FUNDAMENTALS OF $\Gamma$-ALGEBRA AND $\Gamma$-DIMENSION

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In this paper, we generalize the notion of algebra over a field. A $\Gamma$-algebra is an algebraic structure consisting of a vector space $V$, a groupoid $\Gamma$ together with a map from $V \times \Gamma \times V$ to $V$, usually called multiplication. We introduce the notion of $\Gamma$-dimension and give some examples and prove some properties of $\Gamma$-algebras. Then, we give some results about $m \times n$ real matrices. Also, we study the notion of regular $\Gamma$-algebra and we obtain some results in this respect. Finally, we define the notions of $T$-functor and $H$-system over a $\Gamma$-algebra and prove some results. Moreover, we see that there exists a covariant functor between the categories of $\Gamma$-algebras and algebras. We see that this functor is exact.

Keywords: $\Gamma$-algebra, homomorphism, regular $\Gamma$-algebra, $H$-system, $T$-functor.

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1. $\Gamma$-algebra

In [5], Nobusawa introduced the notion of $\Gamma$-ring, as more general than ring. Barnes [2] weakened slightly the conditions in the definition of the $\Gamma$-ring in the sense of Nobusawa. After these two papers are published, many mathematicians made good works on $\Gamma$-ring in the sense of Barnes and Nobusawa. Luh [4] and Kyuno [3] studied the structure of $\Gamma$-rings and obtained various generalization analogous to corresponding parts in ring theory. In [1], Chakraborty and Pau defined an isomorphism, an anti-isomorphism and a Jordan isomorphism in a $\Gamma$-ring and developed some important results relating to these concepts, also see [6, 7].

An algebra over a field is a vector space equipped with a bilinear vector product. That is to say, it is an algebraic structure consisting of a vector space together with an operation, usually called multiplication, that combines any two vectors to form a third vector; to qualify as an algebra, this multiplication must satisfy certain compatibility axioms with the given vector space structure, such as distributivity. In other words, an algebra over a field is a set together with operations of multiplication, addition, and scalar multiplication by elements of the field. Now, we generalize this notion.

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Definition 1.1. Let \( \Gamma \) be a groupoid and \( V \) be a vector space over a field \( F \). Then, \( V \) is called a \( \Gamma \)-algebra over the field \( F \) if there exists a mapping \( V \times \Gamma \times V \to V \) (the image is denoted by \( x \alpha y \) for \( x, y \in V \) and \( \alpha \in \Gamma \)) such that the following conditions hold:

1. \((x + y) \alpha z = x \alpha z + y \alpha z, \\ x \alpha (y + z) = x \alpha y + x \alpha z,\)
2. \(x(\alpha + \beta)y = x \alpha y + x \beta y,\)
3. \((cx) \alpha y = c(x \alpha y),\)
4. \(0 \alpha y = y \alpha 0 = 0,\)
   for all \( x, y, z \in V, \ c \in F \) and \( \alpha \in \Gamma \).

Moreover, a \( \Gamma \)-algebra is called associative if

5. \((x \alpha y) \beta z = x \alpha (y \beta z),\)

and unital if for every \( \alpha \in \Gamma \), there is an element \( 1_\alpha \) in \( V \) such that \( 1_\alpha x = x = x 1_\alpha \) for all non-zero elements of \( V \).

A non-empty subset \( V' \) of a \( \Gamma \)-algebra \( V \) is called a \( \Gamma \)-subalgebra if it is a subspace of \( V \) and for all \( x, y \in V' \) and \( \alpha \in \Gamma \) we have \( x \alpha y \in V' \). A subset \( I \) of a \( \Gamma \)-algebra \( V \) is called a left (right) ideal if it is a \( \Gamma \)-subalgebra of \( V \) and for all \( a \in I \) and \( v \in V \) and \( \alpha \in \Gamma \) we have \( v \alpha a \in I \) (\( a \alpha v \in I \)) and is a (two-sided) ideal if it is both a left and right ideal. It easy to see that \( V \) and \( \{0\} \) are ideals of \( V \). An ideal \( I \) such that \( \{0\} \subset I \subset V \) is called proper.

Let \( X \) be a subset of \( \Gamma \)-algebra \( V \). Then, the smallest left (right, two-sided) ideal of \( V \) containing \( X \) exists and we shall call it the left (right or two-sided) ideal generated by \( X \), and will be denoted by \( \langle X \rangle \) (\( \langle X \rangle_r \) or \( \langle X \rangle_l \)). If \( X = \{x\} \), then we also write \( \langle x \rangle \) instead of \( \langle \{x\} \rangle \).

Example 1.1. Let \( A \) be a vector space and \( \Gamma \) be a groupoid. For every \( x, y \in A \) and \( \alpha \in \Gamma \) we define \( x \alpha y = 0 \). Then, \( A \) is a \( \Gamma \)-algebra.

Example 1.2. Let \( F \) be a field, \( V \) and \( W \) be two vector spaces and \( A = \text{Hom}_F(V, W), \\ \Gamma = \text{Hom}_F(W, V) \). For every \( f, g \in A \) and \( \alpha \in \Gamma \) we define \( f \alpha g = f \circ \alpha \circ g \), where \( \circ \) is the combination operation. Then, \( A \) is an associative \( \Gamma \)-algebra.

Example 1.3. Let \( A \) and \( \Gamma \) be the sets of \( n \times m \) and \( m \times n \) matrices over the field \( F \), respectively. Then, it is easy to see that \( A \) is an associative \( \Gamma \)-algebra.

