U.P.B. Sci. Bull., Series A, Vol. 77, Iss. 4, 2015 ISSN 1223-7027

FUZZY NOETHERIAN AND ARTINIAN RESIDUATED LATTICE

A. Ahadpanah[1], L. Torkzadeh[2]

Residuated lattices play an important role in the study of fuzzy logic. In the present paper, we introduce the notion of Noetherian and Artinian residuated lattices and give some characterizations of them. We prove some important theorems of rings theory in residuated lattices. After that we define the concept of fuzzy Noetherian and Artinian residuated lattices and investigate some properties of them. We determine relationships between Noetherian (Artinian) residuated lattices and fuzzy Noetherian (Artinian) residuated lattices.

Keywords: (fuzzy) Noetherian residuated lattice, (fuzzy) Artinian residuated lattice, prime filter, filter.

MSC2010: 03G10, 03B50, 08A72.

1. Introduction and Preliminaries

Non-classical logic has become a formal and useful tool for computer science to deal with uncertain information and fuzzy information. The algebraic counterparts of some non-classical logics satisfy residuation and those logics can be considered in a frame of residuated lattices [8]. For example, Hajek's BL (basic logic [5]), Lukasiewiczs MV (many-valued logic [1]) and MTL (monoidal t-norm based logic [3]) are determined by the class of BL-algebras, MV-algebras and MTL-algebras, respectively. All of these algebras have lattices with residuation as a common support set. Thus it is very important to investigate properties of algebras with residuation. Residuated lattices were introduced by Ward and Dilworth in [8]. The filter theory of the logical algebras plays an important role in the studying of these algebras and the completeness of the corresponding non-classical logics. At present, the filter theory of residuated lattice has been widely studied, and some important results have been obtained. In particular, in [7, 10], some types of filters in a residuated lattice were introduced, and some of their characterizations and relations were presented. In addition, based on the fuzzy set theory introduced by Zadeh in [9], the related fuzzy structures of filters in residuated lattice were further studied [10].

In the following, some preliminary theorems and definitions are stated from [4, 7, 8, 10]. In section 3, by defining the notion of Noetherian and Artinian residuated lattices, we prove the some theorems of rings theory in residuated lattices. We show that residuated lattice L is Noetherian if and only if every prime filter of the second kind of L is finitely generated. In section 4, we introduce the concept of fuzzy

[1]Department of Mathematics, Science and Research Branch, Islamic Azad University, Kerman, Iran

[2]Department of Mathematics, Kerman Branch, Islamic Azad University, Kerman, Iran, corresponding author e-mail: ltorkzadeh@yahoo.com

Noetherian and Artinian residuated lattices by fuzzy filters and investigate some properties of them.

At first we recall the definition of a residuated lattice. By a residuated lattice, we mean an algebraic structure L=$(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$, where
(LR_1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice,
(LR_2) $(L, \odot, 1)$ is a commutative monoid with a unit element 1,
(LR_3) For all $a, b, c \in L$, $c \leq a \rightarrow b$ if and only if $a \odot c \leq b$.

In [6], residuated lattices are called commutative, integral, residuated l-monoids.

Let L be a residuated lattice. We have the following results.

Theorem 1.1. *The following properties hold for all $x, y, z \in L$:*
(lr_1) $x \rightarrow x = 1, 1 \rightarrow x = x$,
(lr_2) $x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y)$,
(lr_3) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$,
(lr_4) $x \leq y \Leftrightarrow x \rightarrow y = 1$,
(lr_5) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z) = (x \odot y) \rightarrow z$,
(lr_6) $x \odot (x \rightarrow y) \leq y$ *and* $x \leq y \rightarrow x$,
(lr_7) If $x \leq y$, *then* $y \rightarrow z \leq x \rightarrow z$ *and* $z \rightarrow x \leq z \rightarrow y$.

We denote the set of natural numbers by N and define $a^0 = 1$ and $a^n = a^{n-1} \odot a$, for $n \in N - \{0\}$ and $a \in A$.
A nonempty subset F of L is called a filter of L if
(F_1) $x \in F$ and $x \leq y$ imply $y \in F$,
(F_2) for all x and y in F, $x \odot y \in F$.
The set of all filters of L is denoted by $F(L)$.
The smallest filter of L which contains X, is said to be the filter of L generated by X and will be denoted by $\langle X \rangle$. $\langle \emptyset \rangle = \{1\}$ and for a nonempty subset X of L, $F \in F(L)$ and $a \in L$ we have:
(i) $\langle X \rangle = \{x \in L : x_1 \odot x_2 \odot ... x_n \leq x,$ for some $n \geq 1$ and $x_1, x_2, ..., x_n \in X\}$.
(ii) $F(a) = \langle F \cup \{a\} \rangle = \{x \in L : x \geq f \odot a^n,$ for some $f \in F$ and $n \geq 1\}$.
$F \in F(L)$ is called a finitely generated filter, if $F = \langle x_1, x_2, ..., x_n \rangle$, for some $x_1, x_2, ..., x_n \in L$ and $n \in N$

