept that the evolution is via the surface $w = w(x,y)$ described by first order pseudo-linear PDE

$$a(x,y,w)\frac{\partial w}{\partial x} + b(x,y,w)\frac{\partial w}{\partial y} = f(x,y,w),\ w(x,0) = u(x).$$

To solve the problem, suppose we have two first integrals C_1, C_2 for the characteristic system

$$\frac{dx}{a(x,y,w)} = \frac{dy}{b(x,y,w)} = \frac{dw}{f(x,y,w)},$$

associated to the last PDE. The general solution of the PDE is given implicitly by

$$F(C_1(x,y,w), C_2(x,y,w)) = 0,$$

where F is an arbitrary function of class C^1 that we must determine. The function F can be recovered in three steps: (i) we replace $y = 0$, obtaining

$$F(C_1(x,0,u), C_2(x,0,u)) = 0;$$

(ii) taking the derivative with respect to x, it follows

$$\frac{\partial F}{\partial C_1}\left(\frac{\partial C_1}{\partial x} + \frac{\partial C_1}{\partial u}\,\dot{u}\right) + \frac{\partial F}{\partial C_1}\left(\frac{\partial C_1}{\partial x} + \frac{\partial C_1}{\partial u}\,\dot{u}\right) = 0$$

and hence

$$\frac{\partial F}{\partial C_1}\left(\frac{\partial C_1}{\partial x} + \frac{\partial C_1}{\partial u}\,g(x,u)\right) + \frac{\partial F}{\partial C_1}\left(\frac{\partial C_1}{\partial x} + \frac{\partial C_1}{\partial u}\,g(x,u)\right) = 0;$$

(iii) from the algebraic system $C_1(x,0,u) = C_1$, $C_2(x,0,u) = C_2$, we obtain $x = x(C_1,C_2), u = u(C_1,C_2)$, which fix the relation in (ii) as a linear PDE

$$A(C_1,C_2)\frac{\partial F}{\partial C_1} + B(C_1,C_2)\frac{\partial F}{\partial C_2} = 0.$$

REFERENCES

[1] S. Antontsev, S. Shmarev, *Evolution PDEs with Nonstandard Growth Conditions*, Springer, 2015.

[2] O. Calin, C. Udrişte, *Geometric Modeling in Probability and Statistics*, Springer, 2014.

[3] L. C. Evans, *Partial Differential Equations*, Graduate Texts in Mathematics 19, AMS, 1998.

[4] O. C. Schnürer, *Geometric Evolution Equations*, Lecture notes for the Summer School of the International Research Training Group Arithmetic and Geometry, 02.09.2007-07.09.2007, in Alpbach (Tirol), Austria.

[5] P. A. Samuelson, *Foundations of Economic Analysis*, Harward University Press, Cambridge, 1947.

[6] C. Udrişte, *Convex Functions and Optimization methods on Riemannian Manifolds*, Kluwer Academic Publisheres, 1994.

[7] C. Udrişte, *Geometric Dynamics*, Kluwer Academic Publishers, 2000.