U.P.B. Sci. Bull., Series A, Vol. 77, Iss. 4, 2015 ISSN 1223-7027

SOME NOTIONS OF AMENABILITY OF WEIGHTED SEGAL ALGEBRAS

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This paper considers amenability and weak amenability of Beurling Segal algebras, Beurling group algebras and Lebesgue weighted L^p-algebras on locally compact groups. We show that approximate amenability of Beurling Segal algebra implies that amenability of Beurling group algebra $L^1_\omega(G)$. Furthermore, character amenability of the mentioned Banach algebras is considered.

Keywords: amenable group, approximately amenable, Beurling algebra, Segal algebras, weak amenability.

MSC2010: Primary 46H20; Secondary 43A20.

1. Introduction

Let $\mathcal{A}$ be a Banach algebra and X be a Banach $\mathcal{A}$-bimodule. A derivation $D : \mathcal{A} \longrightarrow X$ is a linear map which satisfies $D(ab) = a\cdot D(b)+D(a)\cdot b$ for all $a, b \in \mathcal{A}$. A derivation δ is said to be inner if there is a $x \in X$ such that $\delta(a) = \delta_x(a) = a\cdot x-x\cdot a$ for all $a \in \mathcal{A}$, and it is called approximately inner if there is a net $(x_\alpha)_\alpha$ in X such that $\delta(a) = \lim_\alpha a\cdot x_\alpha-x_\alpha\cdot a$. Let $Z^1(\mathcal{A}, X)$ be the space of all continuous derivations from $\mathcal{A}$ into X and let $N^1(\mathcal{A}, X)$ be the space of all inner derivations from $\mathcal{A}$ into X. We consider the quotient space $H^1(\mathcal{A}, X) = Z^1(\mathcal{A}, X)/N^1(\mathcal{A}, X)$, which is called the *first Hochschild cohomology* group of $\mathcal{A}$ with coefficients in X.

The concept of amenability for Banach algebras was introduced by Johnson in 1972 [19]. The Banach algebra $\mathcal{A}$ is said to be amenable if $H^1(\mathcal{A}, X^*) = \{0\}$ for all Banach $\mathcal{A}$-bimodule X, where X^* is the first dual of X. The concept of weak amenability of Banach algebras was first introduced by Bade, Curtis and Dales in [2] for commutative Banach algebras. The Banach algebra $\mathcal{A}$ is called weakly amenable if $H^1(\mathcal{A}, \mathcal{A}^*) = \{0\}$. The Banach algebra $\mathcal{A}$ is called n-weakly amenable ($n \in \mathbb{N} \cup \{0\}$) or permanently weakly amenable if $H^1(\mathcal{A}, \mathcal{A}^{(n)}) = \{0\}$, where $\mathcal{A}^{(n)}$ is the n-th dual of Banach algebra $\mathcal{A}$. Dales, Ghahramani and Grønbæk in [9], showed that C^*-algebras are permanently weakly amenable. Permanent weak amenability of $L^1(G)$ for any locally compact group G is proved by Choi, Ghahramani and Zhang in [6]. The Banach algebra $\mathcal{A}$ is called approximately weak amenable if every bounded derivation from $\mathcal{A}$ into $\mathcal{A}^{(n)}$ is approximately inner.

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Let G be a locally compact group with identity e. A weight on G is a continuous function $\omega : G \longrightarrow (0, \infty)$ such that

$$\omega(st) \leq \omega(s)\omega(t), \quad \omega(e) = 1,$$

for all $s, t \in G$. For $1 \leq p < \infty$, a Beurling space on a locally compact group G is defined as follows

$$L^p_\omega(G) := \{f \ : \ f \in L^p(G), f \cdot \omega \in L^p(G)\}.$$

For $n = 1$, $L^1_\omega(G)$ is the set of all $\lambda-$measurable (λ is the Haar measure on G) functions f such that $\int_G |f(t)|\omega(t)d\lambda(t) < \infty$. For abbreviation we write dt instead of $d\lambda(t)$. $L^1_\omega(G)$ becomes a Banach algebra with convolution multiplication

