BOUNDARY VALUE PROBLEMS OF A CLASS OF NONLINEAR DIFFERENCE EQUATIONS

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This article is concerned with the existence of solutions of boundary value problems for a class of nonlinear difference equations. We apply the Saddle Point Theorem in the critical point theory and give new results for the existence of solutions.

Keywords: boundary value problems, nonlinear, difference equations, Saddle Point Theorem, variational methods

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1. Introduction

Below **N**, **Z** and **R** denote the sets of all natural numbers, integers and real numbers, respectively. For any a, $b \in \mathbf{Z}$, define $\mathbf{Z}(a) = \{a, a+1, \cdots\}$, $\mathbf{Z}(a, b) = \{a, a+1, \cdots, b\}$ when $a \leq b$. Besides, * denotes the transpose of a vector.

Consider the discrete boundary value problem (BVP, for short):

$$\Delta\left(p_n(\Delta x_{n-1})^{\delta}\right) + f_n(x_n) = 0,\ n \in \mathbf{Z}(1, k), \tag{1}$$

with the boundary value conditions

$$\alpha x_0 - \beta\Delta x_0 = 0 = \gamma x_{k+1} + \sigma\Delta x_k, \tag{2}$$

where Δ is the forward difference operator, $\Delta x_n = x_{n+1} - x_n$, $\delta > 0$ is the ratio of odd positive integers, i.e., for any two odd numbers $\mu \in \mathbf{Z}$, $\nu \in \mathbf{Z}$, $\mu > 0$ and $\nu > 0$, $\delta = \frac{\mu}{\nu}$, p_n is real valued for each $n \in \mathbf{Z}$, k is a given positive integer, α, β, γ and σ are constants, $f_n(y) \in C(\mathbf{R}, \mathbf{R})$. The boundary value problem (1), (2) contains the following special Dirichlet boundary value conditions, Mixed boundary value conditions and Neumann boundary value conditions:

$$x_0 = 0,\ x_{k+1} = 0;$$
$$x_0 = 0,\ \Delta x_k = 0;$$
$$\Delta x_0 = 0,\ x_{k+1} = 0;$$
$$\Delta x_0 = 0,\ \Delta x_k = 0.$$

Eq. (1) can be considered as a discrete analogue of the following second-order differential equation

$$(q(t)\varphi_p(x'))' + f(t, x(t)) = 0,\ t \in \mathbf{R}. \tag{3}$$

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Eq. (3) includes the following equation

$$(q(t)\psi(x'))' + f(t, x(t)) = 0,\ t \in \mathbf{R}, \tag{4}$$

which arises in fluid dynamics, combustion theory, gas diffusion through porous media, thermal self-ignition of a chemically active mixture of gases in a vessel, catalysis theory, chemically reacting systems, and adiabatic reactor [2]. Equations similar in structure to (4) arise in the study of periodic solutions and homoclinic orbits [8, 9, 10, 11] of differential equations.

Difference equations [1, 5, 6, 7, 12, 13, 14, 15, 16, 17, 18, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31], the discrete analogue of differential equations, occur widely in numerous settings and forms, both in mathematics itself and in its applications to finance insurance, biological populations, disease control, genetic study, physical field, and computer application technology, etc. For the general background of difference equations, one can refer to the monographs [1, 7, 21].

By using the Symmetric Mountain Pass Theorem, Chen and Wang [5] established some existence criteria to guarantee that a class of nonlinear difference equation of the type

$$\Delta\left(p_n(\Delta x_{n-1})^{\delta}\right) - q_n\Delta x_n^{\delta} + f_n(x_n) = 0,\ n \in \mathbf{Z}, \tag{5}$$

has infinitely many homoclinic orbits. Here, $f_n(y) \in C(\mathbf{R}, \mathbf{R})$.

Deng [6] established some sufficient conditions for the existence of the solution to boundary value problem of the equation

$$\Delta^2\left(p_{n-1}\Delta^2 x_{n-2}\right) - \Delta\left(q_n\left(\Delta x_{n-1}\right)^{\delta}\right) + r_n x_n^{\delta} = f_n(x_n), \tag{6}$$

and gave some new results by using the critical point theory. Here, $f_n(y) \in C(\mathbf{R}, \mathbf{R})$.

