FORMULAS FOR BINOMIAL SUMS INCLUDING POWERS OF FIBONACCI AND LUCAS NUMBERS

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Recently Prodinger [2] *proved general expansion formulas for sums of powers of Fibonacci and Lucas numbers. In this paper, we will prove general expansion formulas for binomial sums of powers of Fibonacci and Lucas numbers.*

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1. Introduction

The Fibonacci numbers F_n and Lucas numbers L_n are defined by the following recursions: for $n > 0$,

$$F_{n+1} = F_n + F_{n-1} \text{ and } L_{n+1} = L_n + L_{n-1},$$

where $F_0 = 0$, $F_1 = 1$ and $L_0 = 2$, $L_1 = 1$, respectively.

If the roots of the characteristic equation $x^2 - x - 1 = 0$ are α and β, then the Binet formulas for them are

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } L_n = \alpha^n + \beta^n.$$

Wiemann and Cooper [4] mentioned about some conjectures of Melham for the sum:

$$\sum_{k=1}^{n} F_{2k}^{2m+1}.$$

Ozeki [1] considered Melham's sum and then he gave an explicit expansion for Melham's sum as a polynomial in F_{2n+1}.

In general, Prodinger [2] derived the general formula for the sum:

$$\sum_{k=0}^{n} F_{2k+\delta}^{2m+\varepsilon},$$

where $\varepsilon, \delta \in \{0, 1\}$, as well as the evaluations of the corresponding sums for Lucas numbers.

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In [5], the authors gave formulas for the alternating analogues of sums of Melham for Fibonacci and Lucas numbers of the forms

$$\sum_{k=1}^{n}(-1)^k F_{2k+\delta}^{2m+\varepsilon} \quad \text{and} \quad \sum_{k=1}^{n}(-1)^k L_{2k+\delta}^{2m+\varepsilon},$$

where $\varepsilon, \delta \in \{0,1\}$.

In this paper, we consider certain binomial sums given by

$$\sum_{k=0}^{n}\binom{n}{k} F_{(2k+\delta)t}^{2m+\varepsilon}, \qquad \sum_{k=0}^{n}\binom{n}{k} L_{(2k+\delta)t}^{2m+\varepsilon},$$
$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k F_{(2k+\delta)t}^{2m+\varepsilon} \text{ and } \sum_{k=0}^{n}\binom{n}{k}(-1)^k L_{(2k+\delta)t}^{2m+\varepsilon},$$

where t is an positive integer and $\varepsilon, \delta \in \{0,1\}$.

Throughout this paper, we use the *indicator function* $[S]$ defined by 1 if the statement S is true or 0 otherwise.

We recall some facts for the readers convenience in [3]: For any real numbers m and n,

$$(m+n)^t = \sum_{i=0}^{t/2-1}\binom{t}{i}(mn)^i\left(m^{t-2i}+n^{t-2i}\right) + \binom{t}{t/2}(mn)^{t/2}[t \text{ is even}] \tag{1}$$

and

$$\begin{aligned}(m-n)^t &= \sum_{i=0}^{t/2-1}\binom{t}{i}(mn)^i(-1)^i\left(m^{t-2i}+n^{t-2i}\right) \\ &\quad + \binom{t}{t/2}(mn)^{t/2}(-1)^{t/2}[t \text{ is even}]. \end{aligned} \tag{2}$$

From [6], we have the following result:

Lemma 1.1. *Let r and s be arbitrary integers. Then*

i)

$$\sum_{i=0}^{n}\binom{n}{i}F_{r+2si} = \begin{cases} \begin{rcases} 5^{(n-1)/2}F_s^n L_{sn+r} & \text{if } n \text{ is odd,} \\ 5^{n/2}F_s^n F_{sn+r} & \text{if } n \text{ is even,} \end{rcases} \text{if } s \text{ is odd,} \\ L_s^n F_{sn+r} \quad \text{if } s \text{ is even.} \end{cases} \tag{3}$$

ii)

$$\sum_{i=0}^{n}\binom{n}{i}L_{r+2si} = \begin{cases} \begin{rcases} 5^{(n+1)/2}F_s^n F_{sn+r} & \text{if } n \text{ is odd,} \\ 5^{n/2}F_s^n L_{sn+r} & \text{if } n \text{ is even,} \end{rcases} \text{if } s \text{ is odd,} \\ L_s^n L_{sn+r} \quad \text{if } s \text{ is even.} \end{cases} \tag{4}$$

iii)

