U.P.B. Sci. Bull., Series A, Vol. 83, Iss. 1, 2021 ISSN 1223-7027

ON THE LOGARITHM OF θ-CENTRALIZERS WITH SOME RELATED RESULTS

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Let $\mathcal{A}$ be an algebra and let θ be a linear mapping on $\mathcal{A}$. By a left θ-centralizer we mean a linear mapping $\Phi : \mathcal{A} \to \mathcal{A}$ satisfying $\Phi(ab) = \Phi(a)\theta(b)$ $(a, b \in \mathcal{A})$. In this article, we show that if $\theta : \mathcal{A} \to \mathcal{A}$ is a continuous automorphism, $\Phi : \mathcal{A} \to \mathcal{A}$ is a left θ-centralizer such that Φ and θ have their spectrum in the right open halfplane of the complex plane, then $\ln \Phi$ is a continuous generalized derivation associated with the continuous derivation $\ln \theta$. Also, we prove that if $\mathcal{A}$ is a unital C^*-algebra on a Hilbert space $\mathcal{H}$, $\theta : \mathcal{A} \to \mathcal{A}$ is an automorphism and $\Phi : \mathcal{A} \to \mathcal{A}$ is a left θ-centralizer such that Φ and θ have their spectrum in the right open halfplane of the complex plane, then there exist two invertible elements $\bar{x}$ and $\bar{y}$ in the weak closure $\bar{\mathcal{A}}$ of $\mathcal{A}$ on the Hilbert space $\mathcal{H}$ such that $\Phi(a) = \bar{y}a\bar{x}^{-1}$ for all $a \in \mathcal{A}$. Some other related results are also discussed.

Keywords: derivation, generalized derivation, θ-centralizer, C^*-algebra.
MSC2020: 16W20, 47B47, 47B48

1. Introduction and preliminaries

Let us first recall some basic definitions and fix some notations which will be used in what follows. Throughout this article, all algebras are defined over the field of complex numbers. If $\mathcal{A}$ is an algebra with identity, we denote by $\mathbf{1}$ and $\mathrm{Inv}(\mathcal{A})$, the identity element and the set of all invertible elements in $\mathcal{A}$, respectively. The spectrum of $a \in \mathcal{A}$ is the set $\sigma_{\mathcal{A}}(a) = \{\lambda \in \mathbb{C} \ : \ \lambda\mathbf{1} - a \notin \mathrm{Inv}(\mathcal{A})\}$. The spectral radius of a is $r(a) = \sup\{|\lambda| \ : \ \lambda \in \sigma_{\mathcal{A}}(a)\}$. A nonzero linear functional φ on $\mathcal{A}$ is called a *character* if $\varphi(ab) = \varphi(a)\varphi(b)$ holds for every $a, b \in \mathcal{A}$. By $\Phi_{\mathcal{A}}$ we denote the set of all characters on $\mathcal{A}$. It is well known that, $\ker \varphi$, the kernel of φ, is a maximal ideal of $\mathcal{A}$, where φ is an arbitrary element of $\Phi_{\mathcal{A}}$. By $\prod_c(\mathcal{A}^\sharp)$, we denote the set of all primitive ideals $\mathcal{P}$ of $\mathcal{A}^\sharp$ such that the quotient algebra $\frac{\mathcal{A}^\sharp}{\mathcal{P}}$ is commutative, where $\mathcal{A}^\sharp$ is the unitization of $\mathcal{A}$. For more material about characters, primitive ideals, maximal ideals and the unitization of an algebra, see, e.g. [4].

A linear mapping $T : \mathcal{A} \to \mathcal{A}$ is called a left (resp. right) centralizer if $T(ab) = T(a)b$ (resp. $T(ab) = aT(b)$) for all $a, b \in \mathcal{A}$, and T is called a centralizer if it is both a left- and a right centralizer. For example, for $a \in \mathcal{A}$, the left multiplication operator $L_a : \mathcal{A} \to \mathcal{A}$ defined by $L_a(b) = ab$ $(b \in \mathcal{A})$ is a left centralizer; similarly, the right multiplication operator $R_a : \mathcal{A} \to \mathcal{A}$ defined by $R_a(b) = ba$ $(b \in \mathcal{A})$ is a right centralizer. It is straightforward to show that $R_bL_a = L_aR_b$ for all $a, b \in \mathcal{A}$. Let $\theta : \mathcal{A} \to \mathcal{A}$ be a linear mapping. A linear