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)dy,$$

and with the following norm

$$\|f\| = \int_G |f(t)|\omega(t)dt.$$

The augmentation ideal of $L^1_\omega(G)$ is $I_0 = \{f \in L^1_\omega(G) : \int_G f(x)dx = 0\}$ which is of codimension one. The space $L^\infty_{1/\omega}(G)$ is the set of all measurable functions φ on G, such that $\varphi/\omega \in L^\infty_\omega(G)$ and with the norm $\|.\|_\infty$ defined by

$$\|\varphi\|_\infty = \sup_{x \in G} \frac{|\varphi(x)|}{\omega(x)}$$

is a Banach algebra. $L^\infty_{1/\omega}(G)$ is the dual of $L^1_\omega(G)$. For more details see [10]. It is known that $L^1_\omega(G)$ is a closed ideal of

$$M_\omega(G) = \{\mu \ : \ \mu \in M(G), \int_G \omega \ d|\mu| < \infty\}.$$

We denote the set of all continuous functions on G with compact support by $C_C(G)$. Then, for each $f \in C_C(G)$, f is $\lambda-$measurable. Moreover, $C_C(G)$ is norm-dense in $L^1_\omega(G)$.

Weak amenability of Beurling algebras has been studied in [2] and [16]. When $G = \mathbb{Z}$, $\ell^1_\omega(\mathbb{Z})$ is weakly amenable if and only if $\inf_n \frac{\omega(n)\omega(-n)}{n} = 0$ [16]. Dales and Lau showed that if $\omega \geq 1$ is almost invariant and satisfies $\inf_n \frac{\omega(nt)}{n} = 0$ for all $t \in G$, then $L^1_\omega(G)$ is 2-weakly amenable (Theorem 13.8, [10]), and conjectured that $L^1_\omega(G)$ is 2-weakly amenable if one only assumes that $\inf_n \frac{\omega(nt)}{n} = 0$ for all $t \in G$. For a weight $\omega \geq 1$, let $\omega_1(s) = \limsup_{t \longrightarrow \infty} \frac{\omega(t+s)}{\omega(t)}$. Ghahramani and Zabandan proved that $L^1_\omega(G)$ is 2-weakly amenable if $\inf\{\frac{\omega(nt)}{n} | n \in \mathbb{N}\} = 0$ and ω_1 is bounded [15]. Samei [24], showed that there exists weight σ_ω on G which is closely related to ω_1 such that σ_ω is bounded if and only if ω_1 is bounded and he studied 2-weakly amenability of $L^1_\omega(G)$, when $\omega \geq 1$ and σ_ω is bounded.

Approximate amenability of Banach algebras were introduced by Ghahramani and Loy in [13]. The Banach algebra $\mathcal{A}$ is called approximately amenable if for every Banach $\mathcal{A}$-bimodule X and every bounded derivation $D : \mathcal{A} \to X^*$ there is a net (x_α) of in X^* such that $D(a) = \lim_\alpha a \cdot x_\alpha - x_\alpha \cdot a$ for all $a \in \mathcal{A}$, and it is boundedly approximately amenable if (x_α) is bounded net. The Banach algebra $\mathcal{A}$ is boundedly approximately contractible, if for every Banach $\mathcal{A}$-bimodule X and derivation $D : \mathcal{A} \longrightarrow X$, there is a bounded net $(x_\alpha)_\alpha \subseteq X$ such that $D(a) = \lim_\alpha a \cdot x_\alpha - x_\alpha \cdot a$, for every $a \in \mathcal{A}$.

2. Amenability and Approximate Amenability

Cigler [7] introduced the notion of Weighted Segal algebra. Let $S^1_\omega(G)$ be a subalgebra of $L^1_\omega(G)$. Then $S^1_\omega(G)$ is called a weighted Segal algebra, if it satisfies the following conditions:

S1) $S^1_\omega(G)$ is dense in $L^1_\omega(G)$.
S2) $S^1_\omega(G)$ is a Banach algebra under some norm $\|\cdot\|_{S^1_\omega}$, and invariant under translations.
S3) $\|L_a f\|_{S^1_\omega} \leq \omega(a)\|f\|_{S^1_\omega}$ for all $a \in G$ and every $f \in S^1_\omega(G)$.
S4) For every $f \in S^1_\omega(G)$ and $\varepsilon > 0$, there is a neighborhood U of the unit element $e \in G$ such that $\|L_y f - f\|_{S^1_\omega} < \varepsilon$ for every $y \in U$.
S5) $\|f\|_{1,\omega} \leq \|f\|_{S^1_\omega}$ for every $f \in S^1_\omega(G)$.