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^i F_{r+2si} = \begin{cases} \begin{rcases} 5^{n/2}F_s^n F_{sn+r} & \text{if } n \text{ is even,} \\ -5^{(n-1)/2}F_s^n L_{sn+r} & \text{if } n \text{ is odd,} \end{rcases} \text{if } s \text{ is even,} \\ (-1)^n L_s^n F_{sn+r} \quad \text{if } s \text{ is odd.} \end{cases} \tag{5}$$

iv)

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}L_{r+2si}=\begin{cases}\begin{cases}5^{n/2}F_s^nL_{sn+r} & \text{if } n \text{ is even,}\\ -5^{(n+1)/2}F_s^nF_{sn+r} & \text{if } n \text{ is odd,}\end{cases} & \text{if } s \text{ is even,}\\ (-1)^nL_s^nL_{sn+r} \quad \text{if } s \text{ is odd.} & \end{cases} \tag{6}$$

2. Some Binomial Sums for Fibonacci Numbers

Here we consider binomial and alternating binomial sums of powers of Fibonacci numbers.

Theorem 2.1. *i) For* $t>0$,

$$\sum_{k=0}^{n}\binom{n}{k}F_{2kt}^{2m}=\frac{1}{5^m}\sum_{i=0}^{m-1}(-1)^i\binom{2m}{i}L_{2(m-i)t}^{n}L_{2(m-i)tn}+\frac{1}{5^m}\binom{2m}{m}(-1)^m2^n.$$

ii) For odd $t>0$,

$$\sum_{k=0}^{n}\binom{n}{k}F_{2kt}^{2m+1}$$

$$=\begin{cases}5^{\frac{(n-2m)}{2}}\sum_{i=0}^{m}(-1)^i\binom{2m+1}{i}F_{(2m-2i+1)t}^{n}F_{(2m-2i+1)tn} & \text{if } n \text{ is even,}\\[2ex] 5^{\frac{(n-2m-1)}{2}}\sum_{i=0}^{m}(-1)^i\binom{2m+1}{i}F_{(2m-2i+1)t}^{n}L_{(2m-2i+1)tn} & \text{if } n \text{ is odd,}\end{cases}$$

and for even $t>0$,

$$\sum_{k=0}^{n}\binom{n}{k}F_{2kt}^{2m+1}=\frac{1}{5^m}\sum_{i=0}^{m}(-1)^i\binom{2m+1}{i}L_{(2m-2i+1)t}^{n}F_{(2m-2i+1)tn}$$

Proof. i) From the Binet formulas of $\{F_n\}$ and $\{L_n\}$, and by (2), we write

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}F_{2kt}^{2m}\\ =\;&\sum_{k=0}^{n}\binom{n}{k}\left(\frac{\alpha^{2kt}-\beta^{2kt}}{\alpha-\beta}\right)^{2m}\\ =\;&\frac{1}{(\alpha-\beta)^{2m}}\sum_{k=0}^{n}\binom{n}{k}\left(\sum_{i=0}^{m-1}\binom{2m}{i}(-1)^i\left(\alpha^{2(2m-2i)tk}+\beta^{2(2m-2i)tk}\right)\right.\\ &\left.+\binom{2m}{m}(-1)^m\right)\\ =\;&\frac{1}{5^m}\left(\sum_{i=0}^{m-1}(-1)^i\binom{2m}{i}\sum_{k=0}^{n}\binom{n}{k}L_{2(2m-2i)tk}+\binom{2m}{m}(-1)^m\sum_{k=0}^{n}\binom{n}{k}\right),\end{aligned}$$

which, by (4) in Lemma 1.1 and since $\sum_{k=0}^{n}\binom{n}{k}=2^n$, equivalent to

$$\sum_{k=0}^{n}\binom{n}{k}F_{2kt}^{2m}=\frac{1}{5^m}\left(\sum_{i=0}^{m-1}(-1)^i\binom{2m}{i}L_{2(m-i)t}^{n}L_{2(m-i)tn}+\binom{2m}{m}(-1)^m2^n\right),$$

as claimed.

