

# EXTENSION AND DECOMPOSITION OF LINEAR OPERATORS DOMINATED BY CONTINUOUS INCREASING SUBLINEAR OPERATORS

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*We point out new applications of earlier results on constrained extension of linear operators. In section 2, similar results with respect to previous ones on the Riesz decomposition property, but now for arbitrary linear bounded operators are proved. Increasing continuous sublinear dominating operators play a central role in both sections 2 and 3. In section 3, decomposition as differences of positive bounded linear operators is investigated. Under appropriate assumptions, one proves that the space  $\mathfrak{B}(X, Y)$  of all bounded linear operators from  $X$  into  $Y$  is an order complete Banach lattice. Finally, section 4 focuses on a constrained optimization problem.*

**Keywords:** constrained extension of linear operators, decomposition, equicontinuity, moment problem, constrained optimization

**MSC 2010:** 47A57, 46B42, 49J45

## 1. Introduction

Using Hahn – Banach type results in proving several other main theorems in functional analysis and their consequences is a well - known technique. Most of the works in the References of the present article contain more or less direct applications of Hahn-Banach principle, or proofs of extension - results similar to Hahn-Banach theorem. For terminology and results related to Sections 2 and 3 of the present work see [1] - [5]. Constrained extension type results for linear operators have been intensively applied in solving moment problems, especially in solving Markov moment problem and related problems (see [6], [10]-[15] and many other works). Uniqueness and construction of the solutions of Markov moment problems are partially solved in [7] and [11]. In [8], a necessary and sufficient condition for the existence of a linear extension of a linear operator, preserving two constraints is stated and partially proved. Some consequences are formulated in the same work, without proofs. Complete proofs of these results (and of many other related theorems) can be found in [9]. Applications to the abstract moment problem are stated in [10]. Further applications to the classical

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moment problem are proved in [6], [11] – [15]. The article [13] contains a polynomial approximation result valid on unbounded subsets, whose proof is using Hahn – Banach theorem. The first purpose of this paper is to prove similar results to those from [4] (respectively from [5]), but for spaces of bounded linear operators. Namely, one assumes that the target space  $F$  is an order complete (Dedekind complete) normed vector lattice. The domain space  $E$  is an arbitrary normed vector lattice. Thus, one obtains new statements for theorems 2.1, 3.1 [4] and respectively theorems 1, 2 [5] (see section 2). All the theorems in Section 2 refer to applications of Hahn-Banach type results to the Riesz decomposition property in spaces of linear continuous operators. For similar previous results on this subject, see [3] - [5], [16] - [19]. It is possible that some of the extension results of section 2 to be partially known. In this case, the contribution of Section 2 of the present paper is to give new simple proofs for such results, based on a general earlier result mentioned above [8], [9]. The second purpose of this work is to point out the possibility of decomposition of each element of an equicontinuous family of linear operators as a difference of positive linear operators, such that the corresponding families of the latter (positive) linear operators to be equicontinuous too (section 3). Uniform evaluation of the norms is studied too. Both Sections 2 and 3 are based on the idea of the existence of a dominating increasing sublinear continuous operator. In the end of section 3, under appropriate assumptions, one proves that the space  $B(X, Y)$  of all bounded linear operators from  $X$  into  $Y$  is a Dedekind complete Banach lattice. A characteristic of the present work which is new is that of proving results valid for  $B(X, Y)$ . Similar (but not identical) previous results are proved for the spaces  $L^r(X, Y)$  (the space of all linear regular operators from  $X$  into  $Y$ , that is the space of those operators which can be written as a difference of two linear positive operators) and  $\mathcal{L}^r(X, Y)$ , the space of all operators which are differences of linear continuous positive operators. Finally, the third purpose of this work is to solve constrained optimization problems in infinite dimensional spaces, related to special Markov moment problems [7] (Section 4). The background of the present work consists in some chapters from [16] – [19]. The rest of the article is organized as follows. Section 2 is devoted to Riesz decomposition property in spaces of bounded linear operators and related results. In Section 3, decomposition of linear bounded operators as differences of positive linear bounded operators is investigated. The equicontinuity of the resulting families of linear positive operators is investigated too. Under additional assumption on the target space  $Y$ , one proves that the space of all linear bounded operators  $B(X, Y)$  is an order complete Banach lattice. Section 4 focuses on a constrained optimization problem in infinite dimensional spaces. Section 5 concludes the paper.

## 2. On Riesz decomposition property for linear bounded operators

We start this section by recalling some known results on the subject. A conjecture posed by A. W. Wickstead [3] found a positive answer (Theorem 3.1 [4]). This theorem is a consequence of the following extension – type result.