Proposition 2.1. *Let ω be a weight on G, and $\Omega(t) = \omega(t)\omega(t^{-1})$. Then $S^1_\omega(G)$ has a bounded approximate identity if and only if $S^1_\omega(G) = L^1_\omega(G)$. Furthermore, $S^1_\Omega(G)$ has a bounded approximate identity if and only if $S^1_\Omega(G) = L^1_\Omega(G)$.*

Proof. Suppose that $S^1_\omega(G)$ has a bounded approximate identity. Then Burnham's Theorem ([5, Theorem 1.2]) implies that $S^1_\omega(G) = L^1_\omega(G)$ (algebraically). Let $(e_\alpha)_\alpha$ be a bounded approximate identity for $L^1_\omega(G)$. Without loss of generality assume that $(e_\alpha)_\alpha$ is bounded by 1. For every $f, g \in L^1_\omega(G)$, there exists $C > 0$ such that $\|f * g\|_{S^1_\omega} \leq C\|f\|_{1,\omega}\|g\|_{S^1_\omega}$. Then for every $f \in L^1_\omega(G)$ we have

$$\|f\|_{1,\omega} = \lim_\alpha \|e_\alpha * f\|_{1,\omega} \leq C \lim_\alpha \|e_\alpha\|_{1,\omega}\|f\|_{S^1_\omega} \leq C\|f\|_{S^1_\omega}.$$

Therefore $S^1_\omega(G)$ is isomorphic to $L^1_\omega(G)$. The converse is trivial. □

Corollary 2.1. *$S^1_\omega(G)$ is amenable if and only if $S^1_\omega(G) = L^1_\omega(G)$.*

Generally, for every Banach algebra $\mathcal{A}$ and abstract Segal algebra $\mathcal{B}$ in $\mathcal{A}$, if I is a closed left (right) ideal of $\mathcal{A}$, then $I \cap \mathcal{B}$ is a closed left (right) ideal of $\mathcal{B}$. Moreover, if J is a closed left (right) ideal of $\mathcal{B}$, the closure of J in $\mathcal{A}$ is a closed left (right) ideal of $\mathcal{A}$ (for more details see [21, 23]). We rewrite Willis Theorem ([25, Theorem 5.2]) as follows:

Theorem 2.1. *Let ω be a weight on G and $\Omega(t) = \omega(t)\omega(t^{-1})$. If Ω is bounded on non-amenable locally compact group G, then no finite codimensional, closed, left ideal in $L^1_\Omega(G)$ has multiple right approximate identity.*

Theorem 2.2. *Let ω be a weight on G and $\Omega(t) = \omega(t)\omega(t^{-1})$. If $S^1_\Omega(G)$ is approximately amenable (pseudo-amenable) and Ω is bounded on G, then $L^1_\omega(G)$ and $L^1_\Omega(G)$ are amenable.*

Proof. According to Theorem 0 of [17], it is sufficient to show that G is amenable. Assume that G is not amenable. Let I_0 be the augmentation ideal in $L^1_\Omega(G)$. Set $J = I_0 \cap S^1_\Omega(G)$. Therefore J is a maximal ideal in $S^1_\Omega(G)$. Since $S^1_\Omega(G)$ is approximately amenable (pseudo-amenable) then by lemma 2.2 of [13], J has an approximate identity. Then by Proposition 5.4 of [6], I_0 has an approximate identity which by Theorem 2.1 is a contradiction. Therefore G is amenable, and the proof is complete. □

Ghahramani et al. proved that for every boundedly approximately contractible Banach algebra, every closed two-sided ideal of codimension one has a bounded approximate identity [14, Theorem 2.1]. It is known that the augmentation ideal $I_0(G)$ has a bounded approximate identity if and only if G is amenable. Consider the non-amenable locally compact group $SL(2,\mathbb{R})$ (group of all 2×2 matrices on $\mathbb{R}$ with determinant 1). Therefore $L^1(SL(2,\mathbb{R}))$ is not boundedly approximately contractible. Moreover, in light of Corollary 2.2 of [14], for any weight ω on $SL(2,\mathbb{R})$, $L^1_\omega(SL(2,\mathbb{R}))$ is not boundedly approximately contractible. Furthermore by Theorem 2.2 we have the following result.