ii) Consider

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}F_{2kt}^{2m+1}\\
=&\sum_{k=0}^{n}\binom{n}{k}\left(\frac{\alpha^{2kt}-\beta^{2kt}}{\alpha-\beta}\right)^{2m+1}\\
=&\frac{1}{(\alpha-\beta)^{2m+1}}\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{m}\binom{2m+1}{i}(-1)^{i}\left(\alpha^{2(2m+1-2i)tk}-\beta^{2(2m+1-2i)tk}\right)\\
=&\frac{1}{5^{m}}\sum_{i=0}^{m}(-1)^{i}\binom{2m+1}{i}\sum_{k=0}^{n}\binom{n}{k}F_{2(2m+1-2i)kt},
\end{aligned}$$

which, by taking $s=(2m+1-2i)\,t$ and $r=0$ in (3) in Lemma 1.1, gives the claimed results. □

Following the proof way of Theorem 2.1, we have the following result without proof:

Theorem 2.2. *i*) *For* $t>0$,

$$\sum_{k=0}^{n}\binom{n}{k}F_{(2k+1)t}^{2m}=\frac{1}{5^{m}}\sum_{i=0}^{m-1}(-1)^{i(t+1)}\binom{2m}{i}L_{(2m-2i)t}^{n}L_{(2m-2i)t(n+1)}+\frac{1}{5^{m}}\binom{2m}{m}(-1)^{(t+1)m}\,2^{n}.$$

ii)*For odd* $t>0$,

$$\sum_{k=0}^{n}\binom{n}{k}F_{(2k+1)t}^{2m+1}=\begin{cases}5^{(n-2m)/2}\sum\limits_{i=0}^{m}\binom{2m+1}{i}F_{(2m+1-2i)t}^{n}F_{(2m+1-2i)t(n+1)} & \textit{if } n \textit{ is even},\\[2ex] 5^{(n-2m-1)/2}\sum\limits_{i=0}^{m}\binom{2m+1}{i}F_{(2m-2i+1)t}^{n}L_{(2m-2i+1)t(n+1)} & \textit{if } n \textit{ is odd},\end{cases}$$

and, for even $t>0$,

$$\sum_{k=0}^{n}\binom{n}{k}F_{(2k+1)t}^{2m+1}=\frac{1}{5^{m}}\sum_{i=0}^{m}\binom{2m+1}{i}(-1)^{i}\,L_{(2m+1-2i)t}^{n}F_{(2m+1-2i)t(n+1)}.$$

Theorem 2.3. *i) For $t > 0$,*

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k F_{(2k+1)t}^{2m} = \begin{cases} 5^{\frac{(n-2m)}{2}} \sum\limits_{i=0}^{m-1} (-1)^{i(t+1)} \binom{2m}{i} F_{(2m-2i)t}^{n} L_{(2m-2i)t(n+1)} \\ +\frac{1}{5^m} \binom{2m}{m} (-1)^{m(t+1)} [n=0] & \text{if } n \text{ is even,} \\ -5^{\frac{(n-2m+1)}{2}} \sum\limits_{i=0}^{m-1} (-1)^{i(t+1)} \binom{2m}{i} F_{(2m-2i)t}^{n} F_{(2m-2i)t(n+1)} & \text{if } n \text{ is odd.} \end{cases}$$

ii) For odd $t > 0$,

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k F_{(2k+1)t}^{2m+1} = \frac{(-1)^n}{5^m} \sum_{i=0}^{m} \binom{2m+1}{i} L_{(2m+1-2i)t}^{n} F_{(2m+1-2i)t(n+1)},$$

and, for even $t > 0$,

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k F_{(2k+1)t}^{2m+1} = \begin{cases} 5^{\frac{(n-2m)}{2}} \sum\limits_{i=0}^{m} (-1)^i \binom{2m+1}{i} F_{(2m+1-2i)t}^{n} F_{(2m+1-2i)t(n+1)} & \text{if } n \text{ is even,} \\ -5^{\frac{(n-2m-1)}{2}} \sum\limits_{i=0}^{m} (-1)^i \binom{2m+1}{i} F_{(2m+1-2i)t}^{n} L_{(2m+1-2i)t(n+1)} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. *i)* For $t > 0$, by (2), consider