Example 1.4. Consider the pervious example. Let \( A \) be the set of \( 3 \times 2 \) matrices over the field of real numbers \( \mathbb{R} \) and

\[
\Gamma = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{R} \right\}.
\]

Then, \( A \) is an associative \( \Gamma \)-algebra and

\[
B = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : x, y \in \mathbb{R} \right\},
\]

is a \( \Gamma \)-subalgebra of \( A \).
Let $V_1$ and $V_2$ be $\Gamma_1$- and $\Gamma_2$-algebras respectively, $T$ be a linear transformation from $V_1$ to $V_2$, $f$ be a homomorphism from $\Gamma_1$ to $\Gamma_2$. Then, we say that $(T,f)$ is a $(\Gamma_1, \Gamma_2)$- homomorphism (homomorphism) from $(V_1, \Gamma_1)$ to $(V_2, \Gamma_2)$ if $(T,f)(x\alpha y) = T(x)f(\alpha)f(y)$.

Example 1.5. Let $V_1$ be the vector space of $n \times 1$ real matrices generated by $a = (a_{1i})_{n \times 1}$ such that $a_{11} = 1$ and $a_{1i} = 0$ for $i \neq 1$, $\Gamma_1 = \{ (\begin{array}{c} r_1 \\ 0 \\ \vdots \\ 0 \end{array}) : r_1 \in \mathbb{R} \}$, $V_2$ be the vector space of $m \times 1$ real matrices generated by $b = (b_{1i})_{m \times 1}$ such that $b_{11} = 1$ and $b_{1i} = 0$ for $i \neq 1$, $\Gamma_2 = \{ (\begin{array}{c} r_2 \\ 0 \\ \vdots \\ 0 \end{array}) : r_2 \in \mathbb{R} \}$, $T$ be the linear transformation from $V_1$ to $V_2$ with the matrix

$$
\begin{pmatrix}
  k & 0 & \cdots & 0 \\
  0 & 0 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & 0
\end{pmatrix}
$$

where $0 \neq k \in \mathbb{R}$ and $f : \Gamma_1 \longrightarrow \Gamma_2$ defined by $f(X) = \frac{1}{k} \times X$. Then, $(T,f)$ is a homomorphism from $V_1$ to $V_2$.

For non-empty subsets $A$ and $B$ of $\Gamma$-algebra $V$ and non-empty subset $\Gamma_1$ of $\Gamma$. Let

$$
A\Gamma_1 B := \{ a\gamma b : a \in A, b \in B \text{ and } \gamma \in \Gamma_1 \},
$$

$$
A\Gamma_1 \Sigma B := \left\{ \sum_{i=1}^{n} a_i \gamma_i b_i : a_i \in A, b_i \in B, \gamma_i \in \Gamma_1 \text{ and } n \in \mathbb{N} \right\},
$$

$$
\mathbb{Z}X = \left\{ \sum_{i=1}^{n} n_i x_i : n_i \in \mathbb{Z}, x_i \in X \right\}.
$$

If $A = \{a\}$, then we also write $a\Gamma_1 B$ instead of $\{a\}\Gamma_1 B$.

An ideal $P$ is called prime if $A\Gamma_1 \Sigma B \subseteq P$, then $A \subseteq P$ or $B \subseteq P$ and $P$ is called semiprime if $A\Gamma_1 \Sigma A \subseteq P$ then $A \subseteq P$.

Lemma 1.1. Let $V$ be a $\Gamma$-algebra and $X$ be a non-empty subset of $V$. Then,

1. $<X> = \mathbb{Z}X + X\Gamma_1 \Sigma V$,
2. $<X> = \mathbb{Z}X + V\Gamma_1 \Sigma X$,
3. $<X> = \mathbb{Z}X + X\Gamma_1 \Sigma V + V\Gamma_1 \Sigma X + V\Gamma_1 \Sigma X\Gamma_1 \Sigma V$.

Definition 1.2. Let $V$ be a $\Gamma$-algebra. Then, the ordinary dimension of $V$ as a vector space is called the dimension and the dimension of the subspace of $V$ generated by all products of the form $a\alpha b$ is called the $\Gamma$-dimension.

Example 1.6. Let $A$ be the vector space of $2 \times 3$ real matrices with the basis

$$
\left\{ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.
$$
and $\Gamma$ be a groupoid of $3 \times 2$ matrices of the form \[
\begin{pmatrix}
  r & 0 \\
- r & 0 \\
  0  & 0
\end{pmatrix}, \text{ where } r \in \mathbb{Z}.
\] Then, $A$ is a $\Gamma$-algebra and the dimension of $A$ is 4 but the $\Gamma$-dimension is 0. Since
\[
\begin{pmatrix}
  1 & 1 & 0 \\
  0 & 0 & 0 \\
  1 & 1 & 0
\end{pmatrix}\begin{pmatrix}
  r & 0 \\
- r & 0 \\
  0  & 0
\end{pmatrix} = \begin{pmatrix}
  0 & 0 \\
  0  & 0 \\
  0  & 0
\end{pmatrix},
\]
\[
\begin{pmatrix}
  0 & 0 & 0 \\
  1 & 1 & 0 \\
  0 & 0 & 0
\end{pmatrix}\begin{pmatrix}
  r & 0 \\
- r & 0 \\
  0  & 0
\end{pmatrix} = \begin{pmatrix}
  0 & 0 \\
  0  & 0 \\
  0  & 0
\end{pmatrix}.
\]

**Example 1.7.** Suppose that
\[
A = \left\{ \begin{pmatrix}
  a & 0 & 0 & 0 \\
  0 & 0 & b & c
\end{pmatrix} : a, b, c \in \mathbb{R} \right\}
\] and $\Gamma = \left\{ \begin{pmatrix}
  r & 0 \\
  0 & 0
\end{pmatrix} : r \in \mathbb{R} \right\}$.