**Theorem 2.1** (see [2], Th. 3.5 and [4], Th. 2.1). *Let  $X$  and  $Y$  be Banach lattices, such that  $X$  is separable and  $Y$  has the countable interpolation property, and let  $P: X \rightarrow Y$  be a continuous sublinear operator. If  $X_0$  is a vector subspace of  $X$  and  $U: X_0 \rightarrow F$  is a linear continuous operator satisfying  $U(x) \leq P(x)$  for all  $x \in X_0$ , then there exists a linear extension  $\tilde{U}$  of  $U$  to all of  $X$  also satisfying  $\tilde{U}(x) \leq P(x)$  for all  $x \in X$ .*

Using Theorem 2.1, in [4] one proves the following main positive answer to the conjecture mentioned above.

**Theorem 2.2** (see [4], Th. 3.1). *Let  $E, F$  be two Banach lattices such that  $E$  is separable and  $F$  has the countable interpolation property. Then the space of all continuous regular operators  $\mathfrak{L}^r(E, F)$  has the Riesz decomposition property.*

For statements and proofs of the above theorems formulated in a more general setting see [5]. We recall our necessary and sufficient condition on the extension of a linear positive extension, which generalizes H. Bauer’s theorem ([19], Theorem V. 5.4; see also [8], [9], [10]).

**Theorem 2.3** (see [8], Th. 2 and [9], Th. III. 2). *Let  $X$  be a preordered vector space of positive cone  $X_+$ ,  $Y$  an order complete vector lattice,  $P: X \rightarrow Y$  a convex operator,  $X_0 \subset X$  a vector subspace,  $U: X_0 \rightarrow Y$  a linear operator. The following statements are equivalent:*

- (a)  $U$  admits a linear positive extension  $\tilde{U}: X \rightarrow Y$  such that  $\tilde{U}(x) \leq P(x), \forall x \in X$ ;
- (b)  $U(x') \leq P(x)$  for all  $(x', x) \in X_0 \times X$  such that  $x' \leq x$ .

Theorem 2.3 was published firstly in [8], without proof. Its detailed proof and related results can be found in [9] (see [9], p. 978 – 980). In the sequel, we deduce some new results from this theorem.

**Corollary 2.1.** *Let  $X, Y, X_0, U, P$  be as in Theorem 2.3. Assume that  $P$  verifies the additional condition:  $P(x') \leq P(x)$  for all  $(x', x) \in X_0 \times X$  such that  $x' \leq x$ . The following statements are equivalent:*

- (a)  $U$  admits a linear positive extension  $\tilde{U}: X \rightarrow Y$  such that  $\tilde{U}(x) \leq P(x), \forall x \in X$ ;
- (b)  $U(x') \leq P(x')$  for all  $x' \in X_0$ .

*Proof.* The implication (a) $\Rightarrow$ (b) is obvious. To prove the converse, we apply the implication (b) $\Rightarrow$ (a) of Theorem 2.3. Namely, due to the assumption on  $P$ , the following relations hold

$$U(x') \leq P(x') \leq P(x) \text{ for all } (x', x) \in X_0 \times X \text{ such that } x' \leq x.$$

Thus, condition (b) from Theorem 2.3 is verified. Application of the latter theorem leads to the conclusion of the Corollary 2.1. The proof is finished.  $\square$

**Corollary 2.2.** *Corollary 2.1 is valid in the particular case when the following stronger condition on  $P$  holds:  $P(x') \leq P(x)$  for all  $(x', x) \in X \times X$  such that  $x' \leq x$ .*

*Example 2.1.* Let  $X, Y$  be as above. Assume that  $X$  is a vector lattice. Let  $U: X \rightarrow Y$  be a positive linear operator. Then  $P: X \rightarrow Y$  defined by  $P(x) = U(x^+)$ ,  $x \in X$ , is a sublinear operator which verifies the monotony condition from Corollary 2.2.

**Corollary 2.3.** *Let  $X, Y$  be normed vector lattices, such that  $Y$  is order complete,  $P: X \rightarrow Y$  a sublinear continuous operator satisfying the monotony condition from Corollary 2.2. Let  $X_0, U$  be as in the statements of corollaries 2.1, 2.2. The following statements are equivalent*

- (a)  $U$  admits a linear bounded positive extension  $\tilde{U}: X \rightarrow Y$  such that
 
$$\tilde{U}(x) \leq P(x), \forall x \in X;$$
- (b)  $U(x') \leq P(x')$  for all  $x' \in X_0$ .