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mapping $\Phi : \mathcal{A} \to \mathcal{A}$ satisfying

$$\Phi(ab) = \Phi(a)\theta(b) \ \big(\text{resp. } \Phi(ab) = \theta(a)\Phi(b)\big) \ \ (a, b \in \mathcal{A})$$

is called a left θ-centralizer (resp. right θ-centralizer) associated with θ, and Φ is called a θ-centralizer if it is both a left- and a right θ-centralizer. A linear mapping $\theta : \mathcal{A} \to \mathcal{A}$ is said to be a Jordan homomorphism if $\theta(a \circ b) = \theta(a) \circ \theta(b)$ holds for all $a, b \in \mathcal{A}$, where $a \circ b = ab + ba$. Also, a linear mapping $\Phi : \mathcal{A} \to \mathcal{A}$ is called a Jordan θ-centralizer if there exists a Jordan homomorphism θ such that $\Phi(a \circ b) = \Phi(a) \circ \theta(b)$ holds for all $a, b \in \mathcal{A}$. Recall that a linear mapping $d : \mathcal{A} \to \mathcal{A}$ is called a derivation if $d(ab) = d(a)b + ad(b)$ for all $a, b \in \mathcal{A}$. For $c \in \mathcal{A}$, the linear mapping $d_c : \mathcal{A} \to \mathcal{A}$ defined by $d_c(a) = ca - ac$ is a derivation. Such a derivation is called an inner derivation. Brešar [3] introduced the following class of derivation-like mappings on an algebra. Let $d : \mathcal{A} \to \mathcal{A}$ be a derivation. A linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ satisfying

$$\delta(ab) = \delta(a)b + ad(b) \quad (a, b \in \mathcal{A}),$$

is called a *generalized derivation* (associated with d) on $\mathcal{A}$. Letting $T = \delta - d$, we have $T(ab) = \delta(ab) - d(ab) = \delta(a)b + ad(b) - d(a)b - ad(b) = (\delta - d)(a)b = T(a)b$, which means that T is a left centralizer on $\mathcal{A}$. Also, let T be a right centralizer and let d be a derivation on $\mathcal{A}$. Then $\delta = T + d$ satisfies $\delta(ab) = T(ab) + d(ab) = aT(b) + d(a)b + ad(b) = a\delta(b) + d(a)b$ for all $a, b \in \mathcal{A}$, that is δ is a generalized derivation.

Let $\mathcal{A}$ be a Banach algebra and let $d : \mathcal{A} \to \mathcal{A}$ be a continuous derivation. It follows from [1, Proposition 18.7] that $\exp(d) := \sum_{n=0}^{\infty} \frac{d^n}{n!}$ is a continuous automorphism on $\mathcal{A}$. Conversely, Zeller-Meier [12] proved that if θ is an automorphism on $\mathcal{A}$, then $\theta = \exp(d)$ for some continuous derivation d on $\mathcal{A}$ under certain conditions. Indeed, he showed that the logarithm of a certain continuous automorphism is a continuous derivation. So, the derivation d will be denoted by $\ln \theta$. Also, see [1, Theorem 18.15] in this regard.

Kamowitz and Scheinberg [8] proved that, if θ is an automorphism of a commutative semisimple complex Banach algebra, then either $\theta^n = I$, the identity mapping, for some positive integer n, in which case $\sigma(\theta)$ consists of a finite union of finite subgroups of the unit circle Γ, or $\Gamma \subset \sigma(\theta)$. Moreover, combining Singer-Wermer Theorem and the above-mentioned result of Zeller-Meier, they obtain that if the automorphism θ satisfies $r(I-\theta) < 1$, then $\theta = I$. In this article, we get a generalization of this result as follows. Let $\mathcal{A}$ be a semiprime Banach algebra and let θ be a continuous Jordan automorphisms on $\mathcal{A}$ such that $\sigma(\theta) \subset \{z \in \mathbb{C} \ : \ Re(z) > 0\} \subset \mathbb{C} - \mathbb{R}_-$. If $\dim(\bigcap_{\mathcal{P} \in \Pi_c(A^\sharp)} \mathcal{P}) \leq 1$, then $\theta = I$. Note that according to [7, Corollary 2.9], the conditions considered for the Banach algebra $\mathcal{A}$ cause that $\mathcal{A}$ is commutative. But as can be seen, we do not need to assume the Banach algebra $\mathcal{A}$ is semisimple. Examples of commutative semiprime Banach algebras which are not semisimple include certain Banach algebras of formal power series, as discussed in [5].