Then, the dimension of $A$ is 3 and the $\Gamma$-dimension of $A$ is 1.

2. Results about $m \times n$ matrices

**Lemma 2.1.** Let $A$ be the vector space of $m \times n$ real matrices and $\Gamma$ be a set of $n \times m$ real matrices, where the $ij$ entire is a real number and the others are zero. Then, the elements of $\Gamma$-algebra $A$ are $m \times n$-matrices with dependent rows.

**Proof.** The proof is straightforward. $\square$

**Proposition 2.1.** Let $A$ be the vector space of $m \times n$ real matrices and $\Gamma$ is a set of $n \times m$ real matrices with $1 \leq k \leq mn$ non-zero entries. Then, every element of $\Gamma$-algebra $A$ is the sum of $k$, $m \times n$ real matrices with dependent rows.

**Proof.** The proof obtains by Lemma 2.1 and the following relation,
\[
a(a_1 + a_2 + \cdots + a_k)b = a_1b + a_2b + \cdots + a_kb,
\]
where $a, b \in A$ and $a_i \in \Gamma$. $\square$

**Proposition 2.2.** Let $A$ be the vector space of $m \times n$ real matrices and $\Gamma$ is a groupoid of $n \times m$ real matrices with at least one non-zero entire. Then, the dimension and $\Gamma$-dimension of $A$ are equals.

**Proof.** With out loss of generality, suppose that $a_{n \times m} = \begin{pmatrix}
  r & 0 & \cdots & 0 \\
  0 & 0 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & 0
\end{pmatrix}$ is an arbitrary element of $\Gamma$. Then, the basis element $E_{ij}$ obtained from the product $A_{m \times n}a_{n \times m}B_{m \times n}$, where $A_{m \times n} = (a_{i'j'})$, $B_{m \times n} = (b_{i'j'})$
\[
a_{i'j'} = \begin{cases}
  \frac{1}{r} & i' = i, j' = 1 \\
  0 & \text{a.w}
\end{cases}
\]
\[ b_{i', j'} = \begin{cases} 1 & i' = 1, j' = j \\ 0 & \text{o.w} \end{cases} \]

This completes the proof. \[\square\]

3. Regular $\Gamma$-algebra

A $\Gamma$-algebra $V$ is regular if for every $x \in V$, there exists $y \in V$ and $\alpha, \beta \in \Gamma$ such that

\[ x = x\alpha y\beta x. \]

In this case $x$ is called an \((\alpha, \beta)\)-regular element. An ideal $I$ of a $\Gamma$-algebra $V$ is called \((\alpha, \beta)\)-regular if every element of $I$ is \((\alpha, \beta)\)-regular. An element $x$ of a $\Gamma$-algebra $V$ is called $\alpha$-idempotent if $x\alpha x = x$.

**Example 3.1.** Let $F$ be a field, $V = F \times F$ and $\Gamma$ be a sub-groupoid of $F$. For every $\alpha, \beta \in \Gamma$ and $x_1, x_2, x_3, x_4 \in F$, we define

\[ (x_1, x_2) \oplus (x_3, x_4) = (x_1 + x_3, x_2 + x_4), \]

\[ (x_1, x_2)(\alpha, \beta)(x_3, x_4) = (x_1\alpha x_3, x_2\beta x_4). \]

Then, $V$ is a regular $\Gamma$-algebra.

Notice: Let $V$ be a regular $\Gamma$-algebra. Then, $< x >_r = x\Gamma\Sigma V$. Indeed, since $V$ is regular there exist $\alpha, \beta \in \Gamma$ and $y \in V$ such that $x = x\alpha y\beta x$. Hence, $Zx = Z(x\alpha y\beta x) \subseteq x\Gamma\Sigma V$. This implies that $< x >_r = x\Gamma\Sigma V$.

**Proposition 3.1.** Let $V$ be an associative regular $\Gamma$-algebra such that every element is \((\alpha, \beta)\)-regular. Then, every finitely generated right (left) ideal of $V$ is generated by idempotent elements.

**Proof.** Suppose that $x \in V$. Then, there exists $y \in V$ such that $x = x\alpha y\beta x$. We have $(x\alpha y)(\alpha, \beta)(x\alpha y) = (x\alpha y\beta x)\alpha y = x\alpha y$. Hence, $x\alpha y$ is a $\beta$-idempotent element of $V$. We see that

\[ < x\alpha y >_r = (x\alpha y)\Gamma\Sigma V = \left\{ \sum_{i=1}^{n} (x\alpha y)\beta_i v_i : n \in \mathbb{N}, v_i \in V, \beta_i \in \Gamma \right\} \]

\[ = \left\{ \sum_{i=1}^{n} x\alpha (y\beta_i v_i) : n \in \mathbb{N}, v_i \in V, \beta_i \in \Gamma \right\} \]

\[ \subseteq x\Gamma\Sigma V = < x >_r. \]

On the other hand, since

\[ x = x\alpha y\beta x \in (x\alpha y)\Gamma\Sigma V, \]

\[ < x >_r \subseteq < x\alpha y >_r. \] Therefore, $< x >_r = < x\alpha y >_r$.