Observe that Corollary 2.3 follows directly from Corollary 2.2. The continuity of  $\tilde{U}$  is a consequence of continuity of  $P$  and of the relation  $\tilde{U} \leq P$  on  $X$ . Also, completeness with respect to the corresponding norms seems to be not important. In the sequel, we prove the main result of this section. It shows that in the case when  $E, F$  are normed vector lattices, such that  $F$  is order complete, the space  $B(E, F)$  of all bounded linear operators from  $X$  into  $Y$  has the Riesz decomposition property. Observe that contrary to Theorem 3.1 [4], the concerned operators from Theorem 2.4 are not assumed to be regular. The proof is based on Corollary 2.3 and also on the ideas of the proof of Theorem 3.1 [4], without repeating technical details which remain unchanged. In the sequel,  $E$  will be an arbitrary normed vector lattice.

**Theorem 2.4.** *Let  $E, F$  be normed vector lattices, such that  $F$  is order complete. Then the space  $B(E, F)$  has the Riesz decomposition property.*

*Proof.* Let  $T, S_1, S_2$  be linear bounded positive operators from  $E$  into  $F$  such that

$$T \leq S_1 + S_2.$$

We have to prove the existence of two linear operators  $T_1, T_2 \in B(E, F)$  with the properties

$$\begin{aligned} 0 \leq T_j \leq S_j, \quad j = 1, 2, \\ T = T_1 + T_2. \end{aligned}$$

Consider the normed vector lattices  $X := E \times E$  with the canonical order and norm, and  $Y := F$ . Define

$$P: X = E \times E \rightarrow Y = F, P(x_1, x_2) = S_1(x_1^+) + S_2(x_2^+), (x_1, x_2) \in E \times E.$$

Obviously,  $P$  is a sublinear operator. Moreover, it is an increasing operator, satisfying the monotony condition from corollary 2.2 (it is the sum of two such linear operators, defined in example 2.1). Since the lattice operations on  $X$  are continuous and  $S_1, S_2$  are bounded,  $P$  is continuous too. Consider the subspace  $X_0$  of  $X$  defined by  $X_0 := \{(x, x); x \in E\}$  and define  $U: X_0 \rightarrow F$  by  $U(x, x) = T(x), x \in E$ .  $U$  is a linear continuous operator and, as in the proof of Theorem 3.1 [4], one shows that

$$U(x, x) \leq P(x, x), x \in E.$$

Thus  $U \leq P$  on  $X_0$ . Application of Corollary 2.3 leads to the existence of a linear positive (continuous) extension  $\tilde{U}$  of  $U$  to the whole space  $X = E \times E$ , such that

$$\tilde{U}(x_1, x_2) \leq P(x_1, x_2), (x_1, x_2) \in E \times E.$$

Define

$$\begin{aligned} T_1(x) &= \tilde{U}(x, 0), & T_1: E &\rightarrow F, \\ T_2(x) &= \tilde{U}(0, x), & T_2: E &\rightarrow F. \end{aligned}$$

By the proof of theorem 3.1 [4],  $T_1, T_2$  are the desired operators. Moreover, from the previous relations  $0 \leq T_j \leq S_j, j = 1, 2$ , one deduces easily that  $\|T_j\| \leq \|S_j\|, j = 1, 2$  (one uses the fact that the norms on  $F, E$  are solid:  $|x_1| \leq |x_2| \Rightarrow \|x_1\| \leq \|x_2\|$ ). The proof is finished.  $\square$

### 3. Decomposition of a bounded linear operator as a difference of two positive bounded linear operators

In this section we characterize the property of decomposition of a linear operator dominated on the positive cone by an increasing continuous sublinear operator, as a difference of two linear positive continuous operators. Related results on equicontinuous families of operators are mentioned too.

**Theorem 3.1** (see [8], Th. 3 and [9], Th. III. 4). *Let  $X, Y, P$  be as in theorem 2.3,  $U: X \rightarrow Y$  a linear operator. The following statements are equivalent*

- (a)  $U$  admits a decomposition  $U = U_1 - U_2$ , with  $U_1, U_2$  positive linear operators such that  $U_1(x) \leq P(x), x \in X$ ;
- (b)  $U(c') \leq P(c)$  for all  $(c', c) \in X_+ \times X_+$  such that  $c' \leq c$ .

Theorem 3.1 was stated for the first time in [8]. Its detailed proof can be found in [9], p. 982-983.

**Corollary 3.1.** *Let  $X, Y$  be normed vector lattices, such that  $Y$  is order complete,  $P: X \rightarrow Y$  a sublinear continuous increasing operator on the positive cone  $C = X_+$  ( $P(c') \leq P(c)$  for all  $(c', c) \in X_+ \times X_+$  such that  $c' \leq c$ ). Assume additionally*

that  $P(x) = P(-x), x \in X$ . Let  $\{U_j\}_{j \in J}$  be an equicontinuous family of bounded linear operators from  $X$  into  $Y$ . The following statements are equivalent

- (a) there exist two equicontinuous families  $\{U_{1,j}\}_{j \in J}, \{U_{2,j}\}_{j \in J}$  of linear bounded positive operators such that  $U_j = U_{1,j} - U_{2,j}, U_{1,j} \leq P$  on  $X$  for all  $j \in J$ ;
- (b)  $U_j(c) \leq P(c), c \in X_+, j \in J$ .

