

A REMARK ON LIMIT SHADOWING FOR HYPERBOLIC ITERATED FUNCTION SYSTEMS

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This paper provides two kinds of iterated function systems (IFSs), the first type is generated by two linear functions defined on the real line, and does not have the limit shadowing property, another class is generated by a linear function and a constant function defined on the real line, and does not have the exponential limit shadowing property. These two IFSs solve Problems 2 and 3 in [7, M.F. Nia, S.A. Ahmadi, Various shadowing properties for parameterized iterated function systems, U.P.B. Sci. Bull., Series A, 2018]. Further, a continuous linear operator with bounded inverse defined on a Banach space has the exponential shadowing property.

Keywords: Iterated function system (IFS), Limit shadowing property, Exponential limit shadowing property.

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1. Introduction

Fractal geometry provides us with a new way of looking at our world, where the shape of the objects has self-similar form, that is, a shape must be able to be divided into small parts that are smaller copies which are more or less similar to the whole [3]. The iterated function system (IFS) formalism is an important method for generating fractals by iterating a collection of transformations. This method is popularized by Barnsley [2], the strict theoretical work was finished by Hutchinson and Dekking, which have been applied to many fields, such as image compression and image processing, and so on.

On the other hand, the theory of shadowing provides tools for fitting real trajectories near to approximate trajectories. The motivation comes from computer simulations, where we always have a numerical error when calculating a trajectory, no matter how careful we are, but at the same time we want to be sure that what we see on the computer screen is a good approximation of the

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genuine orbit of the system. Shadowing is a classical notion, which originated from works of Anosov, Bowen and others (see [9, 11] for historical remarks and more recent advances). For more results on the IFS and shadowing properties, one is referred to [4, 5, 6, 8, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21] and references therein.

Throughout this paper, let $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$.

An IFS \mathcal{F} is a complete metric space (X, d) together with a family of continuous self-maps $f_\lambda : X \rightarrow X$ ($\lambda \in \Lambda$), where Λ is a nonempty set, and is denoted by $\mathcal{F} = \{X; f_\lambda \mid \lambda \in \Lambda\}$.

For any $\sigma = (\lambda_0, \lambda_1, \lambda_2, \dots) \in \Lambda^{\mathbb{Z}^+}$, denote

$$\mathcal{F}_0 = \text{id}_X \text{ and } \mathcal{F}_{\sigma_n} = f_{\lambda_n} \circ \dots \circ f_{\lambda_0} \text{ for any } n \in \mathbb{Z}^+.$$

A sequence $\{x_n\}_{n=0}^{+\infty} \subset \mathbb{R}$ converges to zero with rate $\theta \in (0, 1)$, and denoted by $x_n \xrightarrow{\theta} 0$, if there exists a constant $L > 0$ such that $|x_n| \leq L\theta^n$ for all $n \in \mathbb{Z}^+$.

Definition 1.1. Let $\theta \in (0, 1)$. A sequence $\{x_i\}_{i=0}^{+\infty} \subset X$ is called

- (1) an *orbit* of \mathcal{F} if there exists $\sigma = (\lambda_0, \lambda_1, \lambda_2, \dots) \in \Lambda^{\mathbb{Z}^+}$ such that $f_{\lambda_i}(x_i) = x_{i+1}$ for all $i \in \mathbb{Z}^+$, i.e., $\mathcal{F}_{\sigma_i}(x_0) = x_{i+1}$ for all $i \in \mathbb{Z}^+$.
- (2) an *asymptotic pseudo-orbit* of \mathcal{F} if there exists $\sigma = (\lambda_0, \lambda_1, \lambda_2, \dots) \in \Lambda^{\mathbb{Z}^+}$ such that

$$\lim_{n \rightarrow +\infty} d(f_{\lambda_n}(x_n), x_{n+1}) = 0.$$

- (3) a *θ -exponentially asymptotic pseudo-orbit* of \mathcal{F} if there exists $\sigma = (\lambda_0, \lambda_1, \lambda_2, \dots) \in \Lambda^{\mathbb{Z}^+}$ such that

$$\lim_{n \rightarrow +\infty} d(f_{\lambda_n}(x_n), x_{n+1}) \xrightarrow{\theta} 0.$$

Definition 1.2. An IFS $\mathcal{F} = \{X; f_\lambda \mid \lambda \in \Lambda\}$ has