Without lose of generality we suppose that $I = < x, y >_r$. Now, $< x >_r = < a >_r$, for some $\beta$-idempotent element and since $y - a\beta y \in < x, y >_r$, we have $< a >_r \subseteq < x >_r$. Therefore, $< a >_r = < x >_r$.
Let $x, y >_r = < a, y - a \beta y >_r$, and there exists a $\beta$-idempotent element $b \in V$ such that $< b >_r = < y - a \beta y >_r$. Consequently, $a \beta b = 0$ and
\[
(b - b \beta a)\beta(b - b \beta a) = b \beta b - (b \beta b)\beta a - (b \beta a)\beta b + (b \beta a)\beta(b \beta a) = b - b \beta a;
\]
\[
b \beta(b - b \beta a) = b \beta b - b \beta(b \beta a) = b \beta b - (b \beta b)\beta a = b - b \beta a.
\]
We conclude that $< b - b \beta a >_r = < b >_r = < y - a \beta y >_r$.

Therefore, $< x, y >_r = < a, b - b \beta a >_r$. This completes the proof. □

**Proposition 3.2.** Let $V$ be a $\Gamma$-algebra, $x_1 = x - x \alpha y \beta x$ and $x_1 = x_1 \alpha a \beta x_1$ for some $a \in V$. Then, $x = x_1 \alpha b \beta x$ for some $b \in V$.

**Proof.** We observe that
\[
x = x_1 + x \alpha y \beta x = x_1 \alpha a \beta x_1 + x \alpha y \beta x
\]
\[
= (x - x \alpha y \beta x)\alpha a \beta(x - x \alpha y \beta x) + x \alpha y \beta x
\]
\[
= x\alpha(a - a \beta x \alpha y - y \beta x \alpha + y \beta x \alpha a x \beta y)\beta x.
\]
This implies that $x = x_1 \alpha b \beta x$ for some, $b \in a - a \beta x \alpha y - y \beta x \alpha + y \beta x \alpha a x \beta y$. This completes the proof. □

**Lemma 3.1.** Let $V_1 \leq V_2$ be ideals in an associative $\Gamma$-algebra $V$. Then, $V_2$ is $(\alpha, \beta)$-regular if and only if $V_1$ and $[V_2 : V_1]$ are both $(\alpha, \beta)$-regular.

**Proof.** Suppose that $V_2$ is $(\alpha, \beta)$-regular. Then, obviously $[V_2 : V_1]$ is $(\alpha, \beta)$-regular.

Let $x \in V_1$. Then, we have $x = x \alpha y \beta x$ for some $y \in V_2$. We set $b = y \beta x \alpha y$. Then, $b$ is an element of $V_1$ such that
\[
x_1 \alpha b \beta x = x\alpha(y \beta x \alpha y)\beta x = (x \alpha y \beta x)\alpha y \beta x = x \alpha y \beta x = x,
\]
Then, $V_1$ is $(\alpha, \beta)$-regular.

Conversely, assume that $V_1$ and $[V_2 : V_1]$ are both $(\alpha, \beta)$-regular and $x \in V_1$. Hence, there exist $\alpha, \beta \in \hat{\Gamma}$ and $y + V_1 \in [V_2 : V_1]$ such that
\[
x + V_1 = (x + V_1)\alpha(y + V_1)\beta(x + V_1) = x \alpha y \beta x + V_1,
\]
where $\hat{\Gamma} = \{ \gamma : \gamma \in \Gamma \}$. Hence, $x - x \alpha y \beta x \in V_1$ for some $y \in V_2$. Since $V_1$ is $(\alpha, \beta)$-regular,
\[
x - x \alpha y \beta x = (x - x \alpha y \beta x)\alpha z \beta(x - x \alpha y \beta x)
\]
for some $z \in V_1$, from which we conclude that $x = x_1 \alpha b \beta x$. Therefore, $V_2$ is $(\alpha, \beta)$-regular. □

**Proposition 3.3.** Let $V$ be a regular associative $\Gamma$-algebra such that every element is $(\alpha, \alpha)$-regular. Then, $\theta = \{ x \in V : x \alpha y = y \alpha x \text{ for all } y \in V, \alpha \in \Gamma : x = x \alpha y \alpha x \}$ is $(\alpha, \alpha)$-regular.

**Proof.** Suppose that $x \in \theta$. There exists $y \in V$ such that $x = x \alpha y \alpha x$. We set $z = y \alpha x \alpha y$. Then, we obtain that
\[
x \alpha z \alpha x = x \alpha(y \alpha x \alpha y)\alpha x = (x \alpha y \alpha x)\alpha y \alpha x = x \alpha y \alpha x = x.
\]
We have
\[z\alpha v = y\alpha x\alpha y = (y\alpha y)\alpha\alpha x\alpha x = y\alpha x\alpha y\alpha x\alpha y = y\alpha x\alpha v.
\]

In the same way, \(v\alpha z = y\alpha x\alpha y\) = \(y\alpha x\alpha v = z\alpha v\), where \(v \in V\). Therefore, \(z \in \theta\) and \(\theta\) is \((\alpha, \alpha)\)-regular. \(\square\)

Let \(V\) be a \(\Gamma\)-algebra. An equivalence relation \(\rho\) on \(V\) is called regular if for every \(a_1, a_2, b_1, b_2\), such that \((a_1, b_1)\) and \((a_2, b_2)\) \(\in \rho\) and for all \(\alpha \in \Gamma\), \((a_1\alpha a_2, b_1\alpha b_2)\) \(\in \rho\) and is called strong regular if \((a_1 + a_2, b_1 + b_2)\) \(\in \rho\) and \((a_1\alpha a_2, b_1\beta b_2)\) \(\in \rho\) for every \(\alpha, \beta \in \Gamma\).