*Proof.* We only have to prove that (b)  $\Rightarrow$  (a), since the converse is obvious. Let  $(c', c) \in X_+ \times X_+$  be such that  $c' \leq c$ .

Then from (b) and the monotony property of  $P$ , we derive:  $U_j(c') \leq P(c') \leq P(c), j \in J$ . Theorem 3.1 ensures the existence of positive linear operators  $U_{1,j}, U_{2,j}$  such that  $U_j = U_{1,j} - U_{2,j}, U_{1,j} \leq P$  on  $X, j \in J$ . Now  $U_{1,j}(x) \leq P(x) \leq |P(x)|, x \in X \Rightarrow -U_{1,j}(x) = U_{1,j}(-x) \leq P(-x) = P(x)$

$$\leq |P(x)|, j \in J \Rightarrow |U_{1,j}(x)| \leq |P(x)|, j \in J, x \in X.$$

Since the norms on  $Y, X$  are solid, one deduces that

$$\|U_{1,j}(x)\| \leq \|P(x)\|, x \in X, j \in J.$$

The continuity of  $P$  and  $\|\cdot\|_Y$  at the origins of  $X$ , respectively  $Y$  leads to the continuity of  $\|\cdot\|_Y \circ P$  at  $0_X$ . Hence, there exists a radius  $r > 0$  such that  $\|P(x)\| < 1$  for all  $x \in X$  with  $\|x\| < r$ . From the relations written two lines above, we infer that  $\|U_{1,j}(x)\| < 1$  for all  $x \in X$  such that  $\|x\| < r$  and for all  $j \in J$ . Thus, the family  $\{U_{1,j}\}_{j \in J}$  is equicontinuous. Since  $\{U_j\}_{j \in J}$  was assumed to be equicontinuous and  $U_{2,j} = U_{1,j} - U_j, j \in J$ , it follows that  $\{U_{2,j}\}_{j \in J}$  is equicontinuous too. This concludes the proof.  $\square$

**Corollary 3.2.** Let  $X, Y, \{U_j\}_{j \in J}$  be as in corollary 3.1, and  $V \in B_+(X, Y)$  a (bounded) positive linear operator applying  $X$  into  $Y$ . The following statements are equivalent

- (a) there exist two equicontinuous families  $\{U_{1,j}\}_{j \in J}, \{U_{2,j}\}_{j \in J}$  of linear bounded positive operators such that  $U_j = U_{1,j} - U_{2,j}, U_{1,j}(x) \leq |V(x)|, x \in X, j \in J$  (in particular  $\|U_{1,j}\| \leq \|V\|, j \in J$ );
- (b)  $U_j(c) \leq V(c), c \in X_+, j \in J$ .

*Proof.* One applies corollary 3.1 to  $P$  defined by  $P(x) = |V(x)|, x \in X$ . It is easy to see that  $P$  verifies all conditions from the statement of corollary 3.1, and  $P = V$  on the positive cone  $X_+$ . The conclusion follows.  $\square$

*Remark 3.1.* The set of all sublinear operators  $P: X \rightarrow Y$  which are monotone on  $X_+$  in the sense specified in corollary 3.1, and have the property  $P(x) = P(-x), x \in X$ , is closed with respect to the addition and “sup” – operations.

A question which appears naturally is the following one: which are concrete sublinear operators  $P$  verifying the two conditions mentioned in corollary 3.1 and how such examples can be applied? A partial answer was given in [9], theorem III. 5, p. 983. Now we prove a similar result, adapted to the case of normed vector lattices setting. In this particular case, an evaluation of a common upper bound for  $\{\|U_{1,j}\|; j \in J\}$  is deduced.

**Corollary 3.3.** *Let  $X, Y$  be normed vector lattices, such that  $Y$  is order complete, it has an order unit  $u_0$  and its unit ball is the order interval  $[-u_0, u_0]$ . Let  $\{U_j\}_{j \in J}$  be an equicontinuous family of bounded linear operators from  $X$  into  $Y$ . There exist two equicontinuous families  $\{U_{1,j}\}_{j \in J}, \{U_{2,j}\}_{j \in J}$  of linear bounded positive operators such that*

$$U_j = U_{1,j} - U_{2,j}, U_{1,j}(x) \leq (1/r)\|x\|u_0, x \in X, j \in J,$$

where  $r > 0$  is sufficiently small such that  $U_j(\bar{B}(0; r)) \subset [-u_0, u_0], j \in J$ . In particular,  $\|U_{1,j}\| \leq 1/r, j \in J$ .