Motivated by Zeller-Meier's result [12], it is natural to ask wether there is a generalization of this characterization in the setting of generalized derivations. Let $\mathcal{A}$ be a unital Banach algebra, let $\theta : \mathcal{A} \to \mathcal{A}$ be a continuous automorphism and let $\Phi : \mathcal{A} \to \mathcal{A}$ be a left θ-centralizer. We show that if both $\sigma(\Phi)$ and $\sigma(\theta)$ are contained in $\{z \in \mathbb{C} \ : \ Re(z) > 0\}$, then $\ln \Phi$ is a continuous generalized derivation associated with the continuous derivation

$\ln\theta$. Moreover, we present a characterization of θ-centralizers on C^*-algebras as follows. Let $\mathcal{A}$ be a unital C^*-algebra on a Hilbert space $\mathcal{H}$. Let $\theta : \mathcal{A} \to \mathcal{A}$ be an automorphism and let $\Phi : \mathcal{A} \to \mathcal{A}$ be a left (or right) θ-centralizer. If $\sigma(\Phi), \sigma(\theta) \subset \{z \in \mathbb{C} \ : \ Re(z) > 0\}$, then there exist two invertible elements $\bar{x}$ and $\bar{y}$ in the weak closure $\overline{\mathcal{A}}$ of $\mathcal{A}$ on the Hilbert space $\mathcal{H}$ such that $\Phi(a) = \bar{y}a\bar{x}^{-1}$ for all $a \in \mathcal{A}$.

2. **Results and Proofs**

Throughout this section, we assume that $\mathcal{A}$ is a unital Banach algebra and $\mathrm{Inv}(\mathcal{A})$ denotes the group of invertible elements in $\mathcal{A}$. We denote by $B(\mathcal{A})$ the set of all bounded linear operators from $\mathcal{A}$ into $\mathcal{A}$. Suppose that $b_1, b_2 \in \mathcal{A}$. A *generalized inner derivation* (corresponding to b_1, b_2) on $\mathcal{A}$ is defined to be a linear mapping $\delta_{b_1,b_2} : \mathcal{A} \to \mathcal{A}$ satisfying $\delta_{b_1,b_2}(a) = b_1 a - a b_2$ for all $a \in \mathcal{A}$. Obviously, $\delta_{b_1,b_2} = L_{b_1} - R_{b_2} \in B(\mathcal{A}) \ \ (b_1, b_2 \in \mathcal{A})$. A straightforward verification shows that δ_{b_1,b_2} is a generalized derivation associated with both the inner derivations d_{b_1} and d_{b_2}. Indeed, we have $\delta_{b_1,b_2}(ab) = \delta_{b_1,b_2}(a)b + a d_{b_2}(b) = d_{b_1}(a)b + a\delta_{b_1,b_2}(b)$ for all $a, b \in \mathcal{A}$. It is evident that, δ_{b_1,b_2} is a bounded linear operator on $\mathcal{A}$. Recall that an automorphism $\alpha : \mathcal{A} \to \mathcal{A}$ is inner if there exists an invertible element $c \in \mathcal{A}$ such that $\alpha(a) = cac^{-1}$ $(a \in \mathcal{A})$. Let $b \in \mathcal{A}$ and $c \in \mathrm{Inv}(\mathcal{A})$. We define $\alpha_{b,c} : \mathcal{A} \to \mathcal{A}$ by $\alpha_{b,c}(a) = bac^{-1}$ $(a \in \mathcal{A})$. Obviously, $\alpha_{c,c}$ is an inner automorphism which will be simply denoted by α_c. It is easy to see that $\alpha_{b,c}$ is a left α_c-centralizer. If b is an invertible element of $\mathcal{A}$, then $\alpha_{b,c}$ is also a right α_b-centralizer.