- (1) the *limit shadowing property* if every asymptotic pseudo-orbit $\{x_i\}_{i=0}^{+\infty}$ is asymptotically shadowed by an orbit of some point $z \in X$, i.e., there exists $\sigma = (\lambda_0, \lambda_1, \lambda_2, \dots) \in \Lambda^{\mathbb{Z}^+}$ such that

$$\lim_{n \rightarrow +\infty} d(\mathcal{F}_{\sigma_n}(z), x_{n+1}) = 0.$$

- (2) the *exponential limit shadowing property with ξ* if there exists $\theta_0 \in (0, 1)$ such that for any θ -exponentially asymptotic pseudo-orbit $\{x_n\}_{n=0}^{+\infty}$ with $\theta \in (\theta_0, 1)$, there exists an orbit $\{y_n\}_{n=0}^{+\infty}$ such that $\{x_n\}_{n=0}^{+\infty}$ can be θ^ξ -exponentially asymptotically shadowed by $\{y_n\}_{n=0}^{+\infty}$, i.e.,

$$\lim_{n \rightarrow +\infty} d(x_n, y_n) \xrightarrow{\theta^\xi} 0.$$

In the case $\xi = 1$, we say that \mathcal{F} has the *exponential limit shadowing property*.

In this work, we construct two kinds of IFSs, the first type is generated by two linear functions defined on the real line, and does not have the limit shadowing property, another class is generated by a linear function and a constant function defined on the real line, and does not have the exponential limit shadowing property (See Examples 2.1 and 2.2 in the next section). Examples 2.1 and 2.2 solve Problems 2 and 3 in [7], respectively. Further, we show that a continuous linear operator with bounded inverse defined on a Banach space has the exponential shadowing property.

2. Main results

In this section, two IFSs defined on the real line are given, one does not have the limit shadowing property, another does not have the exponential limit shadowing property. A continuous linear operator with bounded inverse defined on a Banach space is verified to have the exponential shadowing property.

Example 2.1. Fix any $\alpha > 1$. Let $X = \mathbb{R}$ and define $g_1 : X \rightarrow X$ and $g_2 : X \rightarrow X$ as $g_1(x) = \alpha x$, and $g_2(x) = \frac{1}{\alpha}x$ for any $x \in \mathbb{R}$, respectively. Then, the IFS $\mathcal{G} = \{X; g_1, g_2\}$ does not have the limit shadowing property.

Choose a sequence $\xi = \{x_n\}_{n=0}^{+\infty} \subset X$ as following:

$$x_n = \begin{cases} 1, & n = 0, \\ \alpha + (1 + \frac{1}{2} + \cdots + \frac{1}{k}), & n = 2k - 1 \text{ for some } k \in \mathbb{N}, \\ 1 + \frac{1}{\alpha}(1 + \frac{1}{2} + \cdots + \frac{1}{k}), & n = 2k \text{ for some } k \in \mathbb{N}, \end{cases} \quad (1)$$

i.e., $x_0 = 1$, $x_1 = \alpha + 1$, $x_2 = \frac{1}{\alpha}x_1$, $x_3 = \alpha x_2 + \frac{1}{2}$, \dots , $x_{2k} = \frac{1}{\alpha}x_{2k-1}$, $x_{2k+1} = \alpha x_{2k} + \frac{1}{k+1}$, \dots , and take $\sigma = (\lambda_0, \lambda_1, \lambda_2, \dots) = (1, 2, 1, 2, \dots) \in \{1, 2\}^{\mathbb{Z}^+}$. It can be verified that ξ is an asymptotic pseudo-orbit of \mathcal{G} .

Claim 1. For any $z \in X$ and any sequence $\hat{\sigma} = (\hat{\lambda}_0, \hat{\lambda}_1, \hat{\lambda}_2, \dots) \in \{1, 2\}^{\mathbb{Z}^+}$, $\lim_{n \rightarrow +\infty} |\mathcal{G}_{\hat{\sigma}_n}(z) - x_{n+1}| \not\rightarrow 0$.

Suppose that $\lim_{n \rightarrow +\infty} |\mathcal{G}_{\hat{\sigma}_n}(z) - x_{n+1}| = 0$, noting that $\lim_{n \rightarrow +\infty} x_n = +\infty$ and $\lim_{k \rightarrow +\infty} x_{2k-1} - x_{2k} = \lim_{k \rightarrow +\infty} (1 - \frac{1}{\alpha})x_{2k-1} = +\infty$, it is not difficult to verify that there exists some $N \in \mathbb{N}$ such that for any $n \geq N$, $\hat{\lambda}_n = \lambda_n$. This implies that for any $n \in \mathbb{Z}^+$,

$$\mathcal{G}_{\hat{\sigma}_{N+2n}}(z) = \mathcal{G}_{\hat{\sigma}_N}(z) \text{ and } \mathcal{G}_{\hat{\sigma}_{N+2n+1}}(z) = \mathcal{G}_{\hat{\sigma}_{N+1}}(z). \quad (2)$$

Therefore,

$$\lim_{n \rightarrow +\infty} |\mathcal{G}_{\hat{\sigma}_{N+2n}}(z) - x_{N+2n+1}| = +\infty,$$

and

$$\lim_{n \rightarrow +\infty} |\mathcal{G}_{\hat{\sigma}_{N+2n+1}}(z) - x_{N+2n+2}| = +\infty,$$

which is a contradiction.