*Proof.* For  $x \in X \setminus \{0\}, \left(\frac{r}{\|x\|}\right)x \in \bar{B}(0; r)$ , hence  $\frac{r}{\|x\|}U_j(x) \in [-u_0, u_0]$ , that is

$$\pm U_j(x) \leq \frac{1}{r}\|x\|u_0, j \in J.$$

Apply Corollary 3.1 to  $P(x) := \frac{1}{r}\|x\|u_0, x \in X$ . Since the norm on  $X$  is solid and symmetric, it is clear that the sublinear operator  $P$  has the two properties mentioned in the statement of corollary 3.1. Application of the latter corollary shows that  $U_{1,j}(x) \leq (1/r)\|x\|u_0, x \in X, j \in J$ . Replacing  $x$  by  $-x$  one obtains  $|U_{1,j}(x)| \leq (1/r)\|x\|u_0, x \in X, j \in J$ . Now the relations  $\|U_{1,j}\| \leq 1/r, j \in J$  follow from the fact that the norm of  $Y$  is solid too. This concludes the proof.  $\square$

**Corollary 3.4.** *Under the hypothesis and with the notations from Corollary 3.3, if  $M := \sup_{j \in J} \|U_j\|$ , then  $\|U_{1,j}\| \leq M, j \in J$ .*

*Example 3.1.* Let

$$\varepsilon \in (0, 1), X = Y = L_\infty([- \varepsilon, 1]), U_j(\varphi)(t) = t^{2j+1}\varphi(t), t \in [- \varepsilon, 1], \varphi \in L_\infty([- \varepsilon, 1]), j \in \mathbf{N}.$$

Then it easy to see that  $\sup_{j \in \mathbf{N}} \|U_j\| = 1 (= M)$ . One obtains

$$U_j = U_{1,j} - U_{2,j}, U_{1,j}(\varphi)(t) = 0, t \in [- \varepsilon, 0], U_{1,j}(\varphi)(t) = t^{2j+1}\varphi(t), t \in [0, 1]; \\ U_{2,j}(\varphi)(t) = -t^{2j+1}\varphi(t), t \in [- \varepsilon, 0], U_{2,j}(\varphi)(t) = 0, t \in [0, 1], j \in J := \mathbf{N}.$$

It is easy to see that  $\sup_{j \in \mathbf{N}} \|U_{1,j}\| = 1 (= M)$ ,  $\sup_{j \in \mathbf{N}} \|U_{2,j}\| = \varepsilon < 1$ . So this example shows that the case  $\sup_{j \in J} \|U_{1,j}\| = \sup_{j \in J} \|U_j\|$  may occur, while in general,  $\sup_{j \in J} \|U_{2,j}\| \neq \sup_{j \in J} \|U_j\|$ .

For two normed vector lattices  $X, Y$ , recall that one denotes by  $L^r(X, Y)$  the space of all linear regular operators from  $X$  into  $Y$  (that is the space of those operators which can be written as a difference of two linear positive operators). By  $\mathcal{L}^r(X, Y)$  one denotes the space of all operators which are differences of linear continuous positive operators.

**Theorem 3.2.** *Let  $X, Y$  be Banach lattices, such that  $Y$  is order complete, has an order unit  $u_0$  and its unit ball is equal to the order interval  $[-u_0, u_0]$ . Then we have  $\mathcal{B}(X, Y) = \mathcal{L}^r(X, Y) = L^r(X, Y)$  and the space  $\mathcal{B}(X, Y)$  is an order complete Banach lattice with respect to the operatorial norm.*

*Proof.* Relation  $\mathcal{B}(X, Y) \subset \mathcal{L}^r(X, Y)$  follows from Corollary 3.3 (or from Corollary 3.4). The converse inclusion - relation is obvious, so that the equality  $\mathcal{B}(X, Y) = \mathcal{L}^r(X, Y)$  is proved. On the other hand, Proposition 1.3.5 [17] claims that every positive linear operator applying a Banach lattice  $X$  into a normed vector lattice  $Y$ , is continuous. Hence  $L^r(X, Y) = \mathcal{L}^r(X, Y)$ . Thus,  $\mathcal{B}(X, Y) = \mathcal{L}^r(X, Y) = L^r(X, Y)$ . Application of Theorem 1 from [16], p. 162 – 163, proves that  $L^r(X, Y)$  is an order complete vector lattice. Hence  $\mathcal{B}(X, Y)$  has the same property with respect to the usual order relation. It is also a Banach space with respect to the usual operatorial norm, since  $Y$  is a Banach space, hence it is complete as a metric space. It remains to prove that  $\mathcal{B}(X, Y)$  is a Banach lattice, that is:

