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ON APPROXIMATE-ANALYTICAL SOLUTION OF GENERALIZED BLACK-SCHOLES EQUATION

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In option pricing theory, the Black-Scholes equation is one of the most effective models for pricing options. In this paper, we present an analytical method for the generalized Black–Scholes partial differential equation, which is used for option pricing. The proposed method (LTNHPM) is based on Laplace transform (LT) and new homotopy perturbation method (NHPM). Two test examples have been solved for illustrating the merits of the proposed analytical approximation method. The method can be extended for solving problems arising in financial mathematics.

Keywords: Black-Scholes equation; Laplace transform method; Homotopy perturbation method; European option pricing.

1. Introduction

A financial derivative is a contract which provides to the holder a future payment that depends on the price of the assets such as stocks, currencies, commodities. Options are financial derivative products that give the right, but not the obligation, to engage in a future transaction on some underlying financial instrument. For instance, European-style options can be exercised only at the expiration date, in contrast to American-style options, where the holder can exercise them earlier than the maturity date [1]. The Black–Scholes theory of option pricing, developed by Black, Scholes and Merton is one of the most 2. Paper contents influential theories in finance [2-3]. The Black-Scholes and Merton model hinges on the modeling of stock returns by the geometric Brownian motion [4]. According to the idea of Black-Scholes, the option price can be modeled as a terminal boundary problem for a partial differential equation. Therefore, it is reasonable to adopt the existing theory and methods of partial differential equation as a fundamental approach to the study of the option pricing. This includes designing efficient algorithms for solving option pricing problems from the viewpoint of numerical solutions of partial differential equation problems. Many authors have applied several different methods to solve the Black-Scholes partial differential equation [5-16]. Recently the modified HPMs [17] have been used to solve various functional equations. These methods have been applied to solve nonlinear equations of heat transfer [18], MHD viscous flow

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over a stretching sheet [19], Troesch's problem [20], approximation of normal distribution integral [21] and many other subjects [22-25]. In this work, we present the solution of Black–Scholes partial differential equation by combination of Laplace transform and new homotopy perturbation methods. The paper is divided into five sections. After the introduction, we present the standard Black-Scholes-Merton model in Section 2. Section 3 includes analysis of the new technique for Black-Scholes problem and two examples are given to illustrate the proposed approach in Section 4. Conclusions are present in Section 5.

2. The Black-Scholes-Merton model framework

Let $V(S,t)$ denotes the price of an option at the moment t, where $S=S(x), x=x(t)$ is the price of the underlying asset at time t. The randomness of $V(S(t),t)$ would be fully correlated to that $S(t)$. Thus, we consider a portfolio which contains only S and V, but in opposite position in order to cancel out the randomness. Then this portfolio becomes deterministic. Let us consider the following portfolio [4]

$$\Pi = V - \Delta S.$$

The change of the portfolio in one step time is

$$d\Pi = dV - \Delta dS$$

where Δ is held fixed during the time step. From Ito's lemma [1]

$$d\Pi = \sigma S\left(\frac{\partial V}{\partial S} - \Delta\right)dz + \left(\mu S\frac{\partial V}{\partial S} - \mu\Delta S + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt.$$

Here, S is the current asset price, μ is the growth rate of an asset, and $\sigma = \sigma(S(t),t)$ is the volatility function of the underlying asset. By choosing $\Delta = \frac{\partial V}{\partial S}$, at the starting time of each time step we can write

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt.$$

From the hypothesis of no arbitrage opportunities, the return, $\frac{d\Pi}{\Pi}$, should be the same as Π being invested in a riskless bank with interest rate $r(t)$, i.e. $\frac{d\Pi}{\Pi} = r(t)dt$. Therefore, we must have

$$r(t)\Pi dt = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt$$

Or equivalently,

$$r(t)V - r(t)S\frac{\partial V}{\partial S} - \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right) = 0$$

Hence, the partial differential equation of Black-Scholes for option pricing is written as

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+r(t)S\frac{\partial V}{\partial S}-r(t)V=0\,.$$

In this paper, we consider the partial differential equation of Black-Scholes with the following terminal condition,

$$V(S,t)=c(S,T)=(S-K)^+=\max(S-K,0),\quad \text{for call option,}$$

$$V(S,t)=p(S,T)=(K-S)^+=\max(K-S,0),\quad \text{for put option}$$

where K is the exercise (strike) price, and T is the expiry time of European option. In general, the final condition is

$$V(S,T)=f(S)$$

where f is the payoff function. By setting $S=x$ the Black-Scholes equation becomes