Corollary 2.2. *$S^1_\omega(SL(2,\mathbb{R}))$ and $S^1(SL(2,\mathbb{R}))$ (ordinary Segal algebra) are not approximately amenable.*

Let G be a locally compact group, and ω be a weight on G. Abtahi in [1], introduced a new class of Lebesgue Banach algebras $\mathcal{L}^{1,p}_\omega(G) := L^1(G)\cap L^p_\omega(G)$, $1\leq p<\infty$, and studied amenability of these algebras. He proved that for $1<p<\infty$, amenability of $\mathcal{L}^{1,p}_\omega(G)$ implies that G is discrete and amenable ([1, Theorem 3.1]), and the converse is true when ω is bounded on G ([1, Corollary 3.2]).

As we know amenability of $L^1_\omega(G)$ implies amenability of $L^1_\Omega(G)$ and vice versa, for any locally compact group G, but this not true for $\mathcal{L}^{1,p}_\omega(G)$. Then in light of Theorem 3.1 and Corollary 3.2 of [1], we have

Proposition 2.2. *Let ω be a weight on G, $\Omega(t)=\omega(t)\omega(t^{-1})$ and $1<p<\infty$. Then*

(i) *Let ω be bounded on G and $\mathcal{L}^{1,p}_\Omega(G)$ be amenable; then $\mathcal{L}^{1,p}_\omega(G)$ is amenable.*
(ii) *Let Ω be bounded on G and $\mathcal{L}^{1,p}_\omega(G)$ be amenable; then $\mathcal{L}^{1,p}_\Omega(G)$ is amenable.*

Ghahramani and Loy proved that the group algebra $L^1(G)$ is approximately amenable if and only if G is amenable ([13, Theorem 3.2]). We consider approximate amenability of $\mathcal{L}^{1,p}_\omega(G)$ as follows.

Proposition 2.3. *Let ω be a weight on G, $1<p<\infty$, and let $\mathcal{L}^{1,p}_\omega(G)$ has a bounded approximate identity. If $\mathcal{L}^{1,p}_\omega(G)$ is approximately amenable, then G is amenable. The converse is true when ω is bounded.*

Proof. By Proposition 2.4 of [1], G is discrete and thereby $\mathcal{L}^{1,p}_\omega(G)$ is algebraically isomorphic to $\ell^1(G)$. Thus, $\ell^1(G)$ is approximately amenable, and G is amenable. For the converse, similarly to the proof of Corollary 3.2 of [1], $\mathcal{L}^{1,p}_\omega(G)\simeq \ell^1(G)$. Therefore $\mathcal{L}^{1,p}_\omega(G)$ is approximately amenable. □

According to Theorem 2.1 of [1], the dual of $\mathcal{L}^{1,p}_\omega(G)$ is $L^\infty(G)+L^q_{1/\omega}(G)$; by this fact and Theorem 2.7.1 of [3], we have the following result:

Theorem 2.3. *Let G be a locally compact group, ω be a weight on G, and $1\leq p<\infty$. Then $\mathcal{L}^{1,p}_\omega(G)^{**}=L^1(G)^{**}\cap L^p_\omega(G)$.*

In [12, Theorem 1.3], it is shown $L^1(G)^{**}$ is amenable if and only if G is finite, and Ghahramani and Loy generalized it to approximate amenability [13, Theorem 3.3]. We have a similar result as follows.

Theorem 2.4. *Let G be a locally compact group, ω be a weight on G, and $1 < p < \infty$. Then $\mathcal{L}_{\omega}^{1,p}(G)^{**}$ is amenable if and only if G is finite.*

Proof. Let $\mathcal{L}_{\omega}^{1,p}(G)^{**}$ be amenable. Therefore, in light of Theorem 1.8 of [12], $\mathcal{L}_{\omega}^{1,p}(G)$ is amenable. Thus, G is discrete and amenable. Then the identity map from $\mathcal{L}_{\omega}^{1,p}(G)^{**}$ into $\ell^1(G)^{**}$ is an algebraic isomorphism, thereby $\ell^1(G)^{**}$ is amenable if and only if G is finite. □

3. Weak and n-Weak Amenability

In this section we consider weak and n-weak amenability of Beurling algebras, and by the derivations we mean the bounded derivations.