$$U, V \in \mathcal{B}(X, Y), |U| \leq |V| \Rightarrow \|U\| \leq \|V\|$$

This assertion is equivalent to the fact that the unit ball of the space  $\mathcal{B}(X, Y)$  (which will be denoted by  $\mathcal{B}_1$ ), is a solid subset:  $U, V \in \mathcal{B}(X, Y), V \in \mathcal{B}_1, |U| \leq |V| \Rightarrow U \in \mathcal{B}_1$ . Since  $V \in \mathcal{B}_1$ , also using the assumptions on  $Y$ , as well as the formula for computing  $|V|$  in the space  $L^r(X, Y) = \mathcal{B}(X, Y)$  proved in [16], Proposition 3, p. 164, we derive that

$$\begin{aligned} \|x\| \leq 1 &\Rightarrow |U(x)| \leq |U|(|x|) \leq |V|(|x|) \\ &= \sup_{|x'| \leq |x|} |V(x')| \leq \sup_{\|x''\| \leq 1} |V(x'')| \leq u_0 \end{aligned}$$

(we have used the fact that the norm on  $X$  is solid, and also the relation  $V(\mathcal{B}(0_X; 1)) \subset [-u_0, u_0]$ ). The norm on  $Y$  being also solid, one deduces:

$$\|x\| \leq 1 \Rightarrow \|U(x)\| \leq \|u_0\| = 1,$$

that is  $U \in \mathcal{B}_1$ . Thus  $\mathcal{B}_1$  is solid, and so is the norm on  $\mathcal{B}(X, Y)$ . This concludes the proof.  $\square$

**Corollary 3.5.** *Let  $X, Y$  be as in Theorem 3.2. Then  $\mathcal{B}(X, Y)$  is Archimedean and has the Riesz decomposition property.*



#### 4. A Markov moment problem and related optimization

This Section starts by recalling briefly one of the earlier extension type results [10] and, on the other hand, by formulating one main problem due to Douglas Todd Norris' PhD Thesis, entitled "Optimal Solutions to the L\_infinity Moment Problem with Lattice Bounds" [7], directed by Professor Emeritus Robert Kent Goodrich. The latter work suggested us the results of this section. One proves a result in a general setting, motivated by a similar problem to that considered in [7] (theorem 4.2 from below). A constrained related optimization problem in infinite dimensional spaces is solved too. The next result refers to the abstract moment problem [10], and is based on constrained extension theorems for linear operators [8], [9]. It will be applied in the sequel.

**Theorem 4.1.** *Let  $\tilde{X}$  be a preordered vector space with its positive cone  $\tilde{X}_+$ ,  $Y$  an order complete vector lattice,  $\{x_j\}_{j \in J} \subset \tilde{X}$ ,  $\{y_j\}_{j \in J} \subset Y$  given families,  $U_1, U_2 \in L(\tilde{X}, Y)$  two linear operators. The following statements are equivalent:*

(a) *there exists a linear operator  $U \in L(\tilde{X}, Y)$  such that*

$$U_1(x) \leq U(x) \leq U_2(x), \quad \forall x \in \tilde{X}_+, \quad U(x_j) = y_j, \quad \forall j \in J;$$

(b) *for any finite subset  $J_0 \subset J$  and any  $\{\lambda_j\}_{j \in J_0} \subset R$ , we have:*

$$\left( \sum_{j \in J_0} \lambda_j x_j = \varphi_2 - \varphi_1, \quad \varphi_1, \varphi_2 \in \tilde{X}_+ \right) \Rightarrow \sum_{j \in J_0} \lambda_j y_j \leq U_2(\varphi_2) - U_1(\varphi_1).$$

In particular, using the latter theorem, one obtains a necessary and sufficient condition for the existence of a feasible solution (see theorem 4.2 from below). Under such condition, the existence of an optimal feasible solution follows too. On the other hand, the uniqueness and the construction of the optimal solution seems to be not obtained easily by such general methods. Therefore, we focus mainly on the existence problem. For other aspects of such problems on an optimal solution (uniqueness or non – uniqueness, construction of a unique solution, etc.), see [7]. In the latter work, one considers the following primal problem (P)

$$v = \inf \left\{ \|\varphi\|_\infty : \varphi \in L_\mu^\infty(X), \int_X \varphi \varphi_j d\mu = b_j, \quad j = 1, 2, \dots, n, \quad 0 \leq \alpha \leq \varphi \leq \beta \right\}$$

where  $\alpha, \beta$  are in  $L^\infty_\mu(X)$ ,  $\{\varphi_j\}_{j=1}^n$  is a subset of  $L^1_\mu(X)$  and  $b = (b_1, b_2, \dots, b_n)^T \in \mathbf{R}^n$ . The function  $\varphi$  is unknown, and in general it is not determined by a finite number of moments. The next theorem generalizes some of the above existence – type results for a feasible solution. Here  $(X, \mathcal{S})$  is a measure space endowed with a  $\sigma$  – finite positive measure  $\mu$ , and  $\mathcal{S}$  is the  $\sigma$  – algebra of all measurable subsets of  $X$ .