$$\begin{aligned} & \sum_{k=0}^{n} \binom{n}{k} (-1)^k F_{(2k+1)t}^{2m} \\ = & \sum_{k=0}^{n} \binom{n}{k} (-1)^k \left(\frac{\alpha^{(2k+1)t} - \beta^{(2k+1)t}}{\alpha - \beta} \right)^{2m} \\ = & \frac{1}{(\alpha-\beta)^{2m}} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \sum_{i=0}^{m-1} \binom{2m}{i} (-1)^{i(t+1)} \\ & \times \left(\alpha^{(2m-2i)t(2k+1)} + \beta^{(2m-2i)t(2k+1)} \right) \\ & + \binom{2m}{m} (-1)^m \frac{1}{(\alpha-\beta)^{2m}} \sum_{k=0}^{n} \binom{n}{k} (-1)^k (-1)^{(2k+1)tm} \\ = & \frac{1}{5^m} \sum_{i=0}^{m-1} \binom{2m}{i} (-1)^{i(t+1)} \sum_{k=0}^{n} \binom{n}{k} (-1)^k L_{(2m-2i)t(2k+1)} \\ & + \frac{1}{5^m} \binom{2m}{m} (-1)^{m(t+1)} \sum_{k=0}^{n} \binom{n}{k} (-1)^k , \end{aligned}$$

which, by taking $s = r = (2m - 2i)\,t$ in (6) in Lemma 1.1 and $\sum_{k=0}^{n} \binom{n}{k} (-1)^k = [n = 0]$, gives the claimed result.

ii) For $t > 0$, by (2), consider

$$
\begin{aligned}
&\sum_{k=0}^{n} \binom{n}{k} (-1)^k F_{(2k+1)t}^{2m+1} \\
= &\sum_{k=0}^{n} \binom{n}{k} (-1)^k \left(\frac{\alpha^{(2k+1)t} - \beta^{(2k+1)t}}{\alpha - \beta} \right)^{2m+1} \\
= &\frac{1}{(\alpha-\beta)^{2m+1}} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \sum_{i=0}^{m} \binom{2m+1}{i} (-1)^{i+ti} \\
&\times \left(\alpha^{(2m+1-2i)t(2k+1)} - \beta^{(2m+1-2i)t(2k+1)} \right) \\
= &\frac{1}{5^m} \sum_{i=0}^{m} \binom{2m+1}{i} (-1)^{i(t+1)} \sum_{k=0}^{n} \binom{n}{k} (-1)^k F_{(2m+1-2i)t(2k+1)}
\end{aligned}
$$

which, by taking $s = r = (2m + 1 - 2i)\,t$ in (5) in Lemma 1.1, gives the claimed result. □

Following the proof way of Theorem 2.3, we have the following result:

Theorem 2.4. *i)For* $t > 0$,

$$
\sum_{k=0}^{n} (-1)^k \binom{n}{k} F_{2kt}^{2m}
= \begin{cases}
5^{\frac{(n-2m)}{2}} \sum\limits_{i=0}^{m-1} (-1)^i \binom{2m}{i} F_{2(m-i)t}^{n} L_{2(m-i)tn} + \frac{1}{5^m} \binom{2m}{m} (-1)^m [n=0] & \text{if } n \text{ is even,} \\
-5^{\frac{(n-2m+1)}{2}} \sum\limits_{i=0}^{m-1} (-1)^i \binom{2m}{i} F_{2(m-i)t}^{n} F_{2(m-i)tn} & \text{if } n \text{ is odd,}
\end{cases}
$$

ii) For odd $t > 0$,

$$
\sum_{k=0}^{n} (-1)^k \binom{n}{k} F_{2kt}^{2m+1} = (-1)^n \frac{1}{5^m} \sum_{i=0}^{m} (-1)^i \binom{2m+1}{i} L_{(2m-2i+1)t}^{n} F_{(2m-2i+1)tn},
$$

and for even $t > 0$,

$$
\sum_{k=0}^{n} (-1)^k \binom{n}{k} F_{2kt}^{2m+1}
= \begin{cases}
5^{\frac{(n-2m)}{2}} \sum\limits_{i=0}^{m} (-1)^i \binom{2m+1}{i} F_{(2m-2i+1)t}^{n} F_{(2m-2i+1)tn} & \text{if } n \text{ is even} \\
-5^{\frac{(n-2m-1)}{2}} \sum\limits_{i=0}^{m} (-1)^i \binom{2m+1}{i} F_{(2m-2i+1)t}^{n} L_{(2m-2i+1)tn} & \text{if } n \text{ is odd}
\end{cases}.
$$

3. **Binomial Sums and Binomial Alternating Sums for Lucas Numbers**

Now we consider binomial and alternating binomial sums of powers of Lucas numbers.