A left (resp. right) α-centralizer $\Phi : \mathcal{A} \to \mathcal{A}$ is called an *inner left (resp. right) α-centralizer* whenever α is an inner automorphism. A linear mapping $\Phi : \mathcal{A} \to \mathcal{A}$ is called an *inner (α, β)-centralizer* (or simply an *inner centralizer*) if it is both an *inner left α-centralizer* and an *inner right β-centralizer*, where α and β are inner automorphisms on $\mathcal{A}$. For instance, if $b, c \in \mathrm{Inv}(\mathcal{A})$, then $\alpha_{b,c}$ is an inner centralizer on $\mathcal{A}$.

Suppose that $c_0 \in \mathrm{Inv}(\mathcal{A})$ and that $\Phi : \mathcal{A} \to \mathcal{A}$ is a left α_{c_0}-centralizer. Then for all $a, b \in \mathcal{A}$, we have

$$\Phi(ba) = \Phi(b)\alpha_{c_0}(a) = \Phi(b)c_0 a c_0^{-1}.$$

Putting $b = \mathbf{1}$ in the previous equation, we get

$$\Phi(a) = \Phi(\mathbf{1})c_0 a c_0^{-1} \quad (a \in \mathcal{A}).$$

Letting $b_0 = \Phi(\mathbf{1})c_0$, we obtain

$$\Phi(a) = b_0 a c_0^{-1} \qquad (a \in \mathcal{A}). \tag{1}$$

Also, if $\Phi : \mathcal{A} \to \mathcal{A}$ satisfies (1) for some $b_0 \in \mathcal{A}$ and $c_0 \in \mathrm{Inv}(\mathcal{A})$, then it is a left α_{c_0}-centralizer. Indeed, if Φ has the form (1), then it is a left α_{c_0}-centralizer, and if Φ is a left α_{c_0}-centralizer for some $c_0 \in Inv(\mathcal{A})$, then it has the form (1) for some $b_0 \in \mathcal{A}$.
To achieve our main results, we need two auxiliary results. The proposition below has been proved in [6], but in order to make this paper self contained and for the sake of convenience, we state it with its proof here. Let $\mathcal{A}$ be a unital Banach algebra. It is a well-known fact in the theory of Banach algebras that $\exp(a) = \sum_{n=0}^{\infty} \frac{a^n}{n!}$ for any $a \in \mathcal{A}$.

Proposition 2.1. *If* $b_1, b_2 \in \mathcal{A}$, *then* $\exp(\delta_{b_1,b_2}) = \alpha_{\exp(b_1),\exp(b_2)}$.

Proof. An easy induction shows that

$$\delta^n_{b_1,b_2}(a) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}b_1^{n-k}ab_2^k,$$

for each non-negative integer n and all $a \in \mathcal{A}$. Let a be an arbitrary element in $\mathcal{A}$. Then, we have

$$\begin{aligned}\exp(\delta_{b_1,b_2})(a) &= \sum_{n=0}^{\infty}\frac{\delta^n_{b_1,b_2}(a)}{n!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\left(\frac{b_1^{n-k}}{(n-k)!}\right)a\left(\frac{(-1)^k b_2^k}{k!}\right)\\ &= \left(\sum_{n=0}^{\infty}\frac{b_1^n}{n!}\right)a\left(\sum_{n=0}^{\infty}\frac{(-b_2)^n}{n!}\right)\\ &= \exp(b_1)a\exp(-b_2) = \exp(b_1)a(\exp(b_2))^{-1}\\ &= \alpha_{\exp(b_1),\exp(b_2)}(a),\end{aligned}$$

which means that $\exp(\delta_{b_1,b_2}) = \alpha_{\exp(b_1),\exp(b_2)}$. □

Let S be a subset of $\mathcal{A}$. We recall that the commutant of S in $\mathcal{A}$ is the set

$$S' = \{a \in \mathcal{A} \ : \ as = sa \text{ for all } s \in S\}.$$

We say that S commutes if $S \subseteq S'$. The following well-known properties of the commutant are easily verified.

(i) S' is a closed subalgebra of $\mathcal{A}$,
(ii) $S \subset S''$,
(iii) If S commutes, then so does S''.