Liu, Zhang and Shi [15], by making use of the saddle point theorem in combination with variational technique, solved the existence of periodic solutions of a class of nonlinear difference equation

$$\Delta\left(p_n(\Delta x_{n-1})^{\delta}\right) + f_n(x_{n+1}, x_n, x_{n-1}) = 0,\ n \in \mathbf{Z}, \tag{7}$$

where $f_n(y_1, y_2, y_3) \in C(\mathbf{R}^3, \mathbf{R})$.

The existence of solutions to boundary value problem of difference equations has been a very active area of research in the last twenty years, and for surveys of resent results, we refer the reader to the references [1, 15, 17, 18, 21, 28]. However, it seems that results on the existence of solutions of boundary value problems (1), (2) by critical point method are very scarce in the literature [1, 15, 17, 18, 21, 28]. The main purpose of this paper is to develop a new approach to the above problem by using critical point theory. We transfer the existence of the BVP (1), (2) into the existence of the critical points of some functional. The proof is based on the Saddle Point Theorem in combination with variational technique. The motivation for the present work stems from the recent papers [3, 4, 28].

For basic knowledge of variational methods, the reader is referred to [19, 20].

Let

$$F_n(y) = \int_0^y f_n(s)ds \geq 0, n \in \mathbf{Z}, \tag{8}$$

and

$$p_{\min} = \min\left\{p_n :\ n \in \mathbf{Z}(1, k+1)\right\},\ p_{\max} = \max\left\{p_n :\ n \in \mathbf{Z}(1, k+1)\right\}.$$

For convenience, we identify $x \in \mathbf{R}^k$ with $x = (x_1, x_2, \cdots, x_k)^*$.

2. **Preliminaries**

Our main tool is the critical point theory. We shall establish suitable variational structure to study the existence of the boundary value problem (1), (2). At first, we introduce some basic notations which will be used in the proofs of our main results.

Let $\mathbf{R}^k$ be the real Euclidean space with dimension k. On one hand, we define the inner product on $\mathbf{R}^k$ as follows:

$$\langle x, y\rangle = \sum_{j=1}^{k} x_j y_j,\ \forall x, y \in \mathbf{R}^k, \tag{9}$$

which induces the norm $\|\cdot\|$,

$$\|x\| = \left(\sum_{j=1}^{k} x_j^2\right)^{\frac{1}{2}},\ \forall x \in \mathbf{R}^k. \tag{10}$$

On the other hand, we define the norm $\|\cdot\|_s$ on $\mathbf{R}^k$ as follows:

$$\|x\|_s = \left(\sum_{j=1}^{k} |x_j|^s\right)^{\frac{1}{s}},$$

for all $x \in \mathbf{R}^k$ and $s > 1$.

Since $\|x\|_s$ and $\|x\|_2$ are equivalent, there exist constants $c_1,\ c_2$ such that $k_2 \geq k_1 > 0$, and

$$k_1\|x\|_2 \leq \|x\|_s \leq k_2\|x\|_2,\ \forall x \in \mathbf{R}^k. \tag{11}$$

For the boundary value problem (1), (2), consider the real-valued function J defined on $\mathbf{R}^k$ as follows:

$$J(x) = \frac{1}{\delta+1}\sum_{n=1}^{k+1} p_n(\Delta x_{n-1})^{\delta+1} - \sum_{n=1}^{k} F_n(x_n) + \left(\frac{\gamma}{\sigma}\right)^{\delta}\frac{p_{k+1}x_{k+1}^{\delta+1}}{\delta+1} + \left(\frac{\alpha}{\beta}\right)^{\delta}\frac{p_1 x_0^{\delta+1}}{\delta+1}, \tag{12}$$

where

$$x = \{x_n\}_{n=1}^{k} = (x_1, x_2, \ldots, x_k)^*,\ \alpha x_0 - \beta\Delta x_0 = 0 = \gamma x_{k+1} + \sigma\Delta x_k.$$