Theorem 3.1. *i) For* $t > 0$,

$$\sum_{k=0}^{n} \binom{n}{k} L_{2kt}^{2m} = \sum_{i=0}^{m-1} \binom{2m}{i} L_{2(m-i)t}^{n} L_{2(m-i)tn} + \binom{2m}{m} 2^n.$$

ii) For even $t > 0$,

$$\sum_{k=0}^{n} \binom{n}{k} L_{2kt}^{2m+1} = \sum_{i=0}^{m} \binom{2m+1}{i} L_{(2m-2i+1)t}^{n} L_{(2m-2i+1)tn},$$

and, for odd $t > 0$,

$$\sum_{k=0}^{n} \binom{n}{k} L_{2kt}^{2m+1} = \begin{cases} 5^{n/2} \sum\limits_{i=0}^{m} \binom{2m+1}{i} F_{(2m-2i+1)t}^{n} L_{(2m-2i+1)tn} & \text{if } n \text{ is even,} \\ 5^{(n+1)/2} \sum\limits_{i=0}^{m} \binom{2m+1}{i} F_{(2m-2i+1)t}^{n} F_{(2m-2i+1)tn} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. *i)* For $t > 0$, by the Binet formula of $\{L_n\}$ and (1), we write

$$\begin{aligned} &\sum_{k=0}^{n} \binom{n}{k} L_{2kt}^{2m} \\ =& \sum_{k=0}^{n} \binom{n}{k} \left(\alpha^{2kt} + \beta^{2kt}\right)^{2m} \\ =& \sum_{k=0}^{n} \binom{n}{k} \left(\sum_{i=0}^{m-1} \binom{2m}{i} \left(\alpha^{2(2m-2i)tk} + \beta^{2(2m-2i)tk}\right) + \binom{2m}{m} (\alpha\beta)^{2ktm} \right) \\ =& \sum_{i=0}^{m-1} \binom{2m}{i} \sum_{k=0}^{n} \binom{n}{k} L_{2(2m-2i)tk} + \binom{2m}{m} \sum_{k=0}^{n} \binom{n}{k}, \end{aligned}$$

which, by taking $s = (2m - 2i)\, t$ and $r = 0$ in (4) in Lemma 1.1, gives the claimed result.

ii) For $t > 0$, by the Binet formula of $\{L_n\}$ and (1), we write

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}L_{2kt}^{2m+1}\\
&= \sum_{k=0}^{n}\binom{n}{k}\left(\alpha^{2kt}+\beta^{2kt}\right)^{2m+1}\\
&= \sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{m}\binom{2m+1}{i}\left(\alpha^{2(2m+1-2i)tk}+\beta^{2(2m+1-2i)tk}\right)\\
&= \sum_{i=0}^{m}\binom{2m+1}{i}\sum_{k=0}^{n}\binom{n}{k}L_{2(2m+1-2i)tk}
\end{aligned}$$

which, by taking $s = (2m+1-2i)\,t$ and $r = 0$ in (4) in Lemma 1.1, gives the claimed result. □

Following the proof way of the previous Theorems, we give the following results without proof:

Theorem 3.2. *i*) *For* $t > 0$,

$$\sum_{k=0}^{n}\binom{n}{k}L_{(2k+1)t}^{2m} = \sum_{i=0}^{m-1}\binom{2m}{i}(-1)^{ti}\,L_{2(m-i)t}^{n}L_{2(m-i)t(n+1)} + \binom{2m}{m}(-1)^{tm}\,2^{n}.$$

ii) *For even* $t > 0$,

$$\sum_{k=0}^{n}\binom{n}{k}L_{(2k+1)t}^{2m+1} = \sum_{i=0}^{m}\binom{2m+1}{i}L_{(2m-2i+1)t}^{n}L_{(2m-2i+1)t(n+1)},$$

and, for odd $t > 0$,

$$\sum_{k=0}^{n}\binom{n}{k}L_{(2k+1)t}^{2m+1}$$

$$= \begin{cases} 5^{\frac{n}{2}}\sum\limits_{i=0}^{m}\binom{2m+1}{i}(-1)^{i}\,F_{(2m-2i+1)t}^{n}L_{(2m-2i+1)t(n+1)} & \text{if } n \text{ is even,}\\[2ex] 5^{\frac{(n+1)}{2}}\sum\limits_{i=0}^{m}\binom{2m+1}{i}(-1)^{i}\,F_{(2m-2i+1)t}^{n}F_{(2m-2i+1)t(n+1)} & \text{if } n \text{ is odd.}\end{cases}$$