Form Claim 1, it follows that ξ can not be asymptotically shadowed by any orbit of \mathcal{G} , i.e., \mathcal{G} does not have the asymptotic shadowing property.

Example 2.2. Fix $\alpha > 1$ and $c \in \mathbb{R}$. Let $X = \mathbb{R}$ and define $h_1 : \mathbb{R} \rightarrow \mathbb{R}$ and $h_2 : \mathbb{R} \rightarrow \mathbb{R}$ as $h_1(x) = \alpha x$, and $h_2(x) \equiv c$ for any $x \in \mathbb{R}$, respectively. Then the IFS $\mathcal{H} = \{X; h_1, h_2\}$ does not have the exponential limit shadowing property.

Without loss of generality, assume $c > 0$. For any fixed $\theta_0 \in (0, 1)$, inductively choose an increasing sequence $\{n_k\}_{k=0}^{+\infty} \subset \mathbb{R}$ as $n_0 = 0$ and $n_k \in \mathbb{N}$ with $\alpha^{n_k - n_{k-1} - 2} \theta_0^{n_{k-1} + 1} \geq 2$ for any $k \in \mathbb{N}$. Take a sequence $\xi = \{x_n\}_{n=0}^{+\infty} \subset X$ as following:

$$x_i = \begin{cases} c, & i \in \{n_k : k \in \mathbb{Z}^+\}, \\ \alpha^{i - n_k} c + \frac{\alpha^{i - n_k - 1} \theta_0^{n_k + 1} \cdot [1 - (\frac{\theta_0}{\alpha})^{i - n_k}]}{1 - \frac{\theta_0}{\alpha}}, & i \in (n_k, n_{k+1}) \text{ for some } k \in \mathbb{Z}^+, \end{cases}$$

i.e., $x_0 = c$, $x_1 = \alpha x_0 + \theta_0$, $x_2 = \alpha x_1 + \theta_0^2$, \dots , $x_{n_1 - 1} = \alpha x_{n_1 - 2} + \theta_0^{n_1 - 1}$, $x_{n_1} = c$, \dots , $x_{n_k} = c$, $x_{n_k + 1} = \alpha x_{n_k} + \theta_0^{n_k + 1}$, $x_{n_k + 2} = \alpha x_{n_k + 1} + \theta_0^{n_k + 2}$, \dots , $x_{n_{k+1} - 1} = \alpha x_{n_{k+1} - 2} + \theta_0^{n_{k+1} - 1}$, $x_{n_{k+1}} = c$, \dots and take $\sigma = (\lambda_0, \lambda_1, \lambda_2, \dots) \in \{1, 2\}^{\mathbb{Z}^+}$ with

$$\lambda_i = \begin{cases} 2, & i \in \{n_k - 1 : k \in \mathbb{N}\}, \\ 1, & \text{otherwise.} \end{cases}$$

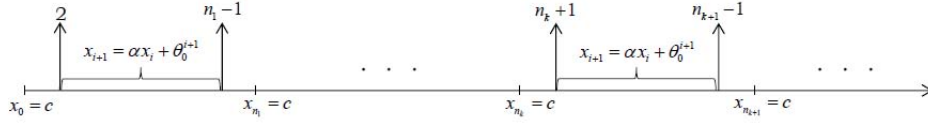


FIGURE 1. The illustration diagram of the construction of the sequence $\xi = \{x_n\}_{n=0}^{+\infty}$

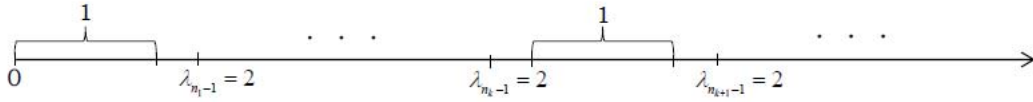


FIGURE 2. The illustration diagram of the construction of $\sigma = (\lambda_0, \lambda_1, \lambda_2, \dots) \in \{1, 2\}^{\mathbb{Z}^+}$