It is easy to see that $J \in C^1(\mathbf{R}^k, \mathbf{R})$ and for any $x = \{x_n\}_{n=1}^{k} = (x_1, x_2, \ldots, x_k)^*$, by using $\alpha x_0 - \beta\Delta x_0 = 0 = \gamma x_{k+1} + \sigma\Delta x_k$ and the summation by parts

$$\sum_{n=1}^{k+1} y_n\Delta x_{n-1} = y_{k+1}x_{k+1} - y_1 x_0 - \sum_{n=1}^{k}\Delta y_n x_n,$$

we can compute the partial derivative as

$$\frac{\partial J}{\partial x_n} = -\Delta\left(p_n(\Delta x_{n-1})^{\delta}\right) - f_n(x_n),\ \forall n \in \mathbf{Z}(1, k).$$

Thus, x is a critical point of J on $\mathbf{R}^k$ if and only if

$$\Delta\left(p_n(\Delta x_{n-1})^{\delta}\right) + f_n(x_n) = 0,\ \forall n \in \mathbf{Z}(1, k).$$

We reduce the existence of the boundary value problem (1), (2) to the existence of critical points of J on $\mathbf{R}^k$. That is, the function J is just the variational framework of the boundary value problem (1), (2).

Let P be the $(k+1)\times(k+1)$ matrix given by

$$P=\begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0\\ -1 & 2 & -1 & \cdots & 0 & 0\\ 0 & -1 & 2 & \cdots & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & \cdots & 2 & -1\\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix}. \tag{13}$$

It is easy to see that 0 is an eigenvalue of P. Let $\lambda_1,\lambda_2,\cdots,\lambda_k$ be the other eigenvalues of P. Applying matrix theory, one can prove that $\lambda_j>0$ for all $j\in\mathbf{Z}(1,k)$.

Denote by Z the real space of P, given by

$$Z=\left\{(x_1,x_2,\cdots,x_k)^*\in\mathbf{R}^k|x_i=c,\ c\in\mathbf{R},\ i\in\mathbf{Z}(1,k)\right\},$$

and let Y be the direct orthogonal complement of Z, i.e., $\mathbf{R}^k=Y\oplus Z$.

3. Main results

In this section, we shall state and prove our main results via the critical point method.

Theorem 3.1. Assume that the following assumptions are satisfied:
(B_1) $\alpha\geq 0,\ \beta>0,\ \gamma\geq 0,\sigma>0$ and $\delta>0$ is the ratio of odd positive integers;
(p) for any $n\in\mathbf{Z}(1,k+1)$, $p_n>0$;
(F_1) there exist constants $c_1>0,\ c_2>0$ and $\vartheta>\delta+1$ such that for any $n\in\mathbf{Z}$,

$$F_n(y)\geq c_1|y|^\vartheta-c_2,\ \forall y\in\mathbf{R},$$

where $F_n(y)$ is referred to (8). Then the boundary value problem (1), (2) possesses at least one solution.

Proof. For any $x=(x_1,x_2,\cdots,x_k)^*\in\mathbf{R}^k$, and with $J(x)$ given by (12), it comes from (F_1) that

$$J(x)\leq\frac{p_{\max}2^{\delta+1}}{\delta+1}\sum_{n=1}^{k+1}\left(x_n^{\delta+1}+x_{n-1}^{\delta+1}\right)+\left(\frac{\gamma}{\sigma}\right)^\delta\frac{p_{\max}\left\|x\right\|_{\delta+1}^{\delta+1}}{\delta+1}+\left(\frac{\alpha}{\beta}\right)^\delta\frac{p_{\max}\left\|x\right\|_{\delta+1}^{\delta+1}}{\delta+1}-c_1\sum_{n=1}^{k}|x_n|^\vartheta+c_2k$$