Lemma 1.1. *Let G, F be filters of L and $x, y, x_1, ..., x_n, y_1, ..., y_m \in L$. Then the following statements hold:*
(1) $x \leq y$ *implies* $\langle y \rangle \subseteq \langle x \rangle$.
(2) $\langle x_1, ..., x_n \rangle \cap \langle y_1, ..., y_m \rangle = \langle x_1 \vee y_1, ..., x_n \vee y_m \rangle$.
(3) *If $F_1, ..., F_k$ are finitely generated filters of L, then $F_1 \cap ... \cap F_k$ is a finitely generated filters of L.*
(4) $F\langle x \rangle \cap F\langle y \rangle = F\langle x \vee y \rangle$.
(5) *If $G \subseteq F$ such that F/G and G are finitely generated filters of L, then F is a finitely generated filter of L.*

A filter F of L is called
(i) a prime filter (PF), if $x \rightarrow y \in F$ or $y \rightarrow x \in F$, for all $x, y \in L$.
(ii) a prime filter of the second kind $(PF2)$, if $x \vee y \in F$, then $x \in F$ or $y \in F$, for all $x, y \in L$
(iii) A prime filter of the third kind $(PF3)$, $(x \rightarrow y) \vee (y \rightarrow x) \in F$, for all $x, y \in L$.
If filter F is PF, then F is $PF2$ and $PF3$. Also filter F of L is $PF2$ if and only if

for all $F_1, F_2 \in F(L)$, $F = F_1 \cap F_2$ implies $F = F_1$ or $F = F2$,
A proper filter M of L is maximal if it is not contained in any other proper filter of L. We shall denote by $Max(L)$ the set of all maximal filters of L.

Let A and B be residuated lattices. $f : A \to B$ is a homomorphism of residuated lattices if f is a homomorphism of bounded lattices and for every $x, y \in A$: $f(x \odot y) = f(x) \odot f(y)$ and $f(x \to y) = f(x) \to f(y)$. Also we have:
(i) If $F \in F(B)$, then $f^{-1}(F) \in F(A)$,
(ii) If f is onto and $G \in F(A)$, then $f(G) \in F(B)$.
A nonempty subset S of L is called $\wedge$-closed system in L if $1 \in S$ and $x, y \in S$ implies $x \wedge y \in S$. On L we consider the relation θ_s defined by $(x, y) \in \theta_s$ if and only if there is $e \in S \cap B(L)$ such that $x \wedge e = y \wedge e$, where $B(L)$ is the set of all Boolean elements of L. The relation θ_s is a congruence relation on L.
For $x \in L$ we denote by x/S the equivalence class of x relative to θ_S and by $L[S] = L/\theta_S$. Clearly, $L[S]$ becomes a residuated lattice. By $p_S : L \to L[S]$, we denote the canonical map defined by $p_S(x) = x/S$, for every $x \in L$. We can see that if S is an $\wedge$-closed system such that $0 \in S$, then $L[S] = 0$.
A fuzzy set μ of L is called a fuzzy filter of L if it satisfies the following conditions, for all $x, y \in L$:
$(FF1):\ \mu(x \odot y) \geq \min\{\mu(x), \mu(y)\}$.
$(FF2):\ x \leq y \Rightarrow \mu(x) \leq \mu(y)$.
The set of all fuzzy filters of L is denoted by FF(L). A fuzzy set μ of L is a fuzzy filter of L if and only if the following hold, for all $x, y \in L$:
$(FF3):\ \mu(1) \geq \mu(x)$.
$(FF4):\ \mu(y) \geq \min\{\mu(x \to y), \mu(x)\}$.
For $t \in [0, 1]$, the set $\mu_t := \{x \in L \mid \mu(x) \geq t\}$ is called a level subset (with level t)of μ. A fuzzy set μ of L is a fuzzy filter of L if and only if for any $t \in [0, 1]$, the level set μ_t is either empty or a filter of L.