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2(x,t)\,x^2\frac{\partial^2 V}{\partial x^2}+r(t)\,x\frac{\partial V}{\partial x}-r(t)V=0,\quad (x,t)\in\mathbb{R}^+\times(0,T)$$

with the terminal and boundary conditions as below

$$V(x,T)=(x-K)^+,\ \ x\in\mathbb{R}^+$$

$$V(0,t)=0,\qquad t\in[0,T].$$

3. Analysis of Method (LTNHPM)

Consider the Black-Scholes equation in the following form

$$\begin{aligned}&\frac{\partial V(x,t)}{\partial t}+\frac{1}{2}\sigma^2x^2\frac{\partial^2 V(x,t)}{\partial x^2}+r(t)\,x\frac{\partial V(x,t)}{\partial x}-r(t)V(x,t)=0,\\ &V(x,0)=f_1(x)\end{aligned}\tag{1}$$

By the new homotopy technique, we construct a homotopy $U:\mathbb{R}^2\times[0,1]\to\mathbb{R}^2$, which satisfies

$$\begin{aligned}H(U(x,t),p)&=\frac{\partial U(x,t)}{\partial t}-v_0+pv_0\\ &\quad+p\left\{\frac{1}{2}\sigma^2x^2\frac{\partial^2 U(x,t)}{\partial x^2}+r(t)\,x\frac{\partial U(x,t)}{\partial x}-r(t)U(x,t)\right\}=0,\end{aligned}\tag{2}$$

where $p\in[0,1]$ is an embedding parameter, v_0 is an initial approximation of the solution of equation (1). By applying Laplace transform on both sides of (2), we have

$$\mathcal{L}\left\{\frac{\partial U}{\partial t}-v_0+pv_0+p\left\{\frac{1}{2}\sigma^2x^2\frac{\partial^2 U}{\partial x^2}+r(t)\,x\frac{\partial U}{\partial x}-r(t)U\right\}\right\}=0\tag{3}$$

Using the differential property of Laplace transform we have

$$s\mathcal{L}\{U(x,t)\}-U(x,0)=\mathcal{L}\left\{v_0-pv_0-p\left\{\frac{1}{2}\sigma^2x^2\frac{\partial^2U}{\partial x^2}+r(t)x\frac{\partial U}{\partial x}-r(t)U\right\}\right\} \quad (4)$$

or

$$\mathcal{L}\{U(x,t)\}=\frac{1}{s}\left\{U(x,0)+\mathcal{L}\left\{v_0-pv_0-p\left\{\frac{1}{2}\sigma^2x^2\frac{\partial^2U}{\partial x^2}+r(t)x\frac{\partial U}{\partial x}-r(t)U\right\}\right\}\right\} \quad (5)$$

By applying inverse Laplace transform on both sides of (5), we have

$$U(x,t)=\mathcal{L}^{-1}\left\{\frac{1}{s}\left\{U(x,0)+\mathcal{L}\left\{v_0-pv_0-p\left\{\frac{1}{2}\sigma^2x^2\frac{\partial^2U}{\partial x^2}+r(t)x\frac{\partial U}{\partial x}-r(t)U\right\}\right\}\right\}\right\} \quad (6)$$

According to the HPM, we use the embedding parameter p as a small parameter, and assume that the solutions of equation (6) can be represented as a power series in p as

$$U(x,t)=\sum_{n=0}^{\infty}p^nU_n(x,t)\,. \quad (7)$$

Now let us write the equation (6) in the following form

$$\sum_{n=0}^{\infty}p^nU_n(x,t)=\mathcal{L}^{-1}\left\{\frac{1}{s}\left\{U(x,0)+\mathcal{L}\left\{v_0-pv_0-p\left\{\frac{1}{2}\sigma^2x^2\frac{\partial^2U}{\partial x^2}+r(t)x\frac{\partial U}{\partial x}-r(t)U\right\}\right\}\right\}\right\} \quad (8)$$