$$\leq\frac{3p_{\max}2^{\delta+1}k_2^{\delta+1}}{\delta+1}\|x\|^{\delta+1}+\left(\frac{\gamma}{\sigma}\right)^\delta\frac{p_{\max}k_2^{\delta+1}\left\|x\right\|^{\delta+1}}{\delta+1}+\left(\frac{\alpha}{\beta}\right)^\delta\frac{p_{\max}k_2^{\delta+1}\left\|x\right\|^{\delta+1}}{\delta+1}-c_1\sum_{n=1}^{k}|x_n|^\vartheta+c_2k$$

$$\leq\frac{1}{\delta+1}\left[3\cdot 2^{\delta+1}+\left(\frac{\gamma}{\sigma}\right)^\delta+\left(\frac{\alpha}{\beta}\right)^\delta\right]k_2^{\delta+1}\left\|x\right\|^{\delta+1}-c_1k_1^\vartheta\|x\|^\vartheta+c_2k\to-\infty$$

as $\|x\|\to+\infty$. By the continuity of $J(x)$, we have from the above inequality that there exist upper bounds for the function J. Classical calculus shows that J attains its maximum value at some point which is just a critical point of J and the result follows. □

Theorem 3.2. Assume that (p), (F_1) and the following assumptions are satisfied:
(B_2) $\alpha=0,\ \beta>0,\ \gamma=0,\ \sigma>0$;
(F_2) there exist a constant $K_1>0$ such that for any $y\in\mathbf{R}$, $|f_n(y)|\leq K_1$;
(F_3) for any $n\in\mathbf{Z}$, $F_n(y)\to+\infty$ as $|y|\to+\infty$,
where $F_n(y)$ is referred to (8). Then the boundary value problem (1), (2) possesses at least one solution.

Remark 3.1. Assumption (F_2) implies that there exists a constant $K_2>0$ such that
(F_2') for any $n\in\mathbf{Z}$, $|F_n(y)|\leq K_2+K_1|y|,\ \forall y\in\mathbf{R}$.

Let V and W be Banach spaces, and $U \subset V$ be an open subset of V. A function $f : U \to W$ is called Fréchet-differentiable at $x \in U$ if there exists a bounded linear operator $A_x : V \to W$ such that

$$\lim_{h\to 0} \frac{\|f(x+h) - f(x) - A_x(h)\|_W}{\|h\|_V} = 0.$$

We write $Df(x) = A_x$ and call it the Fréchet derivative of f at x. Let E be a real Banach space, $J \in C^1(E, \mathbf{R})$, i.e., J is a continuously Fréchet-differentiable functional defined on E. J is said to satisfy the Palais-Smale condition (P.S. condition for short) if any sequence $\{x^{(l)}\}_{l\in\mathbf{N}} \subset E$ for which $\{J(x^{(l)})\}_{l\in\mathbf{N}}$ is bounded and $J'(x^{(l)}) \to 0$ $(l \to \infty)$ possesses a convergent subsequence in E.

Let B_ρ denote the open ball in E about 0 of radius ρ, ∂B_ρ denote its boundary and $\bar{B}_\rho$ denote its closure.

We first prove two basic lemmas useful to the proof of Theorem 3.2.

Lemma 3.1. *(Saddle Point Theorem* [19, 20]*)* Let E be a real Banach space, $E = E_1 \oplus E_2$, where $E_1 \neq \{0\}$ and is finite dimensional. Assume that $J \in C^1(E, \mathbf{R})$ satisfies the P.S. condition and
(J_1) there exist constants $\mu,\ \rho > 0$ such that $J|_{\partial B_\rho \cap E_1} \leq \mu$;
(J_2) there exists $e \in B_\rho \cap E_1$ and a constant $\omega \geq \mu$ such that $J|_{e+E_2} \geq \omega$.
Then J possesses a critical value $c \geq \omega$, where

$$c = \inf_{h\in\Gamma} \max_{x\in B_\rho\cap E_1} J(h(x)),$$

$$\Gamma = \{h \in C(\bar{B}_\rho \cap E_1, E) \mid h|_{\partial B_\rho \cap E_1} = id\}$$

and id denotes the identity operator.