**Theorem 4.2.** *Let  $p \in [1, \infty)$  and  $q$  be the conjugate of  $p$ . Let  $\{\varphi_j\}_{j \in J}$  be an arbitrary family of functions in  $L^p_\mu(X)$ , where the measure  $\mu$  is  $\sigma$  – finite, and  $\{b_j\}_{j \in J}$  a family of real numbers. Assume that  $\alpha, \beta \in L^q_\mu(X)$  are such that  $0 \leq \alpha \leq \beta$ . The following statements are equivalent:*

- (a) *there exists  $\varphi \in L^q_\mu(X)$  such that  $\int_X \varphi \varphi_j d\mu = b_j, j \in J, 0 \leq \alpha \leq \varphi \leq \beta$ ;*
- (b) *for any finite subset  $J_0 \subset J$  and any  $\{\lambda_j\}_{j \in J_0} \subset \mathbf{R}$ , the following implication holds*

$$\sum_{j \in J_0} \lambda_j \varphi_j = \psi_2 - \psi_1, \quad \psi_1, \psi_2 \in (L^p_\mu(X))_+ \Rightarrow \sum_{j \in J_0} \lambda_j b_j \leq \int_X \psi_2 \beta d\mu - \int_X \psi_1 \alpha d\mu$$

Moreover, the set of all feasible solutions  $\varphi$  (satisfying the conditions (a)) is weakly compact with respect the dual pair  $(L^p, L^q)$  and the inferior

$$v := \inf \left\{ \|\varphi\|_q : \varphi \in L^q_\mu(X), \int_X \varphi \varphi_j d\mu = b_j, j \in J, 0 \leq \alpha \leq \varphi \leq \beta \right\} \geq \|\alpha\|_q$$

is attained at an optimal feasible solution  $\varphi_0$  at least.

*Proof.* Since the implication (a)  $\Rightarrow$  (b) is obvious, the next step consists in proving that (b)  $\Rightarrow$  (a). Define the real valued linear positive (continuous) forms  $U_1, U_2$  on  $\tilde{X} := L^p_\mu(X)$ , by

$$U_1(\varphi) := \int_X \varphi \alpha d\mu, \quad U_2(\varphi) := \int_X \varphi \beta d\mu, \quad \varphi \in \tilde{X}.$$

Then condition (b) of the present theorem coincides with condition (b) of theorem 4.1. A straightforward application of the latter theorem, leads to the existence of a linear form  $U$  on  $\tilde{X}$ , such that the interpolation conditions  $U(\varphi_j) = b_j, j \in J$  are verified and

$$\int_X \psi \alpha d\mu \leq U(\psi) \leq \int_X \psi \beta d\mu, \quad \psi \in \tilde{X}_+.$$

In particular, the linear form  $\mathcal{U}$  is positive on  $\tilde{X} = L^p_\mu(X)$ , and this space is a Banach lattice (in particular,  $\tilde{X}$  is a complete metric topological vector space and an ordered vector space, whose positive cone  $\tilde{X}_+$  is closed and generating). It is known that on such spaces, any linear positive functional is continuous (cf. [19], ch. V, sect. 5). The conclusion is that  $\mathcal{U}$  can be represented by means of a nonnegative element  $\varphi \in L^q_\mu(X)$ . From the previous relations, we derive

$$\int_X \psi \alpha d\mu \leq \int_X \psi \varphi d\mu \leq \int_X \psi \beta d\mu, \quad \psi \in \tilde{X}_+.$$

Writing these relations for  $\psi = \chi_B$ , where  $B$  is an arbitrary measurable set of positive measure  $\mu(B)$ , one deduces

$$\int_B (\varphi - \alpha) d\mu \geq 0, \quad \int_B (\beta - \varphi) d\mu \geq 0, \quad B \in S, \quad \mu(B) > 0.$$

Then a standard measure theory argument shows that  $\alpha \leq \varphi \leq \beta$  a.e. This finishes the proof of (b)  $\Rightarrow$  (a). To prove the last assertion of the theorem, observe that the set of all feasible solutions is weakly compact by Alaoglu's theorem (it is a weakly closed subset of the closed ball centered at the origin, of radius  $\|\beta\|_q$ ). On the other hand, the norm of any normed linear space is lower weakly semi-continuous. The conclusion is that the norm  $\|\cdot\|_q$  is weakly lower semi-continuous on the weakly (convex) and compact set described at point (a), so that it attains its minimum at a function  $\varphi_0$  of this set. Hence, there exists at least one optimal feasible solution. This concludes the proof.  $\square$

*Remark 4.1.* If the set  $\{\varphi_j\}_{j \in I}$  is total in the space  $L^p_\mu(X)$ , then the set of all feasible solutions is a singleton, so that there exists a unique solution.