Comparing coefficients of terms with identical powers of p, leads to

$$\begin{aligned}
&p^0: U_0(x,t)=\mathcal{L}^{-1}\left\{\frac{1}{s}\left(U(x,0)+\mathcal{L}\{v_0\}\right)\right\}\\
&p^1: U_1(x,t)=\mathcal{L}^{-1}\left\{-\frac{1}{s}\left(\mathcal{L}\left\{v_0+\frac{1}{2}\sigma^2x^2\frac{\partial^2U_0}{\partial x^2}+r(t)x\frac{\partial U_0}{\partial x}-r(t)U_0\right\}\right)\right\} \qquad (9)\\
&p^{n+1}: U_{n+1}(x,t)=\mathcal{L}^{-1}\left\{-\frac{1}{s}\left(\mathcal{L}\left\{\frac{1}{2}\sigma^2x^2\frac{\partial^2U_n}{\partial x^2}+r(t)x\frac{\partial U_n}{\partial x}-r(t)U_n\right\}\right)\right\},\ n=1,2,\ldots
\end{aligned}$$

Suppose that the initial approximation has the form $U(x,0)=v_0=f_1(x)$, therefore the exact solution may be obtained as following

$$u(x,t)=\lim_{p\to 1}U(x,t)=U_0(x,t)+U_1(x,t)+\cdots. \quad (10)$$

Convergece

Suppose that X and Y be Banach space and $N:X\to Y$ is a contraction nonlinear mapping, that is

$$\forall\, v,\tilde{v}\in X;\ \|N(v)-N(\tilde{v})\|\le\gamma\,\|v-\tilde{v}\|,\ 0<\gamma<1.$$

which according to Banach's fixed point theorem having the fixed point u, that is $N(u)=u$.

Theorem: The sequence generated by LTNHPM will be regarded as

$$V_n=N(V_{n-1}),V_{n-1}=\sum_{i=0}^{n-1}U_i\,,n=1,2,3,\ldots,$$

and suppose that $V_0 = U_0 \in B_r(u)$ where $B_r(u) = \{u^* \in X \,\big|\, \|u^* - u\| < r\}$, then we have the following statements:

$$i)\ \|V_n - u\| \le \gamma^n \|U_0 - u\|,$$
$$ii)\ V_n \in B_r(u),$$
$$iii) \lim_{n\to\infty} V_n = u.$$

Proof: $i)$ By induction method on n, for $n=1$ we have

$$\|V_1 - u\| = \|N(V_0) - N(u)\| \le \gamma \|U_0 - u\|.$$

Assume that $\|V_{n-1} - u\| \le \gamma^{n-1} \|U_0 - u\|$ as induction hypothesis, then

$$\|V_n - u\| = \|N(V_{n-1}) - N(u)\| \le \gamma \|V_{n-1} - u\| \le \gamma\, \gamma^{n-1} \|U_0 - u\| = \gamma^n \|U_0 - u\|.$$

$ii)$ Using (i), we have

$$\|V_n - u\| \le \gamma^n \|U_0 - u\| \le \gamma^n r < r \Rightarrow V_n \in B_r(u).$$

$iii)$ Because of $\|V_n - u\| \le \gamma^n \|U_0 - u\|$, and $\lim_{n\to\infty} \gamma^n = 0$, we derive $\lim_{n\to\infty} \|V_n - u\| = 0$, that is,

$$\lim_{n\to\infty} V_n = u.$$

4. Examples

Example 1. Consider the following Black-Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{11}$$

By setting $x = \ln S$ and $\tau = T - t$, the problem (11) is reduced to a Cauchy problem of a parabolic equation with constant coefficients

$$\begin{aligned} &\frac{\partial W}{\partial \tau} - \frac{1}{2}\sigma^2 \frac{\partial^2 W}{\partial x^2} - (r - \frac{1}{2}\sigma^2)\frac{\partial W}{\partial x} + rW = 0, \\ &W(x,0) = \max(\mathrm{e}^x - 1, 0) \end{aligned} \tag{12}$$

If we set $k = \dfrac{2r}{\sigma^2}$ the equation (12) can be represented as the following form

$$\begin{aligned} &\frac{\partial W}{\partial \tau} = \frac{\partial^2 W}{\partial x^2} + (k-1)\frac{\partial W}{\partial x} - kW = 0, \\ &W(x,0) = \max(\mathrm{e}^x - 1, 0) \end{aligned} \tag{13}$$

The exact solution of above equation was found to be of the form

$$W(x,\tau) = \max(\mathrm{e}^x - 1, 0)\mathrm{e}^{-k\tau} + \max(\mathrm{e}^x, 0)(1 - \mathrm{e}^{-k\tau}). \tag{14}$$

To solve equation (13) by the LTNHPM, we construct the following homotopy

$$\frac{\partial U}{\partial \tau} - u_0 + p\left\{u_0 - \frac{\partial^2 U}{\partial x^2} - (k-1)\frac{\partial U}{\partial x} + kU\right\} = 0, \tag{15}$$