Lemma 3.2. Assume that (p), (B_2), (F_1), (F_2) and (F_3) are satisfied. Then the function J satisfies the P.S. condition.

Proof. By (B_2), we have

$$J(x) = \frac{1}{\delta+1} \sum_{n=1}^{k+1} p_n (\Delta x_{n-1})^{\delta+1} - \sum_{n=1}^{k} F_n(x_n) := H(x) - \sum_{n=1}^{k} F_n(x_n), \tag{14}$$

where

$$H(x) = \frac{1}{\delta+1} \sum_{n=1}^{k+1} p_n (\Delta x_{n-1})^{\delta+1}.$$

Let $\{x^{(l)}\}_{l\in\mathbf{N}} \subset \mathbf{R}^k$ be such that $\{J(x^{(l)})\}_{l\in\mathbf{N}}$ is bounded and $J'(x^{(l)}) \to 0$ as $l \to \infty$. Then there exists a positive constant D such that

$$-D \leq \left| J\left(x^{(l)}\right) \right| \leq D,\ \forall l \in \mathbf{N}.$$

Let

$$\lambda_{\max} = \max\{\lambda_j | j = 1, 2, \cdots, k\}, \tag{15}$$

$$\lambda_{\min} = \min\{\lambda_j | j = 1, 2, \cdots, k\}, \tag{16}$$

whre $\lambda_1, \lambda_2, \cdots, \lambda_k$ are the positive eigenvalues of matrix P defined as (13).

Let $x^{(l)} = y^{(l)} + z^{(l)} \in Y + Z$. First, on one hand, due to $J'(x^{(l)}) \to 0$ as $l \to \infty$, for large enough l, from

$$-\|x\| \leq \left\langle -J'\left(x^{(l)}\right), x \right\rangle = -\left\langle H'\left(x^{(l)}\right), x \right\rangle + \sum_{n=1}^{k} f_n\left(x_n^{(l)}\right) x_n,$$

with (F_2) it follows that

$$\left\langle H'\left(x^{(l)}\right), y^{(l)}\right\rangle \leq \sum_{n=1}^{k} f_n\left(x_n^{(l)}\right) y_n^{(l)} + \left\|y^{(l)}\right\| \leq K_1 \sum_{n=1}^{k}\left|y_n^{(l)}\right| + \left\|y^{(l)}\right\| \leq \left(K_1\sqrt{k}+1\right)\left\|y^{(l)}\right\|.$$

On the other hand, it is easy to see that

$$\left\langle H'\left(x^{(l)}\right), y^{(l)}\right\rangle = \left\langle H'\left(y^{(l)}\right), y^{(l)}\right\rangle \geq p_{\min} k_1^{\delta+1}\left\langle Py^{(l)}, y^{(l)}\right\rangle^{\delta+1} \geq p_{\min}\lambda_{\min}^{\frac{\delta+1}{2}} k_1^{\delta+1}\left\|y^{(l)}\right\|^{\delta+1}.$$

Thus, we have

$$p_{\min}\lambda_{\min}^{\frac{\delta+1}{2}} k_1^{\delta+1}\left\|y^{(l)}\right\|^{\delta+1} \leq \left(K_1\sqrt{k}+1\right)\left\|y^{(l)}\right\|.$$

The above inequality implies that $\left\{y^{(l)}\right\}_{l\in\mathbf{N}}$ is bounded.

Next, we shall prove that $\left\{z^{(l)}\right\}_{l\in\mathbf{N}}$ is bounded. It comes from

$$D \geq -J\left(x^{(l)}\right) = -H\left(x^{(l)}\right) + \sum_{n=1}^{k}\left[F_n\left(x_n^{(l)}\right) - F_n\left(z_n^{(l)}\right)\right] + \sum_{n=1}^{k} F_n\left(z_n^{(l)}\right),$$