2. Noetherian and Artinian residuated lattice

Theorem 2.1. *Let L be a residuated lattice. Then the following conditions are equivalent*
1. *Every nonempty set of filters of L has a maximal element.*
2. *Every filter of L is finitely generated.*
3. *Every ascending sequence of filters of L like $F_1 \subseteq F_2 \subseteq F_3 \subseteq ...$ is stationary.*

Proof. $1 \to 2)$ Let F be a filter of L. Consider

$$T = \{G \in F(L) \mid G \subseteq F \text{ and } G \text{ is finitely generated } \}.$$

Since $\langle 1 \rangle \in T$, by hypothesis T has a maximal element G. Thus $G \subseteq F$ and $G = \langle x_1, ..., x_n \rangle$, for some $x_1, ..., x_n \in L$. We prove that $G = F$. On the contrary there exist $x \in F$ such that $x \notin G$. Hence $G \subset \langle x_1, ..., x_n, x \rangle \subseteq F$, which is a contradiction with maximality G. Therefore $G = F$, and F is a finitely generated filter of L.
$2 \to 3)$ Let $F_1 \subseteq F_2 \subseteq ...$ be an increasing sequence of filters of L. Then $F = \bigcup_{i=1}^{\infty} F_i$ is finitely generated and so $F = \langle x_1, ..., x_n \rangle$, for some $x_1, ..., x_n \in L$. Thus there exist $i_1, ..., i_n \in N$ such that $x_j \in F_{i_j}$, for $1 \leq j \leq n$. Now by hypothesis, there exists $m \in N$, $1 \leq m \leq n$ such that $x_1, ..., x_n \in F_{i_m}$. Hence $F_{i_m} = F$ and we get that

$F_{i_m} = F_k$ for all $k \geq i_m$, Therefore the chain is stationary.
$3 \to 1$) Let T be a nonempty set of filters of L which does not have a maximal element. Then there exists $F_1 \in T$, since T does not have a maximal element, there exists $F_2 \in T$ such that $F_1 \subset F_2$. By continuing this method, we obtain the ascending sequence $F_1 \subset F_2 \subset F_3 \subset ...$, that is not stationary. The proof is complete. □

Definition 2.1. *A residuated L is called Noetherian if it satisfies any one of the conditions of Theorem 2.1.*

Example 2.1.
(*i*) *Any finite residuated lattice is Noetherian.*
(*ii*) *Let $A = [0,1]$. Define $\odot$ and $\to$ as follow, for all $x, y \in A$,*

$$x \odot y = \min\{x, y\},\ \ x \to y = \begin{cases} 1 \ \text{if } x \leq y \\ y \ \text{if } x > y \end{cases}$$

Then $(A, \wedge, \vee, \odot, \to, 0, 1)$ is a residuated lattice and all filters of A are in the from of $[x, 1]$, for $x \in [0,1]$. Let $\{x_n\}_{n=1}^{\infty}$ be a strictly decreasing sequence in $[0,1]$. Put $F_n = [x_n, 1]$ for all $n \in N$. Then the ascending sequence $F_1 \subset F_2 \subset ...$ is not stationary. So A is not a Noetherian residuated lattice.
(*iii*) *Let $A = [0,1]$. Define $\odot$ and $\to$ as follow, for all $x, y \in A$,*

$$x \odot y = \max\{0, x + y - 1\},\ \ x \to y = \min\{1, 1 - x + y\}$$

Then $(A, \wedge, \vee, \odot, \to, 0, 1)$ is a residuated lattice and it is easy to see that the only filters of A are $\{1\}$ and $[0,1]$. So A is a Noetherian residuated lattice.

Proposition 2.1. *Let $1 \to L_1 \to^{\varphi} L_2 \to^{\psi} L_3 \to 1$ be an exact sequence of residuated lattices and L_1, L_3 be two Noetherian residuated lattices. Then L_2 is Noetherian.*