Suppose that \(\rho\) is a regular relation on a \(\Gamma\)-algebra. We define a binary operations on \([V : \rho]\), the set of all equivalence classes, as follows:
\[
\begin{align*}
\rho(a)\hat{\alpha}\rho(b) &= \rho(a\alpha b), \\
\rho(a) \oplus \rho(b) &= \rho(a + b).
\end{align*}
\]

Let \(a_1, a_2, b_1, b_2 \in V\) and \(\rho(a_1) = \rho(b_1)\) and \(\rho(a_2) = \rho(b_2)\). Then,
\[
(a_1, b_1) \in \rho \text{ and } (a_2, b_2) \in \rho \implies (a_1\alpha a_2, b_1\alpha b_2) \in \rho \\
\implies \rho(a_1)\hat{\alpha}\rho(a_2) = \rho(b_1)\hat{\alpha}\rho(b_2)
\]
and \(\rho(a_1) \oplus \rho(a_2) = \rho(b_1) \oplus \rho(b_2)\).

It is easy to see that \([V : \rho]\) is a \(\hat{\Gamma}\)-algebra. Suppose that \(\rho\) is a strong regular relation. Then, for every \(\alpha, \beta \in \Gamma\)
\[
\rho(a)\hat{\alpha}\rho(b) = \rho(a)\hat{\beta}\rho(b).
\]
Hence, \([V : \rho]\) is an algebra.

Suppose that \(V\) is a \(\Gamma\)-algebra and \(a\) is an element of \(V\). We say that \(b\) is an \((\alpha, \beta)\)-inversion of \(a\) if \(a\alpha b\beta a = a\), \(b\beta a\alpha b = b\).

**Example 3.2.** Let \(V = \mathbb{R}^3\) and \(\Gamma = \{(r, 0, 0) : r \in \mathbb{R}\}\). Then, \(V\) is a \(\Gamma\)-algebra with \(\Gamma\)-dimension 1. If \(a = (1, 0, 0), b = (3, 0, 0), \alpha = (2, 0, 0), \beta = (\frac{1}{6}, 0, 0)\), then \(b\) is an \((\alpha, \beta)\)-inversion of \(a\).

Suppose that \(V\) is an associative \(\Gamma\)-algebra and \(a\) is an \((\alpha, \beta)\)-regular. Then, there exist \(\alpha, \beta \in \Gamma\) and \(b \in V\) such that \(a = a\alpha b\beta a\). Let \(x = b\beta a\alpha b\). Then, we observe that
\[
\begin{align*}
a\alpha x\beta a &= a\alpha (b\beta a\alpha b)\beta a = (a\alpha b)\alpha b\beta a = a\alpha b\beta a = a; \\
x\beta a\alpha x &= (b\beta a)\alpha (b\beta a)\alpha b = b\beta (a\alpha b\beta a)\alpha (b\beta a) \\
&= b\beta a\alpha b\beta a = b\beta (a\alpha b\beta a) b = b\beta a b = x.
\end{align*}
\]

**Proposition 3.4.** Let \(\rho\) be a regular relation on a regular associative \(\Gamma\)-algebra and \(\rho(a)\) be an idempotent in \([V : \rho]\). Then, there exists an idempotent element \(e\) in \(V\) such that \(\rho(a) = \rho(e)\).
Proof. Suppose that $\rho(a)$ is a $\gamma$-idempotent element in $[V : \rho]$. Then, there exists $\gamma \in \Gamma$ such that $\rho(a) = \rho(a)\gamma \rho(a) = \rho(a\gamma a)$. Let $x$ be an $(\alpha, \beta)$-inversion of $a\gamma a$. Then,

$$(a\gamma a)\alpha x \beta(a\gamma a) = a\gamma ax \beta(a\gamma a)x = x.$$ 

Let $e = a\alpha x \beta a$. Then,

$$e_\gamma e = (a\alpha x \beta a)\gamma (a\alpha x \beta a) = a\alpha (x \beta a \gamma a\alpha x) \beta a = a\alpha x \beta a = e.$$ 

and so $e$ is $\gamma$-idempotent. We have

$$(a\alpha x \beta a, (a\gamma a)\alpha x \beta (a\gamma a)) \in \rho,$$ 

and $(e, a\gamma a) \in \rho$. Therefore, $\rho(e) = \rho(a\gamma a)$. \qed

**Theorem 3.1.** Let $V$ be an associative $\Gamma$-algebra such that $\{0\}$ is a semiprime ideal, every family of semiprime ideals has a maximal element and $[V : \rho]$ is $(\alpha, \beta)$-regular for all prime ideal of $V$. Then, $V$ is a regular algebra.

Proof. Suppose that $V$ is not regular. Then, there exists $x \in V$ such that $x \notin x\Gamma VT x$. There exists a semiprime ideal $P$ in $V$ such that it is maximal with respect the property $x \notin x\Gamma VT x + P$. If $[V : P]$ is regular, then

$$x + P \in (x + P)\hat{\alpha}[V : P]\hat{\beta}(x + P).$$ 

Hence, there exists $y + P \in [V : P]$ such that

$$x + P \in (x + P)\alpha(y + P)\beta(x + P) = x\alpha y \beta x + P.$$ 

This implies that $x \in x\alpha y \beta x + P \subseteq x\Gamma y \Gamma x + P$, which is a contradiction. Thus, $x \notin x\Gamma VT x + P$. Then, $P$ is not prime. Hence, there exist ideals $A$ and $B$ such that $A\Gamma B \subseteq P$ and $A \notin P$, $B \notin P$. Now, suppose that $T_1 = \{v \in V : v\Gamma \Gamma B \subseteq P\}$ and $T_2 = \{v \in V : T_1 \Gamma \Gamma v \subseteq P\}$. We see that $T_1$ and $T_2$ are semiprime.