that

$$\begin{aligned}\sum_{n=1}^{k} F_n\left(z_n^{(l)}\right) &\leq D + H\left(x^{(l)}\right) + \sum_{n=1}^{k}\left[F_n\left(x_n^{(l)}\right) - F_n\left(z_n^{(l)}\right)\right] \\ &\leq D + \frac{p_{\max}}{\delta+1} k_2^{\delta+1}\left\langle Py^{(l)}, y^{(l)}\right\rangle^{\delta+1} + \sum_{n=1}^{k}\left|f_n\left(z_n^{(l)} + \alpha y_n^{(l)}\right) y_n^{(l)}\right| \\ &\leq D + \frac{p_{\max}}{\delta+1} k_2^{\delta+1}\lambda_{\max}^{\frac{\delta+1}{2}}\left\|y^{(l)}\right\|^{\delta+1} + K_1 k\left\|y^{(l)}\right\|,\end{aligned}$$

where $\alpha \in (0,1)$. It is not difficult to see that

$$\sum_{n=1}^{k} F_n\left(z_n^{(l)}\right) \tag{17}$$

is bounded.

Assumption (F_3) implies that $\left\{z^{(l)}\right\}$ is bounded. Assume by contradiction (17) that $\left\|z^{(l)}\right\| \to +\infty$ as $l \to \infty$. Since there exist $c^{(l)} \in \mathbf{R}$, $l \in \mathbf{N}$, such that $z^{(l)} = \left(c^{(l)}, c^{(l)}, \cdots, c^{(l)}\right)^* \in \mathbf{R}^k$, then

$$\left\|z^{(l)}\right\| = \left(\sum_{n=1}^{k}\left|z_k^{(l)}\right|^2\right)^{\frac{1}{2}} = \left(\sum_{n=1}^{k}\left|c^{(l)}\right|^2\right)^{\frac{1}{2}} = \sqrt{k}\left|c^{(l)}\right| \to +\infty$$

as $l \to \infty$. Since

$$F_n\left(z_n^{(l)}\right) = F_n\left(c^{(l)}\right),\ 1 \leq n \leq k,$$

then $F_n\left(z_n^{(l)}\right) \to +\infty$ as $l \to \infty$. This contradicts the fact that the sum (17) is bounded and hence the proof of Lemma 3.2 is completed. □

Proof of Theorem 3.2. The proof of Theorem 3.2 is similar to that of [[18], Theorem 1.2], but for the sake of completeness, we give the details.

First, it follows from Lemma 3.2 that J satisfies the P.S. condition. Next, we shall prove the conditions (J_1) and (J_2). For any $y \in Y$, by (F_2') we have

$$-J(y) = -H(y) + \sum_{n=1}^{k} F_n(y_n)$$

$$\le -\frac{p_{\min}}{\delta+1}k_1^{\delta+1}\lambda_{\min}^{\frac{\delta+1}{2}}\|y\|^{\delta+1} + kK_2 + K_1\sum_{n=1}^{k}|y_n|$$
$$\le -\frac{p_{\min}}{\delta+1}k_1^{\delta+1}\lambda_{\min}^{\frac{\delta+1}{2}}\|y\|^{\delta+1} + kK_2 + K_1k\|y\| \to -\infty$$

as $\|y\| \to +\infty$. Therefore, one can see that the condition (J_1) is satisfied. For any $z \in Z$, $z = (z_1, z_2, \cdots, z_k)^*$, there exists $c \in \mathbf{R}$ such that $z_n = c$, for all $n \in \mathbf{Z}(1, k)$. From (F_3), we have that there exists a constant $\rho > 0$ such that $F_n(c) > 0$ for $n \in \mathbf{Z}(1, k)$ and $|c| > \rho$. Let $K_3 = \min_{n\in\mathbf{Z}(1,k),\ |c|\le\rho} F_n(c)$, $K_4 = \min\{0, K_3\}$. Then

$$F_n(c) \ge K_4,\ \forall (n, c) \in \mathbf{Z}(1, k) \times \mathbf{R}^2.$$

Therefore, we have

$$-J(z) = \sum_{n=1}^{k} F_n(z_n) = \sum_{n=1}^{k} F_n(c) \ge TK_4,\ \forall z \in Z.$$

So all the assumptions of the Saddle Point Theorem are satisfied and the proof of Theorem 3.2 is complete. □

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