## 5. Conclusions

We have proved new results or gave modified statements and proofs for theorems similar to previous ones, by means of earlier theorems on extension and decomposition of linear operators. It is possible that further related applications can be found particularizing the theorems proved above to concrete spaces. An optimization problem related to Markov moment problem is discussed as well.

## Acknowledgement

The authors would like to thank Professor Douglas Todd Norris and Professor Emeritus Robert Kent Goodrich for their kind support in allowing access to the article [7].

## REFERENCES

- [1] *Y. A. Abramovich and A. W. Wickstead*, Regular operators from and into a small Riesz space. *Indag. Mathem.*, N.S. **2** (1991), 3, 257-274.
- [2] *Y. A. Abramovich and A. W. Wickstead*, The regularity of ordered bounded operators into  $C(K)$ . II, *Quart. J. Math. Oxford (2)*, **44** (1993), 257-270.
- [3] *A. W. Wickstead*, Spaces of operators with the Riesz separation property. *Indag. Mathem.*, N.S. **6** (1995), 2, 235-245.
- [4] *N. Dăneş*, The Riesz decomposition property for the space of regular operators. *Proc. Amer. Math. Soc.* **129** (2001), 539-542.
- [5] *N. Dăneş*, The space of regular operators with the Riesz decomposition property. *Rendiconti del Circolo Matematico di Palermo, Serie II, Suppl.* 68 (2002), 373-380.
- [6] *L. Lemnete - Ninulescu and O. Olteanu*, Applications of classical and functional analysis. LAMBERT Academic Publishing, Saarbrücken, 2017.
- [7] *D. T. Norris*, Optimal Solutions to the  $L_{\infty}$  Moment Problem with Lattice Bounds, PhD Thesis. (2002), University of Colorado, Mathematics Department, 2003.
- [8] *O. Olteanu*, Convexité et prolongement d'opérateurs linéaires. *C. R. Acad. Sci. Paris, Série A*, **286**(1978), 511-514.
- [9] *O. Olteanu*, Théorèmes de prolongement d'opérateurs linéaires. *Rev. Roumaine Math. Pures Appl.*, **28**, (1983), 10, 953-983.
- [10] *O. Olteanu*, Application des théorèmes de prolongement d'opérateurs linéaires au problème des moments et à une généralisation d'un théorème de Mazur-Orlicz. *C. R. Acad. Sci. Paris, Série I*, **313**(1991), 739-742.
- [11] *O. Olteanu*, New results on Markov moment problem, *International Journal of Analysis*, Vol. 2013, Article ID 901318, 17 pages, <http://dx.doi.org/10.1155/2013/901318>
- [12] *O. Olteanu*, Moment problems on bounded and unbounded domains, *International Journal of Analysis*, Vol. 2013, Article ID 524264, 7 pages, <http://dx.doi.org/10.1155/2013/524264>
- [13] *O. Olteanu*, Approximation and Markov moment problem on concrete spaces. *Rendiconti del Circolo Matematico di Palermo*, **63** (2014), 161-172, DOI: 10.1007/s12215-014-0149-7
- [14] *O. Olteanu and J. - M. Mihăilă*, Operator – valued Mazur – Orlicz and moment problems in spaces of analytic functions, *U. P. B. Sci. Bull., Series A*, **79**, 1 (2017), 175-184.
- [15] *O. Olteanu*, Markov moment problems in concrete spaces, *Poincare Journal of Analysis & Applications*, Vol. 2017 (1), 35 – 42.
- [16] *R. Cristescu*, Ordered Vector Spaces and Linear Operators. Academiei, Bucharest, Romania and Abacus Press, Tunbridge Wells, Kent, England, 1976.
- [17] *P. Meyer-Nieberg*, Banach Lattices. Springer-Verlag, Berlin Heidelberg New York, 1991
- [18] *C. Niculescu and N. Popa*, Elemente de Teoria Spațiilor Banach (Elements of Theory of Banach Spaces). Ed. Academiei, Bucharest, 1981.
- [19] *H. H. Schaefer*, Topological Vector Spaces. Third Printing Corrected. Springer – Verlag, New York Heidelberg Berlin, 1971.