It can be verified that

- (1) for any $i \in [n_k, n_{k+1} - 1)$, $x_i \geq c$ and $x_{i+1} - x_i \geq (\alpha - 1)c$;

(2) for any $k \in \mathbb{N}$,

$$\begin{aligned} & x_{n_{k+1}-1} - h_1^{n_{k+1}-n_k-1}(x_{n_k}) \\ &= x_{n_{k+1}-1} - \alpha^{n_{k+1}-n_k-1} \cdot c > \alpha^{n_{k+1}-n_k-2} \theta_0^{n_k+1} \geq 2; \end{aligned}$$

(3) for any fixed $\theta \in (\theta_0, 1)$,

$$|h_{\lambda_n}(x_n) - x_{n+1}| \leq \theta_0^{n+1} < \theta^{n+1},$$

i.e., ξ is a θ -exponentially asymptotic pseudo-orbit of \mathcal{G} .

Claim 2. For any $z \in X$ and any sequence $\sigma' = (\lambda'_0, \lambda'_1, \lambda'_2, \dots) \in \{1, 2\}^{\mathbb{Z}^+}$, $|\mathcal{H}_{\sigma'_i}(z) - x_{i+1}|$ does not converge to zero with rate θ .

In fact, suppose on contrary that $|\mathcal{H}_{\sigma'_i}(z) - x_{i+1}|$ converges to zero with rate θ , i.e., there exists a constant $L > 0$ such that for any $i \in \mathbb{Z}^+$,

$$|\mathcal{H}_{\sigma'_i}(z) - x_{i+1}| \leq L \cdot \theta^{i+1}, \quad (3)$$

implying that there exists $N \in \mathbb{N}$ such that for any $i \geq N$,

$$|\mathcal{H}_{\sigma'_i}(z) - x_{i+1}| < (\alpha - 1)c. \quad (4)$$

For any $i > N$, consider the following two cases:

- (1) if $i = n_k - 1$ for some $k \in \mathbb{N}$ and $\lambda_i \neq \lambda'_i = 1$, applying (1) and (4) yields that

$$\begin{aligned} \mathcal{H}_{\sigma'_i}(z) - x_{i+1} &= h_1(\mathcal{H}_{\sigma'_{i-1}}(z)) - c = \alpha \cdot \mathcal{H}_{\sigma'_{i-1}}(z) - c \\ &> \alpha(x_i - (1 - \alpha)c) - c \geq \alpha^2 c - c > (\alpha - 1)c, \end{aligned}$$

which is a contradiction and implies that $\lambda'_i = \lambda_i$.

- (2) if $i \in [n_k, n_{k+1} - 1)$ for some $k \in \mathbb{N}$ and $\lambda_i \neq \lambda'_i = 2$, it can be verified that $x_{i+1} \geq \alpha c$ and $\mathcal{H}_{\sigma'_i}(z) = h_{\lambda'_i}(\mathcal{H}_{\sigma'_{i-1}}(z)) = c$, implying that

$$|\mathcal{H}_{\sigma'_i}(z) - x_{i+1}| > (\alpha - 1)c,$$

which is a contradiction. Then, $\lambda_i = \lambda'_i$.

Summing up (a) and (b) yields that for any $i > N$, $\lambda'_i = \lambda_i$. Thus, for any $k > N + 1$, it follows from (2) that

$$\begin{aligned} & x_{n_{k+1}-1} - \mathcal{H}_{\sigma'_{n_{k+1}-2}}(z) \\ &= x_{n_{k+1}-1} - h_{\lambda_{n_{k+1}-2}} \circ \dots \circ h_{\lambda_{n_k}}(h_{\lambda_{n_k-1}}(\mathcal{H}_{\sigma'_{n_k-2}}(z))) \\ &= x_{n_{k+1}-1} - h_1^{n_{k+1}-n_k-1}(c) = x_{n_{k+1}-1} - \alpha^{n_{k+1}-n_k-1} \cdot c \geq 2, \end{aligned}$$

which contradicts to (3).

Therefore, for any $\theta_0 \in (0, 1)$, there exists a θ -exponentially asymptotic pseudo-orbit which can not be θ -exponentially asymptotically shadowed by any orbit of \mathcal{H} , i.e., \mathcal{H} does not have the exponential limit shadowing property.