Theorem 3.3. *i*) *For* $t > 0$,

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{k}\,L_{2kt}^{2m}$$

$$= \begin{cases} 5^{\frac{n}{2}}\sum\limits_{i=0}^{m-1}\binom{2m}{i}F_{(2m-2i)t}^{n}L_{(2m-2i)tn} + \binom{2m}{m}[n=0] & \text{if } n \text{ is even,}\\[2ex] -5^{\frac{(n+1)}{2}}\sum\limits_{i=0}^{m-1}\binom{2m}{i}F_{(2m-2i)t}^{n}F_{(2m-2i)tn} & \text{if } n \text{ is odd.}\end{cases}$$

ii) For odd $t > 0$,

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k L_{2kt}^{2m+1} = (-1)^n \sum_{i=0}^{m}\binom{2m+1}{i} L_{(2m+1-2i)t}^{n} L_{(2m+1-2i)tn},$$

and, for even $t > 0$,

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k L_{2kt}^{2m+1} = \begin{cases} 5^{\frac{n}{2}} \sum\limits_{i=0}^{m}\binom{2m+1}{i} F_{(2m+1-2i)t}^{n} L_{(2m+1-2i)tn} & \text{if } n \text{ is even,} \\ -5^{\frac{(n+1)}{2}} \sum\limits_{i=0}^{m}\binom{2m+1}{i} F_{(2m+1-2i)t}^{n} F_{(2m+1-2i)tn} & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 3.4. *i) For* $t > 0$,

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k L_{(2k+1)t}^{2m} = \begin{cases} 5^{n/2} \sum\limits_{i=0}^{m-1}\binom{2m}{i}(-1)^{ti} F_{(2m-2i)t}^{n} L_{(2m-2i)t(n+1)} + \binom{2m}{m}(-1)^{tm}[n=0] & \text{if } n \text{ is even,} \\ -5^{(n+1)/2} \sum\limits_{i=0}^{m-1}\binom{2m}{i}(-1)^{ti} F_{(2m-2i)t}^{n} F_{(2m-2i)t(n+1)} & \text{if } n \text{ is odd.} \end{cases}$$

ii) For odd $t > 0$,

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k L_{(2k+1)t}^{2m+1} = (-1)^n \sum_{i=0}^{m}\binom{2m+1}{i}(-1)^i L_{(2m+1-2i)t}^{n} L_{(2m+1-2i)t(n+1)},$$

and, for even $t > 0$,

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k L_{(2k+1)t}^{2m+1} = \begin{cases} 5^{\frac{n}{2}} \sum\limits_{i=0}^{m}\binom{2m+1}{i} F_{(2m+1-2i)t}^{n} L_{(2m+1-2i)t(n+1)} & \text{if } n \text{ is even,} \\ -5^{\frac{(n+1)}{2}} \sum\limits_{i=0}^{m}\binom{2m+1}{i} F_{(2m+1-2i)t}^{n} F_{(2m+1-2i)t(n+1)} & \text{if } n \text{ is odd.} \end{cases}$$

REFERENCES

[1] *K. Ozeki*, On Melham's sum,The Fibonacci Quarterly, **46-47(2)** (2008-2009),107–110.

[2] *H. Prodinger*, On a sum of Melham and its variants, The Fibonacci Quarterly, **46-47(3)** (2008-2009), 207–215.

[3] *S. Vajda*, Fibonacci & Lucas numbers, and the golden section. Theory and applications. John Wiley & Sons, Inc., New York, 1989.

[4] *M. Wiemann and C. Cooper*, Divisibility of an F–L type convolution. Applications of Fibonacci Numbers, **9** (2004), 267–287.
[5] *E. Kılıç, N. Ömür and Y. T. Ulutaş*, Alternating sums of the powers of Fibonacci and Lucas numbers, Miskolc Math. Notes, Vol. **12** (2011), No. 1, 87–103.
[6] *E. Kılıç, N. Ömür and Y. T. Ulutaş*, Binomial sums whose coefficients are products of terms of binary sequences, Utilitas Math., **84** (2011), 45–52.