For more details in this regard see subsection 11.21 of [9]. In the following proposition $\sigma_{B(\mathcal{A})}(\delta_{b_1,b_2})$ denotes the spectrum of δ_{b_1,b_2} in $B(\mathcal{A})$, similarly for α_{b_1,b_2}.

Proposition 2.2. *Let* b_1, b_2 *be two arbitrary elements in* $\mathcal{A}$. *Then*

(i) $\sigma_{B(\mathcal{A})}(\delta_{b_1,b_2}) \subseteq \{z - w \ : \ z \in \sigma(b_1), w \in \sigma(b_2)\}$,
(ii) *For* $b_2 \in \mathrm{Inv}(\mathcal{A})$, $\sigma_{B(\mathcal{A})}(\alpha_{b_1,b_2}) \subseteq \{zw^{-1} \ : \ z \in \sigma(b_1), w \in \sigma(b_2)\}$.

Proof. (i) Put $S = \{L_{b_1}, R_{b_2}\} \subset B(\mathcal{A})$. Then S commutes and it follows from [9, Theorem 11.12] that $\mathfrak{B} = S''$ is a commutative Banach algebra, $S \subset \mathfrak{B}$ and $\sigma_{\mathfrak{B}}(T) = \sigma_{B(\mathcal{A})}(T)$ for any $T \in \mathfrak{B}$. In view of [9, Theorem 11.23], we get that $\sigma(L_{b_1} + R_{b_2}) \subset \sigma(L_{b_1}) + \sigma(R_{b_2})$. Using [1, Theorem 15.4 and Propositions 16.9, 5.4], we obtain that

$$\begin{aligned}\sigma_{B(\mathcal{A})}(\delta_{b_1,b_2}) &= \sigma_{B(\mathcal{A})}(L_{b_1} - R_{b_2})\\ &= \sigma_{\mathfrak{B}}(L_{b_1} - R_{b_2})\\ &\subset \{z - w \ : \ z \in \sigma_{\mathfrak{B}}(L_{b_1}), w \in \sigma_{\mathfrak{B}}(R_{b_2})\}\\ &= \{z - w \ : \ z \in \sigma_{B(\mathcal{A})}(L_{b_1}), w \in \sigma_{B(\mathcal{A})}(R_{b_2})\}\\ &= \{z - w \ : \ z \in \sigma_{\mathcal{A}}(b_1), w \in \sigma_{\mathcal{A}}(b_2)\}.\end{aligned}$$

(ii) Let $S = \{L_{b_1}, R_{b_2^{-1}}\} \subset B(\mathcal{A})$ and $\mathfrak{B} = S''$. Upon applying [9, Theorem 11.23], we get that $\sigma(L_{b_1} R_{b_2^{-1}}) \subset \sigma(L_{b_1})\sigma(R_{b_2^{-1}})$. Like above, we have

$$\begin{aligned}
\sigma_{B(\mathcal{A})}(\alpha_{b_1,b_2}) &= \sigma_{B(\mathcal{A})}(L_{b_1} R_{b_2^{-1}}) \\
&= \sigma_{\mathfrak{B}}(L_{b_1} R_{b_2^{-1}}) \\
&\subset \{zw^{-1} \ : \ z \in \sigma_{\mathfrak{B}}(L_{b_1}), w \in \sigma_{\mathfrak{B}}(R_{b_2})\} \\
&= \{zw^{-1} \ : \ z \in \sigma_{\mathcal{A}}(b_1), w \in \sigma_{\mathcal{A}}(b_2)\}.
\end{aligned}$$

□

In the following, we present a lemma which will be used extensively to prove Theorem 2.1.