Proof. Let $F_1 \subseteq F_2 \subseteq ...$ be an ascending sequence of filters of L_2. Since ψ is onto, then $\psi(F_1) \subseteq \psi(F_2) \subseteq ...$ is an ascending sequence of filters of L_3 and $\varphi^{-1}(F_1) \subseteq \varphi^{-1}(F_2) \subseteq ...$ is an ascending sequence of filters of L_1. Now L_1, L_3 are Noetherian, there exists $m, n \in N$ such that $\psi(F_i) = \psi(F_m)$ and $\varphi^{-1}(F_j) = \varphi^{-1}(F_n)$, for all $i \geq m$, $j \geq n$. Put $k = max\{m, n\}$. Then $F_k \subseteq F_i$, for all $i \geq k$. It is sufficient to prove that $F_i \subseteq F_k$, for all $i \geq k$. Let $x \in F_i$, for $i \geq k$. Then $\psi(x) \in \psi(F_i) = \psi(F_k)$, and so $\psi(x) = \psi(a)$, for some $a \in F_k$. We get $\psi(a \to x) = \psi(a) \to \psi(x) = 1$, that is $a \to x \in \mathrm{Ker}(\psi) = \mathrm{Im}(\varphi)$. Hence there exists $\acute{a} \in L_1$ such that $a \to x = \varphi(\acute{a})$.
On the other hand, $x \in F_i$ implies that $a \to x \in F_i$, by Theorem 1.1. Then

$$\varphi(\acute{a}) \in F_i \Rightarrow \acute{a} \in \varphi^{-1}(F_i) = \varphi^{-1}(F_k) \Rightarrow \varphi(\acute{a}) \in F_k \Rightarrow a \to x \in F_k.$$

Now since $a \in F_k$, we get that $x \in F_k$. Hence $F_i \subseteq F_k$, that is $F_k = F_i$, for all $i \geq k$. Therefore L_2 is Noetherian. □

Proposition 2.2. *Any homomorphic image of a Noetherian residuated lattice is Noetherian.*

Proof. The proof is easy. □

By the following example we show that the converse of Proposition 2.2 may not be true.

Example 2.2. *Consider the residuated lattice A in Example 2.1 part(ii)and $S = B(A)$. Then $p_S : A \to A[S]$ is onto, $A[S]$ is Noetherian, while A is not Noetherian.*

Corollary 2.1. *Let L be a Noetherian residuated lattice and $F \in F(L)$. Then L/F is Noetherian.*

Corollary 2.2. *If L is a Noetherian residuated lattice, then $L[S]$ is Noetherian.*

By Propositions 2.1 and 2.2, we have the following corollary.

Corollary 2.3. *L_1, L_2 are two Noetherian residuated lattices if and only if $L_1 \times L_2$ is a Noetherian residuated lattice.*

Lemma 2.1. *Let F be a finitely generated filter of L and L/F be Noetherian. Then every $G \in F(A)$ containing F is finitely generated.*

Proof. Let $G \in F(A)$ and $F \subseteq G$. Then G/F is a filter of L/F and so it is finitely generated. Therefore G is a finitely generated filter, by Lemma 1.1. □

Corollary 2.4. *If $L/\langle a\rangle$ is Noetherian, for all $1 \neq a \in F$, then L is Noetherian.*

Theorem 2.2. *In a Noetherian residuated lattice L every filter is a finite intersection of prime filters of the second kind of L.*

Proof. Suppose not; then the set T of filters of L for which the theorem is false is not empty, hence has a maximal element F. Since F is not $PF2$, there exist $F_1, F_2 \in F(L)$ such that $F = F_1 \bigcap F_2$, $F_1 \supset F$ and $F_2 \supset F$. Hence each of F_1 and F_2 is a finite intersection of $PF2s$ and therefore $F \notin T$, which is a contraction. □

In the following example, we show that the above theorem for prime filters and prime filters of the third kind may not be true.

Example 2.3. *Let $L = \{0, a, b, c, 1\}$ with $0 < a, b < c < 1$, but a, b are incomparable. L becomes a residuated lattice relative to the following operations:*

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

The set of all PFs and $PF3s$ are $\{\{a, c, 1\}, \{b, c, 1\}\}$ and $\{\{a, c, 1\}, \{b, c, 1\}, \{c, 1\}\}$ respectively.
$\{1\}$ is a filter of L while we can not write it in the form of finite intersection of PFs or $PF3s$ of L.

Theorem 2.3. *L is a Noetherian residuated lattice if and only if every prime filter of the second kind of L is finitely generated.*