Applying Laplace transform on both sides of (15), we have

$$\mathcal{L}\left\{\frac{\partial U}{\partial \tau}-u_0+p\left\{u_0-\frac{\partial^2 U}{\partial x^2}-(k-1)\frac{\partial U}{\partial x}+kU\right\}\right\}=0 \tag{14}$$

Using the differential property of Laplace transform we have

$$s\mathcal{L}\{U(x,\tau)\}-U(x,0)=\mathcal{L}\left\{u_0-p\left\{u_0-\frac{\partial^2 U}{\partial x^2}-(k-1)\frac{\partial U}{\partial x}+kU\right\}\right\} \tag{16}$$

or

$$\mathcal{L}\{U(x,\tau)\}=\frac{1}{s}\left\{U(x,0)+\mathcal{L}\left\{u_0-p\left\{u_0-\frac{\partial^2 U}{\partial x^2}-(k-1)\frac{\partial U}{\partial x}+kU\right\}\right\}\right\} \tag{17}$$

By applying inverse Laplace transform on both sides of (17), we have

$$U(x,\tau)=\mathcal{L}^{-1}\left\{\frac{1}{s}\left\{U(x,0)+\mathcal{L}\left\{u_0-p\left\{u_0-\frac{\partial^2 U}{\partial x^2}-(k-1)\frac{\partial U}{\partial x}+kU\right\}\right\}\right\}\right\} \tag{18}$$

Suppose the solution of equation (18) to have the following form

$$U(x,\tau)=U_0(x,\tau)+pU_1(x,\tau)+p^2U_2(x,\tau)+\cdots, \tag{19}$$

where $U_i(x,\tau)$ are unknown functions which should be determined. Substituting equation (19) into equation (18), collecting the same powers of p and equating each coefficient of p to zero, results in

$$\begin{aligned}
&p^0: U_0(x,\tau)=\mathcal{L}^{-1}\left\{\frac{1}{s}\big(U(x,0)+\mathcal{L}\{u_0\}\big)\right\}\\
&p^1: U_1(x,\tau)=\mathcal{L}^{-1}\left\{-\frac{1}{s}\left(\mathcal{L}\left\{u_0-\frac{\partial^2 U_0}{\partial x^2}-(k-1)\frac{\partial U_0}{\partial x}+kU_0\right\}\right)\right\} \qquad (20)\\
&p^{n+1}: U_n(x,\tau)=\mathcal{L}^{-1}\left\{\frac{1}{s}\left(\mathcal{L}\left\{\frac{\partial^2 U_n}{\partial x^2}+(k-1)\frac{\partial U_n}{\partial x}-kU_n\right\}\right)\right\},\ n=1,2,\ldots
\end{aligned}$$

Assuming $u_0=U(x,0)=\max(\mathrm{e}^x-1,0)$, and solving the above equation for $U_j(x,\tau),\ j=0,1,\cdots$ leads to the result

$$\begin{aligned}
&U_0(x,\tau)=\max(\mathrm{e}^x-1,0)(1+\tau),\\
&U_1(x,\tau)=\frac{k\tau^2}{2}\left\{\max(\mathrm{e}^x-1,0)+\max(\mathrm{e}^x,0)\right\}+\tau\left\{-(k+1)\max(\mathrm{e}^x-1,0)+k\max(\mathrm{e}^x,0)\right\},\\
&U_2(x,\tau)=\frac{k^2\tau^3}{6}\left\{\max(\mathrm{e}^x-1,0)-\max(\mathrm{e}^x,0)\right\}+\frac{k(k+1)\tau^2}{2}\left\{\max(\mathrm{e}^x-1,0)-\max(\mathrm{e}^x,0)\right\},\\
&U_3(x,\tau)=\frac{k^3\tau^4}{24}\left\{-\max(\mathrm{e}^x-1,0)+\max(\mathrm{e}^x,0)\right\}+\frac{k^2(k+1)\tau^3}{6}\left\{-\max(\mathrm{e}^x-1,0)+\max(\mathrm{e}^x,0)\right\},\\
&U_4(x,\tau)=\frac{k^4\tau^5}{120}\left\{\max(\mathrm{e}^x-1,0)-\max(\mathrm{e}^x,0)\right\}+\frac{k^3(k+1)\tau^4}{24}\left\{\max(\mathrm{e}^x-1,0)-\max(\mathrm{e}^x,0)\right\},\\
&U_5(x,\tau)=\frac{k^5\tau^6}{720}\left\{-\max(\mathrm{e}^x-1,0)+\max(\mathrm{e}^x,0)\right\}+\frac{k^4(k+1)\tau^5}{120}\left\{-\max(\mathrm{e}^x-1,0)+\max(\mathrm{e}^x,0)\right\},\\
&\vdots
\end{aligned}$$