Lemma 2.1. *Let $a_1, a_2 \in \mathcal{A}$ with $\sigma(a_1), \sigma(a_2) \subset \{z \in \mathbb{C} \ : \ Re(z) > 0\}$, and let $b_1 = \ln a_1$ and $b_2 = \ln a_2$. Then $\delta_{b_1,b_2} = \ln \alpha_{a_1,a_2}$, and δ_{b_1,b_2} is an operator norm limit of polynomials in α_{a_1,a_2}.*

Proof. Since $\sigma(a_1), \sigma(a_2) \subseteq D := \{z \in \mathbb{C} \ : \ Re(z) > 0\}$, it follows from holomorphic functional calculus that that $a_1 = \exp(b_1)$ and $a_2 = \exp(b_2)$ for some $b_1, b_2 \in \mathcal{A}$. So $b_1 = \ln a_1$, $b_2 = \ln a_2$ and by the spectral mapping theorem we get $\sigma(b_1), \sigma(b_2) \subset \{z \in C \ : \ \frac{-\pi}{2} < \mathrm{Im}(z) < \frac{\pi}{2}\}$. Now, Proposition 2.2(i) implies that $\sigma(\delta_{b_1,b_2}) \subset \{z \in \mathbb{C} \ : \ -\pi < \mathrm{Im}(z) < \pi\}$. Applying [1, Proposition I.8.3(i)] to the Banach algebra $B(\mathcal{A})$, we get that $\ln(\exp(\delta_{b_1,b_2})) = \delta_{b_1,b_2}$. However, by Proposition 2.1, we observe that $\exp(\delta_{b_1,b_2}) = \alpha_{\exp(b_1),\exp(b_2)} = \alpha_{a_1,a_2}$, which means that $\delta_{b_1,b_2} = \ln \alpha_{a_1,a_2}$. According to Proposition 2.2 (ii), we have

$$\sigma(\alpha_{a_1,a_2}) \subset \{zw^{-1} \ : \ z \in \sigma(a_1), w \in \sigma(a_2)\} \subset \{zw^{-1} \ : \ z, w \in D\} \subset \mathbb{C} - \mathbb{R}_-.$$

It follows from [1, Proposition I.8.3(i)] that there exists a sequence $\{P_n\}$ of polynomials such that

$$\lim_{n\to\infty} P_n(\alpha_{a_1,a_2}) = \ln(\alpha_{a_1,a_2}) = \delta_{b_1,b_2}.$$

□

Now, we are ready to prove our first main result. Note that if $\mathcal{A}$ is a unital normed algebra, θ is a continuous linear mapping on $\mathcal{A}$ and Φ is a left (or right) θ-centralizer on $\mathcal{A}$, then for each $a \in \mathcal{A}$,

$$\|\Phi(a)\| = \|\Phi(\mathbf{1}a)\| = \|\Phi(\mathbf{1})\theta(a)\| \leq \|\Phi(\mathbf{1})\|\|\theta\|\|a\|,$$

which shows that Φ is also continuous.

Theorem 2.1. *Let $\theta : \mathcal{A} \to \mathcal{A}$ be a continuous automorphism and let $\Phi : \mathcal{A} \to \mathcal{A}$ be a left θ-centralizer. If $\sigma(\Phi), \sigma(\theta) \subset \{z \in \mathbb{C} \ : \ Re(z) > 0\}$, then $\ln \Phi$ is a continuous generalized derivation associated with the continuous derivation $\ln \theta$.*

Proof. It follows from [1, Theorem II.18.15] that $d = \ln \theta$ is a continuous derivation on $\mathcal{A}$. Let $\delta = \ln \Phi$, $d = \ln \theta$ and let $\mathcal{B} = \{L_a \ : \ a \in \mathcal{A}\}$. We are going to show that $\mathcal{B}$ is a closed subalgebra of $B(\mathcal{A})$. To see that, let $T \in \overline{\mathcal{B}}$. So, there exists a sequence $\{L_{a_n}\} \subseteq \mathcal{B}$ such that $L_{a_n} \to T$. Since $\mathcal{A}$ is unital, $a_n \to T(\mathbf{1})$. Moreover, for any arbitrary element $b \in \mathcal{A}$, we have $a_n b = L_{a_n}(b) \to T(b)$, and so $T(b) = T(\mathbf{1})b$. It means that $T = L_{T(\mathbf{1})} \in \mathcal{B}$ and consequently, $\mathcal{B}$ is a closed subalgebra of $B(\mathcal{A})$. Notice that

$$\Phi L_a \theta^{-1}(x) = \Phi(a)\theta(\theta^{-1}(x)) = \Phi(a)x = L_{\Phi(a)}(x)$$