Proof. Let every prime filter of the second kind of L be finitely generated. Consider $T = \{F \in F(A) \mid F \text{ is not finitely generated}\}$. If T is a nonempty set, we show that T has a maximal element, by the Zorn's Lemma. Let $\{F_i\}_{i\in I}$ be a sequence of elements of T. Put $F = \cup_{i\in I} F_i$, now we show that $F \in T$. If F is finitely generated, then $F = \langle x_1, x_2, ..., x_k\rangle$, $x_1, x_2, ..., x_k \in L$. Since $\{F_i\}_{i\in I}$ is a sequence, we can obtain F_j such that $x_1, x_2, ..., x_k \in F_j$. Thus $F = \langle x_1, x_2, ..., x_k \rangle \subseteq F_j \subseteq F$, and so $F_j = \langle x_1, x_2, ..., x_k \rangle$ that is a contraction. Hence F is an upper bound for $\{F_i\}_{i \in I}$. Therefore by the Zorn's Lemma T has a maximal element P. Now we show that P is $PF2$. Let P don't be $PF2$, then there exist $P_1, P_2 \in F(L)$ such that $P = P_1 \bigcap P_2$, $P_1 \supset P$ and $P_2 \supset P$. Hence each of P_1 and P_2 is finitely generated and so P is finitely generated, by Lemma 1.1, which is a contraction. Then P is $PF2$, by hypothesis we get that P is finitely generated and so $P \notin T$. Therefore T is empty. The proof of the converse is clear by Theorem 2.1. □

Lemma 2.2. *Let L be a Noetherian residuated lattice and $\varphi : L \to L$ be an onto homomorphism. Then φ is isomorphism.*

Proof. We have $\mathrm{Ker}(\varphi) \subseteq \mathrm{Ker}(\varphi^2) \subseteq ...$ an increasing sequence of filters of L. So by hypothesis, there exists $m \in N$ such that $\mathrm{Ker}(\varphi^i) = \mathrm{Ker}(\varphi^m)$, $\forall\ i \geq m$. Let $a \in \mathrm{Ker}(\varphi)$. Then $\varphi(a) = 1$. Since φ is onto, there is $b \in A$ such that $a = \varphi^m(b)$. Hence $\varphi^{m+1}(b) = \varphi(a) = 1$, so $b \in \mathrm{Ker}(\varphi^{m+1}) = \mathrm{Ker}(\varphi^n)$. Thus $a = \varphi^n(b) = 1$, therefore $\mathrm{Ker}(\varphi) = 1$. i.e φ is one to one, by hypothesis we get that φ is isomorphism. □

Theorem 2.4. *Let L be a residuated lattice. Then the following conditions are equivalent*
1. *Every nonempty set of filters of L has a minimal element,*
2. *Every decreasing sequence of filters of L like $F_1 \supseteq F_2 \supseteq ...$ is stationary.*

Proof. The proof is similar to the proof of Theorem 2.1. □

Definition 2.2. *A residuated L is called Artinian if it satisfies any one of the conditions of Theorem 2.4.*

Consider Example 2.1. The residuated lattices of part (i) and (iii) are Artinian, while the residuated lattice of part (ii) is not Artinian.

Note: Propositions 2.1 and 2.2 and Corollaries 2.1, 2.2 and 2.3 are true for Artinian.

Theorem 2.5. *Let L be a Artinian residuated lattice. Then $Max(L)$ is a finite set.*

Proof. Let

$$T = \{F \in F(L) \mid F \text{ is a finite intersection of maximal filters of } L\}.$$

Since $Max(L) \neq \emptyset$, by hypothesis T has a minimal element F, so there exist $M_1, M_2, ..., M_k$ such that $F = M_1 \cap ... \cap M_k$. Consider $M \in Max(L)$. Since F is minimal element of T, so $M_1 \cap ... \cap M_k = F = M \cap F \subseteq M$. By $M \in Max(L)$, we obtain M is $PF2$. Therefore there exist $i \in N$ Such that $M_i \subseteq M$, and we conclude $M = M_i$. Thus $Max(L) = \{M_1, ..., M_k\}$, that is $Max(L)$ is finite. □

We recall the notion of an ordinal sum of residuated lattices. For two residuated lattices A_1 and A_2 with $A_1 \cap A_2 = \{1\}$, we set $A = A_1 \cup A_2$. On A we define the operations $\odot$ and $\to$ as follows

$$x \odot y = \begin{cases} x \odot_i y \text{ if } x, y \in A_i, i = 1, 2, \\ x \text{ if } x \in A_1 \setminus \{1\}, y \in A_2, \\ y \text{ if } y \in A_1 \setminus \{1\}, x \in A_2. \end{cases}$$

$$x \to y = \begin{cases} x \to_i y \text{ if } x, y \in A_i, i = 1, 2, \\ y \text{ if } y \in A_1 \setminus \{1\}, x \in A_2, \\ 1 \text{ if } x \in A_1 \setminus \{1\}, y \in A_2. \end{cases}$$

Then A is a residuated lattice and we denote the ordinal sum $A = A_1 \oplus A_2$.
It is easy to check that $G \in F(A_1 \oplus A_2)$ if and only if $G \in F(A_2)$ or $G = F \cup A_2$, for some $F \in F(A_1)$.
It is easy to check that A_1 and A_2 are Artinian (Noetherian) if and only if $A_1 \oplus A_2$ is Artinian (Noetherian).