Now, let $A_1$ and $A_2$ be two ideals such that $A_1 \Gamma A_2 \subseteq T_1$. Then, $(A_1 \Gamma A_2) \Gamma B$ and $A_1 \Gamma (A_1 \Gamma B) \subseteq P$. Since $P$ is prime and $B \notin P$, implies that $A_1 \subseteq P$. In the same way, one can see that $T_2$ is a semiprime ideal. On the other hand

$$(T_1 \cap T_2)\Gamma \Gamma (T_1 \cap T_2) \subseteq T_1 \Gamma \Gamma T_2 \subseteq P.$$ 

Hence, $T_1 \cap T_2 \subseteq P$. Since $A \notin P$ and $B \notin P$, $T_1$ and $T_2$ properly contain $P$. Because the maximality of $P$, $[V : T_1]$ and $[V : T_2]$ are regular. Thus, there exist $x_1, x_2 \in V$ such that

$$x + P = (x + P)\hat{\alpha}(x_1 + P)\hat{\beta}(x + P),$$ 

$$x + P = (x + P)\hat{\alpha}(x_2 + P)\hat{\beta}(x + P).$$ 

Thus, $x - x\alpha x_1 \beta x \in T_1$ and $x - x\alpha x_2 \beta x \in T_2$. This implies that

$$x - x\alpha(x_1 + x_2 - x_1 \beta x)\alpha x_2 \beta x = (x - x\alpha x_1 \beta x) - (x - x\alpha x_1 \beta x)\alpha x_2 \beta x \in T_1$$ 

and

$$x - x\alpha(x_1 + x_2 - x_1 \beta x)\alpha x_2 \beta x = (x - x\alpha x_2 \beta x) - x\alpha x_1 \beta (x - x\alpha x_2 \beta x) \in T_2.$$
We conclude that $x \in x\Gamma VTx + T_1 \cap T_2 \subseteq x\Gamma VTx + P$, which is a contradiction. Therefore, $V$ must be regular. \hfill \square

**Proposition 3.5.** Let $V$ be an associative unital $\Gamma$-algebra and set

$$\Theta = \left\{ x \in V : V\Sigma (x \Gamma \Sigma V) \text{ is an } (\alpha, \beta)\text{-regular ideal} \right\}.$$  

Then, $\Theta$ is an $(\alpha, \beta)$-regular ideal and $[V : \Theta]$ has no non-zero $(\alpha, \beta)$-regular ideal.

**Proof.** Suppose that $x, y \in \Theta$. Then, $V\Sigma (x \Gamma \Sigma V)$ and $V\Sigma (y \Gamma \Sigma V)$ are $(\alpha, \beta)$-regular ideals. By Lemma 4.3, $V\Sigma (x \Gamma \Sigma V) + V\Sigma (y \Gamma \Sigma V)$ is a regular ideal. Since

$$V\Sigma (x + y) \Gamma \Sigma V \subseteq V\Sigma (x \Gamma \Sigma V) + V\Sigma (y \Gamma \Sigma V),$$

$V\Sigma (x + y) \Gamma \Sigma V$ is regular. In the same way, we can see that $\Theta \cap V, V\Theta \subseteq \Theta$. Let $J$ be an $(\alpha, \beta)$-regular ideal of $V$ and $x \in J$. Then,

$$V\Sigma (x \Gamma \Sigma V) \subseteq V\Sigma (J \Gamma \Sigma V) \subseteq J.$$

Hence, $V\Sigma (x \Gamma \Sigma V)$ is $(\alpha, \beta)$-regular and $J \subseteq \Theta$. Let $[J : \Theta]$ be an $(\alpha, \beta)$-regular ideal of $[V : \Theta]$. Since $\Theta$ is $(\alpha, \beta)$-regular, $J$ is $(\alpha, \beta)$-regular and $J \subseteq \Theta$. This implies that $[V : \Theta]$ has not non-zero $(\alpha, \beta)$-regular ideal. \hfill \square

**Proposition 3.6.** Let $V$ be a regular $\Gamma$-algebra. Then, the dimension and the $\Gamma$-dimension of $V$ are equal.

**Proof.** Let $x \in V$. Since $V$ is regular there exist $\alpha, \beta \in \Gamma$ and $y \in V$ such that $x = x\alpha y\beta x$. This completes the proof. \hfill \square

4. $T$-functor and $H$-system

The category $\Gamma AL$ is the category whose objects are $\Gamma$-algebras. For $\Gamma_1$-algebra $V_1$ and $\Gamma_2$-algebra $V_2$, $Mor(V_1, V_2)$ is the set of all $(\Gamma_1, \Gamma_2)$-epimorphisms. The composition of morphisms denotes the usual composition of homomorphisms and so satisfies the associative law. $(Id_V, Id_V) : (V, \Gamma) \rightarrow (V, \Gamma)$ is the identity map satisfies the required property $(Id_V, Id_V) \circ (\varphi, f) = (\varphi, f)$ for every $(\varphi, f) \in Mor(V', V)$ and $(\varphi, f) \circ (Id_V, Id_V) = (\varphi, f)$, for every $(\varphi, f) \in Mor(V, V')$. The category $AL$ is the category whose objects are algebras and $Mor(A_1, A_2)$ is the set of all algebra homomorphisms from $A_1$ to $A_2$ and it satisfies the associative law.