According to Ahmadi and Molaei [1], a dynamical system (X, f) has the *exponential limit shadowing property* (ELmSP) if there exist constants $L > 0$ and $0 < \lambda < 1$ such that for any sequence $\xi = \{x_n\}_{n=0}^{+\infty}$ with

$$d(f(x_n), x_{n+1}) < \lambda^n,$$

there exists a point $z \in X$ such that for all $n \in \mathbb{Z}^+$,

$$d(f^n(z), x_n) < L\lambda^{n/2}.$$

Theorem 2.1. *Let X be a Banach space and $T : X \rightarrow X$ be a continuous linear operator such that T^{-1} is continuous. Then, (X, T) has the exponential limit shadowing property.*

Proof. Fix $\lambda \in (0, 1)$ such that $\sqrt{\lambda} < \frac{1}{(\|T^{-1}\|+2)(\|T\|+2)}$ and let $L = \frac{(\|T^{-1}\|+2)}{1-(\|T^{-1}\|+2)\cdot\lambda}$. For any sequence $\xi = \{x_n\}_{n=0}^{+\infty} \subset X$ with $\|e_n\| < \lambda^n$, where $e_n = x_{n+1} - Tx_n$, take

$$z_n := x_0 + T^{-1}e_0 + \cdots + T^{-n}e_{n-1}, \quad \forall n \in \mathbb{N}.$$

Clearly, $x_n = T^n x_0 + T^{n-1}e_0 + \cdots + e_{n-1}$.

Now, we shall show that the limit of $\{z_n\}$ exists, denoted by z . Observe that $\lambda \cdot (\|T^{-1}\| + 2) < \frac{1}{\|T\|+2} < \frac{1}{2}$. Then, for any $n \in \mathbb{N}$ and any $p \in \mathbb{N}$,

$$\begin{aligned} & \|z_{n+p} - z_n\| \\ &= \|T^{-(n+1)}e_n + \cdots + T^{-(n+p)}e_{n+p-1}\| \\ &\leq (\|T^{-1}\| + 2)^{n+1} \cdot \lambda^n + \cdots + (\|T^{-1}\| + 2)^{n+p} \cdot \lambda^{n+p-1} \\ &< (\|T^{-1}\| + 2) \cdot \left[\left(\frac{1}{2}\right)^n + \cdots + \left(\frac{1}{2}\right)^{n+p-1} \right] \\ &< \frac{\|T^{-1}\| + 2}{2^{n-1}} \rightarrow 0 \quad (n \rightarrow +\infty), \end{aligned} \tag{5}$$

implying that $\{z_n\}$ is a Cauchy sequence, i.e., the above limit z exists.

Applying (5) also yields that for any $n \in \mathbb{N}$,

$$\begin{aligned} \|z_n - z\| &\leq (\|T^{-1}\| + 2)^{n+1} \cdot \lambda^n + \cdots + (\|T^{-1}\| + 2)^{n+p} \cdot \lambda^{n+p-1} + \cdots \\ &= \frac{(\|T^{-1}\| + 2)^{n+1} \cdot \lambda^n}{1 - (\|T^{-1}\| + 2) \cdot \lambda}. \end{aligned}$$

This implies that

$$\begin{aligned} & \|T^n(z) - x_n\| \\ &= \|T^n z - T^n z_n\| \leq \|T\|^n \cdot \|z - z_n\| \leq \frac{(\|T^{-1}\| + 2)^{n+1} \cdot \lambda^n}{1 - (\|T^{-1}\| + 2) \cdot \lambda} \cdot \|T\|^n \\ &< \frac{(\|T^{-1}\| + 2)^{n+1} \cdot \lambda^n}{1 - (\|T^{-1}\| + 2) \cdot \lambda} \cdot (\|T\| + 2)^n < \frac{(\|T^{-1}\| + 2)}{1 - (\|T^{-1}\| + 2) \cdot \lambda} \cdot \left(\frac{1}{\sqrt{\lambda}}\right)^n \cdot \lambda^n \\ &= \frac{(\|T^{-1}\| + 2)}{1 - (\|T^{-1}\| + 2) \cdot \lambda} \cdot \lambda^{n/2} = L \cdot \lambda^{n/2}. \end{aligned}$$

Therefore, T has the exponential limit shadowing property. \square

Corollary 2.1. *Let A be a non-singular matrix. Then, the linear system $\mathcal{A}(x) = Ax$ on \mathbb{C}^n has the exponential limit shadowing property.*

Proof. Applying Theorem 2.1, this holds trivially. \square

Remark 2.1. [1, Example 4.1] shows that limit shadowing does not imply exponential limit shadowing. We [20, Corollary 2.1] proved that the linear system \mathcal{A} in Corollary 2.1 have the limit shadowing property if and only if A is hyperbolic. This, together with Corollary 2.1, implies that exponential limit shadowing also does not imply limit shadowing.

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