Therefore we gain the solution of equation (11) as

$$\begin{aligned} W(x,\tau) &= \lim_{p\to 1} U(x,\tau) = U_0(x,\tau) + U_1(x,\tau) + \cdots \\ &= \max(\mathrm{e}^x - 1, 0)\left(1 - k\tau + \frac{k^2\tau^2}{2} - \frac{k^3\tau^3}{6} + \frac{k^4\tau^4}{24} + \cdots\right) \\ &\quad + \max(\mathrm{e}^x, 0)\left(k\tau - \frac{k^2\tau^2}{2} + \frac{k^3\tau^3}{6} - \frac{k^4\tau^4}{24} + \cdots\right) \\ &= \max(\mathrm{e}^x - 1, 0)\,\mathrm{e}^{-k\tau} + \max(\mathrm{e}^x, 0)(1 - \mathrm{e}^{-k\tau}) \end{aligned}$$

which is exact solution.

Example 2. Consider the following Black-Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r(t) S \frac{\partial V}{\partial S} - r(t) V = 0, \tag{21}$$

where $\sigma = k(2 + \sin x), S = x, r(t) = \alpha, k, \alpha > 0$ and initial condition as following $V(x,0) = \max(x - 25\mathrm{e}^{\alpha}, 0)$

The exact solution of above equation was found to be of the form

$$V(x,t) = x(1 - \mathrm{e}^{-\alpha t}) + \max(x - 25\mathrm{e}^{\alpha}, 0)\mathrm{e}^{-\alpha t}. \tag{22}$$

To solve equation (21) by the LTNHPM, we construct the following homotopy

$$\frac{\partial U}{\partial t} - u_0 + p\left\{u_0 + \frac{k}{2}(2+\sin x)^2 x^2 \frac{\partial^2 U}{\partial x^2} + \alpha x \frac{\partial U}{\partial x} - \alpha U\right\} = 0, \tag{23}$$

Applying Laplace transform on both sides of (23), we have

$$\mathcal{L}\left\{\frac{\partial U}{\partial t} - u_0 + p\left\{u_0 + \frac{k}{2}(2+\sin x)^2 x^2 \frac{\partial^2 U}{\partial x^2} + \alpha x \frac{\partial U}{\partial x} - \alpha U\right\}\right\} = 0 \tag{24}$$

Using the differential property of Laplace transform we have

$$s\mathcal{L}\{U(x,t)\} - U(x,0) = \mathcal{L}\left\{u_0 - p\left\{u_0 + \frac{k}{2}(2+\sin x)^2 x^2 \frac{\partial^2 U}{\partial x^2} + \alpha x \frac{\partial U}{\partial x} - \alpha U\right\}\right\} \tag{25}$$

or

$$\mathcal{L}\{U(x,t)\} = \frac{1}{s}\left\{U(x,0) + \mathcal{L}\left\{u_0 - p\left\{u_0 + \frac{k}{2}(2+\sin x)^2 x^2 \frac{\partial^2 U}{\partial x^2} + \alpha x \frac{\partial U}{\partial x} - \alpha U\right\}\right\}\right\} \tag{26}$$

By applying inverse Laplace transform on both sides of (26), we have

$$U(x,t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\left\{U(x,0) + \mathcal{L}\left\{u_0 - p\left\{u_0 + \frac{k}{2}(2+\sin x)^2 x^2 \frac{\partial^2 U}{\partial x^2} + \alpha x \frac{\partial U}{\partial x} - \alpha U\right\}\right\}\right\}\right\} \tag{27}$$