Let $V$ be a $\Gamma$-algebra and

$$\Delta_V = \left\{ \prod_{i=1}^n (x_i, \alpha_i) : \alpha_i \in \Gamma, \ x_i \in V, \ n \in \mathbb{N} \right\}.$$  

Then, the relation $\theta$ on $\Delta_V$ defined by

$$\left( \prod_{i=1}^n (x_i, \alpha_i) \right) \theta \left( \prod_{j=1}^m (y_j, \beta_j) \right) \text{ if and only if } \sum_{i=1}^n x_i \alpha_i x = \sum_{j=1}^m y_j \beta_j x, \ \forall x \in V,$$
is an equivalence relation. We denote the equivalence class containing $\prod_{i=1}^{n}(x_i, \alpha_i)$ by $\theta\left(\prod_{i=1}^{n}(x_i, \alpha_i)\right)$. Then, $[\Delta_V : \theta]$ forms a vector space. Now, we define a multiplication on $[\Delta_V : \theta]$ as follows:

$$\theta\left(\prod_{i=1}^{n}(x_i, \alpha_i)\right) \theta\left(\prod_{j=1}^{n}(y_j, \beta_j)\right) = \theta\left(\prod_{i,j}(x_i \alpha_i y_j, \beta_j)\right).$$

We denote this algebra by $V_L$ and is called the left operator algebra. In the same way, we can define the right operator algebra.

**Proposition 4.1.** Let $V_1$ and $V_2$ be $\Gamma_1$- and $\Gamma_2$- algebras, respectively. If $(\varphi, f) : (V_1, \Gamma_1) \rightarrow (V_2, \Gamma_2)$ is an epimorphism, then there exists a unique homomorphism $(\varphi, f) : [\Delta_{V_1} : \theta_1] \rightarrow [\Delta_{V_2} : \theta_2]$ such that the following diagram is commutative:

\[
\begin{array}{ccc}
(V_1, \Gamma_1) & \xrightarrow{(\varphi, f)} & (V_2, \Gamma_2) \\
\downarrow & & \downarrow \\
[\Delta_{V_1}, \theta_1] & \xrightarrow{(\varphi, f)} & [\Delta_{V_2}, \theta_2]
\end{array}
\]

Moreover, if $(\varphi, f)$ is an isomorphism, then $(\varphi, f)$ is an isomorphism.

**Proof.** We define $(\varphi, f) : [\Delta_{V_1}, \theta_1] \rightarrow [\Delta_{V_2}, \theta_2]$ by

$$((\varphi, f))(\theta(\prod_{i=1}^{n}(x_i, \alpha_i))) = \theta(\prod_{i=1}^{n}(\varphi(x_i), f(\alpha_i))),$$

for every $\theta(\prod_{i=1}^{n}(x_i, \alpha_i)) \in [\Delta_{V_1}, \theta_1]$. It is easy to see that this function is well-defined and homomorphism. One can see that if $(\varphi, f)$ is an isomorphism, then induced homomorphism $(\varphi, f)$ is an isomorphism. \(\square\)

**Corollary 4.1.** There is a covariant functor between the subcategory of $\Gamma$-algebras and the category of algebras.

**Proof.** By Proposition 4.1, it is straightforward. \(\square\)

Let $(\varphi_1, f_1) : (V_1, \Gamma_1) \rightarrow (V_2, \Gamma_2)$ and $(\varphi_2, f_2) : (V_1, \Gamma_1) \rightarrow (V_2, \Gamma_2)$ be homomorphisms. We define

$$S(\varphi_1, \varphi_2) = \left\{ \sum_{i=1}^{n} \varphi_1(v_r)f_j(\alpha_r)v : v_r \in V_1, \alpha_r \in \Gamma_1, n \in \mathbb{N}, 1 \leq i, j \leq 2, i \neq j \right\}.$$  

This homomorphism is said to be $S$-conjugate if $S(\varphi_1, \varphi_2) = 0$.

Let $V_1, V_2, \ldots V_n$ and $V$ be $\Gamma_1$, $\Gamma_2$, \ldots, $\Gamma_n$- and $\Gamma = \Gamma_1 \times \Gamma_2 \ldots \Gamma_n$- algebras, respectively, and suppose that we are given $(\Gamma_i, \Gamma)$- homomorphisms $(\sigma_i, \chi_i) : (V_i, \Gamma_i) \rightarrow (V, \Gamma)$, $(1 \leq i \leq n)$ and $(\Gamma, \Gamma)$- homomorphism $(\pi_i, \vartheta_i) : (V, \Gamma) \rightarrow (V_i, \Gamma_i)$, $(1 \leq i \leq n)$ such that $\pi_j \sigma_i = \delta_{ij}$ and $\sum \sigma_i \pi_i = Id_V$. Then, $V$ is called an $H$- system.
Proposition 4.2. Let $V$ be an $H$-system and $(\varphi_i, f_i) : (V_i, \Gamma_i) \rightarrow (W, \Gamma)$, $(1 \leq i \leq n)$ be given. Then, there exists a unique homomorphism $(\varphi, f) : (V, \Gamma) \rightarrow (W, \Gamma)$ such that $(\varphi, f) \circ (\sigma, \chi_i) = (\varphi_i, f_i)$. If $(\psi_i, g_i) : (W, \Gamma) \rightarrow (V, \Gamma_i)$, then there exists a unique homomorphism $(\psi, g) : (W, \Gamma) \rightarrow (V, \Gamma)$ such that $(\psi, g) \circ (\psi, g_i) = (\psi_i, g_i)$.