for all $x \in \mathcal{A}$, which means that $\Phi L_a \theta^{-1} = L_{\Phi(a)}$. This equation implies that $\alpha_{\Phi,\theta}(L_a) = L_{\Phi(a)}$ for all $a \in \mathcal{A}$ and so $\alpha_{\Phi,\theta}(\mathcal{B}) \subset \mathcal{B}$. Now we introduce $\Delta_{\delta,d} : B(\mathcal{A}) \to B(\mathcal{A})$ by $\Delta_{\delta,d}(T) = \delta T - Td$ for any $T \in B(\mathcal{A})$. It follows from Lemma 2.1 that $\Delta_{\delta,d} = lim_{n\to+\infty} P_n(\alpha_{\Phi,\theta})$, and since $\mathcal{B}$ is a closed subalgebra of $B(\mathcal{A})$, $\Delta_{\delta,d}(\mathcal{B}) \subset \mathcal{B}$. Thus, for each $a \in \mathcal{A}$, there exists $a' \in \mathcal{A}$ such that $\Delta_{\delta,d}(L_a) = L_{a'}$, i.e. $\delta L_a - L_a d = L'_a$. So, for any $b \in \mathcal{A}$ we have

$$\delta(ab) - ad(b) = a'b. \tag{2}$$

Putting $b = \mathbf{1}$ in (2), we get that $\delta(a) - ad(\mathbf{1}) = a'$ and since $d(\mathbf{1}) = 0$, we have $a' = \delta(a)$. Hence, we deduce that $\delta(ab) = ad(b) + a'b = \delta(a)b + ad(b)$ for all $a, b \in \mathcal{A}$. It means that $\delta = \ln \Phi$ is a generalized derivation associated with the derivation $d = \ln \theta$. Our next task is to show that δ is a continuous linear mapping. By use of [1, Proposition I.8.3] there exists a sequence $\{P_n\}$ of complex polynomials such that $lim_{n\to+\infty} P_n(z) = \ln z$ uniformly on a neighbourhood of $\sigma(\Phi)$. Applying the functional calculus to the Banach algebra $B(\mathcal{A})$, we obtain that $lim_{n\to+\infty} P_n(\Phi) = \ln \Phi = \delta$. A direct consequence of the uniform boundedness principle implies that δ is a continuous linear mapping on $\mathcal{A}$. □

Remark 2.1. *Using the argument of Theorem 2.1, we can obtain that if $\theta : \mathcal{A} \to \mathcal{A}$ is a continuous automorphism and $\Phi : \mathcal{A} \to \mathcal{A}$ is a right θ-centralizer such that $\sigma(\Phi), \sigma(\theta) \subset \{z \in \mathbb{C} : Re(z) > 0\}$, then $\delta = \ln \Phi$ is a continuous generalized derivation associated with the continuous derivation $d = \ln \theta$ in the sense that $\delta(ab) = d(a)b + a\delta(b)$ for all $a, b \in \mathcal{A}$.*

Lemma 2.2. *Let $\mathcal{A}$ be a semiprime Banach algebra with or without identity and let θ be a continuous Jordan automorphism on $\mathcal{A}$ such that $\sigma(\theta) \subset \{z \in \mathbb{C} : Re(z) > 0\} \subset \mathbb{C} - \mathbb{R}_-$. Then θ is an automorphism.*

Proof. According to what has been mentioned in [11], if $\mathcal{A}$ does not have an identity, we adjoin an identity and extend θ to the algebra with identity by defining $\theta(\mathbf{1}) = \mathbf{1}$. Similar to the proof of [1, Theorem II.18.15], we can see that $d = \ln \theta$ is a continuous Jordan derivation on $\mathcal{A}$ (see [11]). It follows from [2, Theorem 1] that d is a derivation. Hence, we have

$$\theta(ab) = \exp(d)(ab) = \exp(d)(a)\exp(d)(b) = \theta(a)\theta(b)$$

for all $a, b \in \mathcal{A}$, which means that θ is an automorphism. □

In the following, we provide some conditions where the only continuous Jordan automorphism satisfying those conditions is the identity mapping.