3. **Fuzzy Artinian and Fuzzy Noetherian residuated lattice**

Definition 3.1. *A residuated lattice L is said to be a fuzzy Noetherian (Artinian) if for every ascending sequence $\mu_1 \subseteq \mu_2 \subseteq ...$(decreasing sequence $\mu_1 \supseteq \mu_2 \supseteq ...$) of fuzzy filters of L there is a natural number n such that for all $i \geq n$, $\mu_i = \mu_n$. L is said to satisfy the fuzzy maximal(minimal) condition if every nonempty subset of fuzzy filters of L has a maximal(minimal) element.*

Theorem 3.1. *If L is a fuzzy Noetherian (Artinian) residuated lattice, then L is Noetherian (Artinian).*

Proof. Let $F_1 \subseteq F_2 \subseteq ...$ be an ascending sequence of filters of L. We obtain $\chi_{F_1} \subseteq \chi_{F_2} \subseteq ...$ and so by hypothesis there is $n \in N$ such that $\chi_{F_i} = \chi_{F_n}$ for all $i \geq n$. Thus we can conclude that $F_i = F_n$, for all $i \geq n$. □

The converse of the above theorem is not true general.

Example 3.1. *Consider the residuated lattice of Example 2.1 part (iii). Define for all $i \in N$ $\mu_i : A \to A$ by:*

$$\mu_i(x) = \begin{cases} 1 - \frac{1}{i} & \textit{if } x = 1, \\ 0 & \textit{if } x \neq 1. \end{cases}$$

It is clear that μ_i is a fuzzy filter of A for all $i \geq 1$, and $\mu_1 \subset \mu_2 \subset ...$. So A is Noetherian while it is not fuzzy Noetherian.

Theorem 3.2. *L is a fuzzy Noetherian (Artinian) residuated lattice, if and only if L satisfies the fuzzy maximal (minimal) condition.*

Proof. If L does not satisfy the fuzzy maximal condition, then there is a nonempty set ϑ of fuzzy filters of L such that ϑ dose not have a maximal element. Thus we can select μ_i, $i = 1, 2, ...$ satisfying $\mu_1 \subseteq ... \subseteq \mu_n \subseteq ...$ and for all $i \in N$, $\mu_i \neq \mu_{i+1}$, this is a contradiction.
Conversely, let $\mu_1 \subseteq \mu_2 \subseteq ...$ be a sequence of fuzzy filters of L. Since L satisfies the fuzzy maximal condition, L has a maximal element μ_n. Then $\mu_i = \mu_n$ for all $i \geq n$. Therefore L is a fuzzy Noetherian residuated lattice. □

Corollary 3.1. *If L satisfies the fuzzy maximal (minimal) condition, then L is Noetherian (Artinian).*

Theorem 3.3. *L is Noetherian if and only if for all $\mu \in FF(L)$, $\emptyset \neq \mu_t$ is finitely generated, for all $t \in [0, 1]$.*

Proof. Let L be Noetherian. Then for all $\mu \in FF(L)$, $\emptyset \neq \mu_t \in F(L)$, and so μ_t is finitely generated, for all $t \in [0,1]$. Conversely, let F be a filter of L. We show that F is finitely generated. Consider

$$\mu(x) = \begin{cases} \frac{1}{2} \text{ if } x \in F, \\ 0 \text{ if } x \notin F. \end{cases}$$

Thus $\mu \in FF(L)$ and $\mu_{\frac{1}{2}} = F$. Then by hypothesis F is finitely generated and so L is Noetherian. □

Theorem 3.4. *The following statements are equivalent.*
(i) L is Noetherian,
(ii) Im(μ) is well ordered.