Proof. Suppose that $(\varphi, g) : (W, \Gamma) \rightarrow (V, \Gamma)$ defined by $\varphi = \sum_{j=1}^{n} \varphi_j \phi_j$. Then,

$$\varphi \sigma_i = \left( \sum_{j=1}^{n} \varphi_j \phi_j \right) \sigma_i = \sum_{j=1}^{n} \varphi_j \phi_j \sigma_i = \sum_{j=1}^{n} \varphi_j \phi_j \delta_{ij} = f_i.$$ 

It is easy to see that this homomorphism is unique.

Now, we define $\psi : W \rightarrow V$ by $\psi = \sum_{j=1}^{n} \sigma_j \psi_j$. This is a unique homomorphism such that $\pi \psi = \psi_i$. This completes the proof.

Theorem 4.1. Let $\Omega$ be a subcategory of $\Gamma AL$ such that for every $H$-system $V$ of $\Omega$, $\Delta_V$ is an $H$-system in $AL$. Then, for every morphism $\varphi_1$ and $\varphi_2$ in $\Omega$, $T(\varphi_1 + \varphi_2) = T(\varphi_1) + T(\varphi_2)$.

Proof. Suppose that $(\varphi_i, f_i) : (V_i, \Gamma_i) \rightarrow (W_i, \Gamma_i)$, $(1 \leq i \leq 2)$ are morphisms. Since $\Delta_V$ is an $H$-system of $AL$, we have $T(\pi_1)T(\sigma_1 + \sigma_2)$ and $T(\pi_2)T(\sigma_1 + \sigma_2)$ are identity morphisms. Hence,

$$T(\sigma_1 + \sigma_2) = T(\sigma_1)T(\pi_1)T(\sigma_1 + \sigma_2) + T(\sigma_2)T(\phi_2)T(\sigma_1 + \sigma_2).$$

We define $\varphi : W \rightarrow V_2$ by $\varphi = \varphi_1 \pi_1 + \varphi_2 \pi_2$. Then, $\varphi \sigma_1 = \varphi_1$ and $\varphi \sigma_2 = \varphi_2$. Moreover, $\varphi(\sigma_1 + \sigma_2) = \varphi_1 + \varphi_2$. Hence,

$$T(\varphi_1 + \varphi_2) = T(\varphi_1) + T(\varphi_2).$$

This completes the proof.

Theorem 4.2. Let $0 \rightarrow (V_1, \Gamma_1) \xrightarrow{\pi_1} (V, \Gamma) \xrightarrow{\pi_2} (V_2, \Gamma_2) \rightarrow 0$ be an exact sequence in $\Gamma AL$. Then, the following statements are equivalent:

1. There exists $(\Gamma_2, \Gamma)$-homomorphism $(\sigma, f) : (V, \Gamma) \rightarrow (V, \Gamma)$ and $(\Gamma, \Gamma_1)$-homomorphism $(\pi_1, g) : (V, \Gamma) \rightarrow (V_1, \Gamma_1)$ such that $V$ is an $H$-system.

2. There exists a subalgebra of $V_1$ such that $V = (\sigma_1, f_1)(V_1, \Gamma_1) \oplus V_1$.

Proof. The proof is straightforward.

Proposition 4.3. Let $0 \rightarrow (V_1, \Gamma_1) \xrightarrow{\pi_1} (V, \Gamma) \xrightarrow{\pi_2} (V_2, \Gamma_2) \rightarrow 0$ be a split exact sequence in $\Gamma AL$. Then, $0 \rightarrow \Delta_{V_1} \xrightarrow{T(\pi_1)} \Delta_V \xrightarrow{T(\pi_2)} \Delta_{V_2} \rightarrow 0$ is a split exact sequence in $AL$.

Proof. The proof is straightforward.

Proposition 4.4. Let for every split exact sequence

$$0 \rightarrow (V_1, \Gamma_1) \rightarrow (V, \Gamma) \rightarrow (V_2, \Gamma_2) \rightarrow 0$$

implies that $0 \rightarrow \Delta_{V_1} \rightarrow \Delta_V \rightarrow \Delta_{V_2} \rightarrow 0$ is a split exact sequence. Then, for every homomorphism $\varphi_1, \varphi_2$, we have $T(\varphi_1 + \varphi_2) = T(\varphi_1) + T(\varphi_2)$. 

Proof. Suppose that $V$ is an $H$-system. This implies that
$$0 \longrightarrow (V_1, \Gamma_1) \xrightarrow{(\sigma_1, f_1)} (V, \Gamma) \xrightarrow{(\pi_2, g_2)} (V_2, \Gamma_2) \longrightarrow 0$$
is a split exact sequence. By hypothesis
$$0 \longrightarrow \Delta V_1 T(\sigma_1, f_1) \Delta V T(\pi_2, g_2) \Delta V_2 \longrightarrow 0$$
is a split exact sequence. In the same way
$$0 \longrightarrow \Delta V_2 T(\sigma_2, f_2) \Delta V T(\pi_1, g_1) \Delta V_1 \longrightarrow 0$$
is a split exact sequence. Hence,
\[ T(\pi_2, g_2)T(\sigma_1, f_1) = T((\pi_2, g_2)(\sigma_1, f_1)) = \text{Id}, \]
\[ T(\pi_1, g_1)T(\sigma_2, f_2) = T((\pi_1, g_1)(\sigma_2, f_2)) = \text{Id}. \]
By a routine process, $T(V, \Gamma)$ is an $H$-system. This completes the proof. □

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