Theorem 3.1. *Let G be an abelian group, and let ω be a weight on G such that ω is almost invariant and $\inf_{n\in N} \omega(nt)/n = 0$ for each $t \in G$. Then $S_{\omega}^1(G)$ is 2-weakly amenable.*

Proof. Let $D : S_{\omega}^1(G) \longrightarrow S_{\omega}^1(G)^{**}$ be a derivation bounded by 1. Consider the canonical projection $\jmath : S_{\omega}^1(G)^{**} \longrightarrow S_{\omega}^1(G)$. Then for every $f, g \in S_{\omega}^1(G)$ we have

$$\jmath \circ D(f * g) = f \jmath \circ D(g) + \jmath \circ D(f)g,$$

this means that $\jmath \circ D$ is a bounded derivation. Since $L_{\omega}^1(G)$ is a semisimple Banach algebra, $S_{\omega}^1(G)$ is too [5, Theorem 2.1]. Then Corollary 2.7.20 of [8], $\jmath \circ D = 0$. Therefore $D(S_{\omega}^1(G)) \subseteq \ker \jmath$. The rest of the proof is nothing else but the proof of Theorem 13.1 of [10]. □

Proposition 3.1. *Let ω be a weight on locally compact group G. Then if $L_{\omega}^1(G)^{**}$ is weakly amenable, then $M_{\omega}(G)$ is weakly amenable. Moreover, if G is discrete then weak amenability of $\ell_{\omega}^1(G)^{**}$ implies weak amenability of $\ell_{\omega}^1(G)$.*

Proof. By Proposition 7.17 of [10], we have $LUC_{\frac{1}{\omega}}(G) = L_{\omega}^1(G) \cdot L_{\frac{1}{\omega}}^{\infty}(G)$. Then

$$LUC_{\frac{1}{\omega}}^*(G) = \left(L_{\omega}^1(G) \cdot L_{\frac{1}{\omega}}^{\infty}(G)\right)^*.$$

Since $L_{\omega}^1(G)^{**}$ is weakly amenable, by Corollary 4.2 of [18], $LUC_{\frac{1}{\omega}}^*(G)$ is weakly amenable. Moreover, by the similar method of [11], we have

$$LUC_{\frac{1}{\omega}}^*(G) = M_{\omega}(G) \oplus C_{0,\frac{1}{\omega}}(G)^{\perp},$$

where $C_{0,\frac{1}{\omega}}(G)^{\perp}$ is the space of all functionals in $LUC_{\frac{1}{\omega}}^*(G)$ that annihilate $C_{0,\frac{1}{\omega}}(G)$. Lemma 2.3 of [20], implies that $M_{\omega}(G)$ is weakly amenable. □

Amenability and weak amenability of Beurling Banach sequence algebras have been studied in [2, 10, 16] and [24]. By these references and above Proposition we have the following results.

Corollary 3.1. *Let ω be a weight on $\mathbb{Z}$, such that*

$$\sup_{m,n\in\mathbb{Z}} \frac{\omega(m+n)(1+|n|)}{\omega(m)\omega(n)(1+|m+n|)} < \infty,$$

then $\ell_{\omega}^1(\mathbb{Z})^{(2n)}$ is not weakly amenable.

Proof. Apply Proposition 3.1 and Theorem 2.3 of [2]. □

Corollary 3.2. *Let $\omega_\alpha(n) = (1+|n|)^\alpha$ be a weight on $\mathbb{Z}$, then for $\alpha \geq \frac{1}{2}$, $\ell^1_{\omega_\alpha}(\mathbb{Z})^{(2n)}$ is not weakly amenable.*

Proof. Apply Proposition 3.1 and Theorem 2.4 of [2]. □

The proof of the following Theorem is similar to the proof of Theorem 5.7 of [6].