Proof. $(i) \to (ii)$ If Im(μ) is not well ordered, then there is $\{s_i\}_{i\in N}$ of Im(μ) such that $... < s_2 < s_1$. We get the ascending sequence $\mu_{s_1} \subset \mu_{s_2} \subset ...$, which is not stationary, it is not true by (i).
$(ii) \to (i)$ Suppose there is a strictly increasing sequence $G_1 \subset G_2 \subset ...$ of filters of L which is not stationary. Define μ on L by:

$$\mu(x) = \begin{cases} \frac{1}{k} \text{ if } k = \min\{r \in N \mid x \in G_r\}, \\ 0 \text{ if } x \notin G_n, \ \forall n \geq 1. \end{cases}$$

We prove that $\mu \in FF(L)$. Since $1 \in G_n$, for all $n \geq 1$ we have $1 = \mu(1) \geq \mu(x)$ for all $x \in L$, so $(FF3)$ hold. For proof of $(FF4)$, we consider two cases for $y \in L$.
Case 1: $y \notin G_n$, for all $n \geq 1$, Case 2: $y \in G_n$, where $n = \min\{r \in N \mid x \in G_r\}$.
In case 1, $\mu(y) = 0$. Now we consider the following two cases:
(i) $x \to y \notin G_n$, for all $n \geq 1$. Thus $\mu(x \to y) = 0$, we get that $\mu(y) = 0 = \min\{\mu(x \to y), \mu(x)\}$.
(ii) $x \to y \in G_m$, $m = \min\{r \in N \mid x \in G_r\}$. Since $y \notin G_n$, for all $n \geq 1$ and G_n is a filter of L, then $x \notin G_n$, for all $n \geq 1$. Thus $\mu(y) = 0 = \min\{\mu(x \to y), \mu(x)\}$.
In case 2, $\mu(y) = \frac{1}{n}$. Since $y \leq x \to y$, so $x \to y \in G_m$, for all $m \geq n$. Thus $\mu(x \to y) = \frac{1}{t}$, for $t \leq n$. If $x \notin G_n$ for all n, then $\mu(y) = \frac{1}{n} \geq 0 = \min\{\mu(x \to y), \mu(x)\}$. Now let $x \in G_s$ where $s = \min\{r \in N \mid x \in G_r\}$. If $t = n$, then by considering $s \leq n$ or $s > n$, we have $\mu(y) = \frac{1}{n} \geq \min\{\mu(x \to y), \mu(x)\}$. Now let $t < n$. We show that $s \geq n$. Suppose $s < n$, we have $x \to y \in G_t$ and $x \in G_s$. Thus $y \in G_k$ for $k < n$, which is not true. Hence $s \geq n$, since $n > t$, we obtain $\mu(y) = \frac{1}{n} \geq \frac{1}{s} = \min\{\mu(x \to y), \mu(x)\}$. Therefore $\mu \in FF(L)$ and Im(μ) is not well ordered. □

By using Proposition 3.16 of [2], we have the following Theorem.

Theorem 3.5. *L is Noetherian and Artinian, if and only if Im(μ) is finite, for all $\mu \in FF(L)$.*

Proof. Let $\mu \in FF(L)$ and Im(μ) be infinite. Then there exists a strictly increasing or decreasing sequence $\{s_i\}_{i\in N}$, where $s_i \in$ Im(μ), for all $i \in N$. Suppose $\{s_i\}_{i\in N}$ is strictly increasing sequence. It is clear that μ_{s_i} are filters of L for $i \geq 1$ and $\mu_{s_1} \supset \mu_{s_2} \supset ...$. This sequence is not stationary, which is a contradiction with Artinian. If $\{s_i\}_{i\in N}$ is a decreasing sequence, similar to the above argument, by Noetherian, we can prove theorem.

Conversely, let L don't be Artinian. Then there is a strictly descending sequence $L = A_0 \supset A_1 \supset A_2 \supset ...$ of filters of L which is not stationary. Define μ on L by:

$$\mu(x) = \begin{cases} \frac{r}{r+1} \text{ if } x \in A_r \setminus A_{r+1},\ r = 0, 1, 2, .. \\ 1 \text{ if } x \in \bigcap_{r=0}^{\infty} A_r. \end{cases}$$