Suppose the solution of equation (27) to have the following form

$$U(x,t) = U_0(x,t) + pU_1(x,t) + p^2 U_2(x,t) + \cdots, \tag{28}$$

where $U_i(x,t)$ are unknown functions which should be determined. Substituting equation (28) into equation (27), collecting the same powers of p and equating each coefficient of p to zero, results in

$$
\begin{aligned}
&p^0: U_0(x,t)=\mathcal{L}^{-1}\left\{\frac{1}{s}\big(U(x,0)+\mathcal{L}\{u_0\}\big)\right\}\\
&p^1: U_1(x,t)=\mathcal{L}^{-1}\left\{-\frac{1}{s}\left(\mathcal{L}\left\{u_0+\frac{k}{2}(2+\sin x)^2x^2\frac{\partial^2U_0}{\partial x^2}+\alpha x\frac{\partial U_0}{\partial x}-\alpha U_0\right\}\right)\right\}\\
&p^{n+1}: U_{n+1}(x,t)=\mathcal{L}^{-1}\left\{-\frac{1}{s}\left(\mathcal{L}\left\{\frac{k}{2}(2+\sin x)^2x^2\frac{\partial^2U_n}{\partial x^2}+\alpha x\frac{\partial U_n}{\partial x}-\alpha U_n\right\}\right)\right\},\ n=1,2,\ldots
\end{aligned}
\tag{29}
$$

Assuming $u_0=U(x,0)=\max(x-25\mathrm{e}^{\alpha},0)$, and solving the above equation for $U_j(x,\tau),\ j=0,1,\cdots$ leads to the result

$$
\begin{aligned}
&U_0(x,t)=\max(x-25\mathrm{e}^{\alpha},0)(1+t),\\
&U_1(x,t)=\left(-\frac{a}{2!}\max(x-25\mathrm{e}^{\alpha},0)+\frac{\alpha}{2!}x\right)t^2+\big(-(a+1)\max(x-25\mathrm{e}^{\alpha},0)+\alpha x\big)t,\\
&U_2(x,t)=\left(\frac{\alpha^2}{3!}\max(x-25\mathrm{e}^{\alpha},0)-\frac{\alpha^2}{3!}x\right)t^3+\left(\frac{\alpha(\alpha+1)}{2!}\max(x-25\mathrm{e}^{\alpha},0)-\frac{\alpha(\alpha+1)}{2!}x\right)t^2,\\
&U_3(x,t)=\left(-\frac{\alpha^3}{4!}\max(x-25\mathrm{e}^{\alpha},0)+\frac{\alpha^3}{4!}x\right)t^4\\
&\qquad+\left(-\frac{\alpha^2(\alpha+1)}{3!}\max(x-25\mathrm{e}^{\alpha},0)+\frac{\alpha^2(\alpha+1)}{3!}x\right)t^3,\\
&U_4(x,t)=\left(\frac{\alpha^4}{5!}\max(x-25\mathrm{e}^{\alpha},0)-\frac{\alpha^4}{5!}x\right)t^5\\
&\qquad+\left(\frac{\alpha^3(\alpha+1)}{4!}\max(x-25\mathrm{e}^{\alpha},0)-\frac{\alpha^3(\alpha+1)}{4!}x\right)t^4,\\
&U_5(x,t)=\left(-\frac{\alpha^5}{6!}\max(x-25\mathrm{e}^{\alpha},0)+\frac{\alpha^5}{6!}x\right)t^6\\
&\qquad+\left(-\frac{\alpha^4(\alpha+1)}{5!}\max(x-25\mathrm{e}^{\alpha},0)+\frac{\alpha^4(\alpha+1)}{5!}x\right)t^5,\\
&\vdots
\end{aligned}
$$

Therefore the exact solution of equation (21) as follows

$$
\begin{aligned}
V(x,t)&=\lim_{p\to1}U(x,t)=U_0(x,t)+U_1(x,t)+\cdots\\
&=\max(x-25\mathrm{e}^{\alpha},0)\left(1-\alpha t+\frac{\alpha^2t^2}{2!}-\frac{\alpha^3t^3}{3!}+\frac{\alpha^4t^4}{4!}+\cdots\right)\\
&+x\left(\alpha t-\frac{\alpha^2t^2}{2!}+\frac{\alpha^3t^3}{3!}-\frac{\alpha^4t^4}{4!}+\cdots\right)\\
&=\max(x-25\mathrm{e}^{\alpha},0)\,\mathrm{e}^{-\alpha t}+x(1-\mathrm{e}^{-\alpha t})
\end{aligned}
$$

5. Conclusion

The main goal of this paper is to obtain an analytical solution of the generalized Black-Scholes option pricing equation by LTNHPM. The LTNHPM, a combination of Laplace transform method and new homotopy perturbation method, was applied successfully to find the exact solution of Black-Scholes equation. By applying this method on two examples we conclude that LTNHPM is essential tool for solving other partial differential equations of mathematical finance.

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