Theorem 2.2. *Let $\mathcal{A}$ be a semiprime Banach algebra and let θ be a continuous Jordan automorphism on $\mathcal{A}$ such that $\sigma(\theta) \subset \{z \in \mathbb{C} : Re(z) > 0\} \subset \mathbb{C} - \mathbb{R}_-$. If $\dim(\bigcap_{\mathcal{P}\in\Pi_c(A^\sharp)} \mathcal{P}) \leq 1$, then $\theta = I$.*

Proof. According to Lemma 2.2, θ is an automorphism and it follows from [1, Theorem II.18.15] that $d = \ln\theta$ is a continuous derivation on $\mathcal{A}$. In view of [7, Theorem 2.3.], $d = 0$ and it implies that $\theta = I$, as desired. $\square$

In the next theorem, we present a characterization of left θ-centralizers on a C^*-algebra acting on a Hilbert space.

Theorem 2.3. *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{A}$ be a unital C^*-algebra in $B(\mathcal{H})$. Let $\theta : \mathcal{A} \to \mathcal{A}$ be an automorphism and let $\Phi : \mathcal{A} \to \mathcal{A}$ be a left θ-centralizer. If $\sigma(\Phi), \sigma(\theta) \subset \{z \in \mathbb{C} \ : \ Re(z) > 0\}$, then there exist two invertible elements $\bar{x}$ and $\bar{y}$ in the weak closure $\bar{\mathcal{A}}$ of $\mathcal{A}$ on the Hilbert space $\mathcal{H}$ such that $\Phi(a) = \bar{y}a\bar{x}^{-1}$ for all $a \in \mathcal{A}$.*

Proof. It follows from [10, Lemma 4.1.12] that θ is continuous. By Theorem 2.1, $\delta = \ln\Phi$ is a continuous generalized derivation associated with the continuous derivation $d = \ln\theta$. By [10, Corollary 4.1.7], there exists an element $\bar{b} \in \bar{\mathcal{A}}$ such that $d(a) = [\bar{b}, a]$ for all $a \in \mathcal{A}$. Since δ is a generalized derivation associated with the derivation d, there exists a left centralizer T such that $\delta = d + T$. Thus, we have

$$\begin{aligned}\delta(a) &= d(a) + T(a) = [\bar{b}, a] + T(\mathbf{1})a \\ &= (\bar{b} + T(\mathbf{1}))a - a\bar{b},\end{aligned}$$

for all $a \in \mathcal{A}$. Taking $\bar{c} = \bar{b} + T(\mathbf{1})$, we see that $\delta = \delta_{\bar{c},\bar{b}}$. Using Proposition 2.1, we have

$$\Phi(a) = \exp(\delta_{\bar{c},\bar{b}})(a) = \alpha_{\exp(\bar{c}),\exp(\bar{b})}(a) = (\exp(\bar{c}))a(\exp(-\bar{b})),$$

for all $a \in \mathcal{A}$. Taking $\bar{y} = \exp(\bar{c})$ and $\bar{x} = \exp(\bar{b})$ in the previous equation, we have $\Phi(a) = \bar{y}a\bar{x}^{-1}$ for all $a \in \mathcal{A}$. Since $\bar{x}$ and $\bar{y}$ are invertible elements of $\bar{\mathcal{A}}$, Φ is both a left $\alpha_{\bar{x}}$-centralizer and a right $\alpha_{\bar{y}}$-centralizer. Meanwhile, note that $\Phi(\mathbf{1})$ is an invertible element of $\mathcal{A}$. $\square$

An immediate corollary is as follows.

Corollary 2.1. *Let $\mathcal{A}$ be a von-Neumann algebra. Let $\theta : \mathcal{A} \to \mathcal{A}$ be an automorphism and let $\Phi : \mathcal{A} \to \mathcal{A}$ be a left θ-centralizer. If $\sigma(\Phi), \sigma(\theta) \subset \{z \in \mathbb{C} \ : \ Re(z) > 0\}$, then there exist two invertible elements b and c in $\mathcal{A}$ such that $\Phi(a) = cab^{-1}$ for all $a \in \mathcal{A}$. Indeed, Φ is an inner centralizer on $\mathcal{A}$.*

Remark 2.2. *Theorem 2.3 and Corollary 2.1 are also valid for right θ-centralizers and we leave it to the reader.*

Acknowledgment. The author thanks the referee for his/her careful reading of the paper and suggesting valuable comments that improved the quality of this work. He also thanks Professor Shirin Hejazian for her comments on the manuscript.

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