We prove that $\mu \in FF(L)$. Since $1 \in \bigcap_{r=0}^{\infty} A_r$, we have $1 = \mu(1) \geq \mu(x)$ for all $x \in L$, so $(FF3)$ hold. For proof of $(FF4)$, we consider two cases for $y \in L$.
Case 1: $y \in \bigcap_{r=0}^{\infty} A_r$, Case 2: $y \in A_r \setminus A_{r+1}$ for some $r \geq 0$.
In case 1, $\mu(y) = 1$ and so $\mu(y) \geq \min\{\mu(x \to y), \mu(x)\}$.
In case 2, $\mu(y) = \frac{r}{r+1}$. Now we consider the following two cases:
(i) $x \to y \notin A_{r+1}$. Since $y \leq x \to y$ and $y \in A_r$, so $x \to y \in A_r$, that is $\mu(x \to y) = \frac{r}{r+1}$.
If $x \in \bigcap_{r=0}^{\infty} A_r$, then $\mu(y) \geq \min\{\mu(x \to y), \mu(x)\}$.
If $x \in A_j \setminus A_{j+1}$ for some $j \geq 0$, then by considering $j \leq r$ or $j > r$, we can conclude that $\mu(y) \geq \min\{\mu(x \to y), \mu(x)\}$.
(ii) $x \to y \in A_{r+1}$. Since $y \notin A_{r+1}$ and A_{r+1} is a filter of L, then $x \notin A_{r+1}$. Thus there is $j \leq r$, such that $x \in A_j \setminus A_{j+1}$.
If $x \to y \in \bigcap_{r=0}^{\infty} A_r$, then $\mu(x \to y) = 1$. Hence $\mu(y) = \frac{r}{r+1} \geq \frac{j}{j+1} = \min\{\mu(x \to y), \mu(x)\}$.
If $x \to y \in A_k \setminus A_{k+1}$, then $k > r$ and so $k > j$. We can get that $\mu(y) = \frac{r}{r+1} \geq \frac{j}{j+1} = \min\{\mu(x \to y), \mu(x)\}$.
Therefore $\mu \in FF(L)$, while $\mathrm{Im}(\mu)$ is infinite, which is a contradiction.
Now let L don't be Noetherian. Then there is a strictly increasing sequence $\{1\} = A_0 \subset A_1 \subset A_2 \subset ...$ of filters of L which is not stationary. Define ν on L by:

$$\nu(x) = \begin{cases} \frac{1}{r+1} \text{ if } x \in A_{r+1} \setminus A_r,\ r = 0, 1, 2, .., \\ 1 \text{ if } x = 1, \\ 0 \text{ if } x \in L \setminus \bigcup_{r=0}^{\infty} A_r. \end{cases}$$

We can check that $\nu \in FF(L)$. Since $\mathrm{Im}(\nu)$ is infinite, we obtain a contradiction. □

4. **Conclusion**

Filters theory play an important role in studying logical systems and the related algebraic structures. In this paper, we introduce Noetherian and Artinian residuated lattices, by filters, and derive some of their characterizations. We prove that in an Artinian residuated lattice the set of all maximal filters is finite. Then we define fuzzy Noetherian and Artinian residuated lattices and investigate relations between fuzzy Noetherian (Artinian) residuated lattices and Noetherian (Artinian) residuated lattices. It is our hope that this work would other foundations for further study of the theory of residuated lattices.
In our future work we are going to find the relation between fuzzy Noetherian residuated lattices and fuzzy Artinian residuated lattices. Also we will investigate Noetherian and Artinian in other algebraic structures.

REFERENCES

[1] *R. Cingnoli, I.M.L. D'Ottaviano and D. Mundici*, Algebraic Foundations of Many-valued Reasoning, Kluwer Academic publ, Dordrecht, (2000).

[2] *W.A. Dudek, Y.B. Jun and Z. Stojakovic*, On fuzzy ideals in BCC-algebras, Fuzzy Sets and Systems, 123 (2001) 251-258.

[3] *F. Esteva and L. Godo*, Monoidal t-norm based logic, towards a logic for left-continuous t-norm, Fuzzy sets and systems, 124 (2001) 271-288.

[4] *N. Galatos, P. Jipsen, T. Kowalski and H. Ono*, Residueted Lattices, An Algebraic Glimpes at Submstractural Logic, volume 151 studies in Logics and the Foundations of Mathematics, Elsevier, (2007).

[5] *P. Hajek*, Metamathematics of Fuzzy Logic, Kluwer Academic Publishers, (1998).

[6] *U. Hohle, Commutative*, residuated l-monoids, in: U. Hohle, E.P. Klement (Eds.), Non-Classical Logics and Their Applications to Fuzzy Subsets, Kluwer Academic Publishers, Boston, Dordrecht, (1995), pp. 53-106.

[7] *B. Van Gasse, G. Deschrijver, C. Cornelis and E.E. Kerre*, Filters of residuated lattices and triangle algebras, Information Sciences, 180 (2010) 3006-3020.

[8] *M. Ward, R. P. Dilworth*, Residueted Lattices, Trans. Am. Math. Soc. 45 (1939) 335-354.

[9] *L.A. Zadeh*, Fuzzy sets, Inform. Control 8 (1965) 338-353.

[10] *Y. Zhu, Y. Xu*, On filter theory of residuated latticees, Information Sciences, 180 (2010) 3614-3632.