Theorem 3.2. *Let G be a locally compact group, ω be a weight on G and $S^1_\omega(G)$ has a central approximate identity that is bounded in $L^1_\omega(G)$. If $L^1_\omega(G)$ is n-weakly amenable for $n \in \mathbb{N}$, then $S^1_\omega(G)$ is approximately n-weakly amenable.*

4. **The Character Amenability**

Let G be a locally compact group and ω be a weight on G. A character on G is a nonzero map $\varphi : G \longrightarrow \mathbb{C}$ such that $\varphi(st) = \varphi(s)\varphi(t)$ for all $s, t \in G$. We denote the space of characters on G by $\Delta_G(G)$. Each character of $L^p_\omega(G)$ $(1 \leq p < \infty)$ can be expressed via the corresponding function φ by the formula

$$\widehat{\varphi}(f) = \int_G f\varphi d\lambda \qquad (f \in L^p_\omega(G)).$$

The character $\varphi \in \Delta_G(G)$ is called an augmentation character if $\varphi(s) = 1$ for all $s \in G$. We denote identity map from G into G by id_G, and the augmentation character on G by $\mathbf{1}$. Then we define the augmentation character on $L^p_\omega(G)$ by

$$\widehat{\mathbf{1}}(f) = \int_G f\mathbf{1}d\lambda = \int_G fd\lambda \qquad (f \in L^p_\omega(G)).$$

Note, that the kernel of the augmentation character is the augmentation ideal I_0 which is defined in the first section and $\mathbf{1} \in L^1_\omega(G)^*$. Clearly, any $\widehat{\varphi} \in \Delta(L^1_\omega(G))$ is a character on $S^1_\omega(G)$. On the other hand, if $\psi \in \Delta(S^1_\omega(G))$, then there is an element $f_0 \in S^1_\omega(G)$ such that $\psi(f_0) = 1$. We extend ψ to $L^1_\omega(G)$ by $\widetilde{\psi}(f) = \psi(f * f_0)$. It is easy to see that $\widetilde{\psi}$ is a unique extension of ψ and $\widetilde{\psi} \in \Delta(L^1_\omega(G))$. Therefore,

$$\Delta(S^1_\omega(G)) = \{\psi|_{S^1_\omega(G)} : \psi \in \Delta(L^1_\omega(G))\}.$$

Proposition 4.1. *Let G be a locally compact group, ω be a weight on G and $\Omega(t) = \omega(t)\omega(t^{-1})$.*

(i) *If $L^1_\omega(G)$ is character amenable, then G is amenable.*
(ii) *If $S^1_\omega(G)$ is character amenable, then $S^1_\omega(G) = L^1_\omega(G)$ and $L^1_\omega(G)$ is character amenable.*
(iii) *If Ω is bounded on G, then $L^1_\omega(G)$ is character amenable if and only if G is amenable*

Proof. (i) Let $\widehat{\mathbf{1}}$ be the augmentation character on $L^1_\omega(G)$. Since $L^1_\omega(G)$ is character amenable, Theorem 2.3 of [22] implies that there is a $m \in (L^1_\omega(G))^{**}$ such that $m(\widehat{\mathbf{1}}) = 1$. This means that G is amenable.

(ii) If $S^1_\omega(G)$ is character amenable, then by means of Theorem 2.3 of [22], $S^1_\omega(G)$ has a bounded approximate identity. Now, apply Proposition 2.1.

(iii) Apply the case (i) and Theorem 0 of [17]. □

Proposition 4.2. *Let G be a locally compact group, and ω be a weight on G. If the Lebesgue Banach algebras $\mathcal{L}_\omega^{1,p}(G) := L^1(G) \cap L_\omega^p(G)$, $1 < p < \infty$ is character amenable, then G is amenable and discrete.*

Proof. Character amenability of $\mathcal{L}_\omega^{1,p}(G)$ implies that it has a bounded approximate identity and due to Proposition 2.4 of [1], G is discrete and this follows that $\mathcal{L}_\omega^{1,p}(G)$ is algebraically isomorphic with $\ell^1(G)$. Thus $\ell^1(G)$ is character amenable, so that G is amenable [22, Corollary 2.4]. □

Acknowledgments

The authors wish to thank referee for his/her invaluable comments.

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