WILLMORE TOTALLY REAL SUBMANIFOLDS OF COMPLEX SPACE FORMS $\tilde{M}^{n+p}(4c)$

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Let M be an n-dimensional compact Willmore totally real submanifold of complex space forms $\tilde{M}^{n+p}(4c), (p > 0)$. In this paper, we obtain some integral inequalities of Simons' type and characterization theorems of n-dimensional compact Willmore totally real submanifolds of $\tilde{M}^{n+p}(4c)$, which are connected with the squared norm of the second fundamental form and the mean curvature as well as the sectional curvature and Ricci curvature of M.

Keywords: Willmore totally real submanifolds, complex space forms, totally umbilical, totally geodesic

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1. Introduction

Let $\tilde{M}^{n+p}$ be a Kaehler manifold of complex dimension $n+p$, $p \geq 0$, and M be a Riemannian manifold of real dimension n. If the Kaehler manifold $\tilde{M}^{n+p}$ has constant holomorphic sectional curvature $4c$, we call it the complex space form, which is denoted by $\tilde{M}^{n+p}(4c)$. When $c = 0$, $c > 0$ and $c < 0$, we call $\tilde{M}^{n+p}(4c)$ the complex Euclidean space, complex projective space and complex hyperbolic space, which are denoted by C^{n+p}, $CP^{n+p}(4c)$ and $CH^{n+p}(4c)$, respectively. Let J be the almost complex structure of M. We call M the totally real submanifold of $\tilde{M}^{n+p}$ if M admits an isometric immersion into $\tilde{M}^{n+p}$ such that $JT_x(M) \subset T_x(M)^{\perp}$, where $T_x(M)$ and $T_x(M)^{\perp}$ denote the tangent space and the normal space of M at x respectively. If $p = 0$, the totally real submanifold of $\tilde{M}^n$ is called the Lagrangian submanifold.

Let M be an n-dimensional compact totally real submanifold of complex space forms $\tilde{M}^{n+p}(4c)$. If the mean curvature of M identically vanishing, we call M a minimal totally real submanifold of $\tilde{M}^{n+p}(4c)$. When $p = 0$ (i.e., the minimal Lagrangian submanifold) or $p > 0$, we know that many interesting

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results of minimal totally real submanifolds had been obtained by different authors (see [2], [12], [13]).

Let $h_{ij}^{\alpha}, S, \vec{H}$ and H be the second fundamental form, the squared norm of the second fundamental form, the mean curvature vector and the mean curvature of M. We denote by $W(x) = \int_M \rho^n dv = \int_M (S - nH^2)^{\frac{n}{2}} dv$ the Willmore functional on M (see [1], [10]). A totally real submanifold of $\tilde{M}^{n+p}(4c)$ is called a Willmore totally real submanifold if it is an extremal submanifold to the Willmore functional (see [6]). In [6], Hu and Li obtained that every minimal totally real surface or every minimal and Einstein totally real submanifold of $\tilde{M}^{n+p}(4c)$ is Willmore. We notice, when $n = 2$, the minimal is equivalent to Willmore, but, when $n > 2$, it is not true. In recent years, Willmore submanifolds in a Riemannian manifold have been intensively studied by many authors (see [8], [9]).

Let M be an n-dimensional compact Willmore totally real submanifold of complex space forms $\tilde{M}^{n+p}(4c)$. When $p = 0$, in [11], we obtained some interesting results of such submanifolds; when $p > 0$, in this paper, we shall also obtain some interesting results, see Theorem 4.1–Theorem 4.3.

2. **Preliminaries**

Let $x : M \mapsto \tilde{M}^{n+p}(4c)$ be an n-dimensional totally real submanifold of $\tilde{M}^{n+p}(4c)$. We choose a local field of orthonormal frames $e_1, \cdots, e_n$, $e_{n+1}, \cdots, e_{n+p}$, $e_{1^*} = Je_1, \cdots, e_{n^*} = Je_n$, $e_{(n+1)^*} = Je_{n+1}, \cdots, e_{(n+p)^*} = Je_{n+p}$ in $\tilde{M}^{n+p}(4c)$, such that, restricted to M, the vectors $e_1, \cdots, e_n$ are tangent to M, where J is the complex structure of $\tilde{M}^{n+p}(4c)$. Let $\omega_1, \cdots, \omega_n$, $\omega_{n+1}, \cdots, \omega_{n+p}$, $\omega_{1^*}, \cdots, \omega_{n^*}$, $\omega_{(n+1)^*}, \cdots, \omega_{(n+p)^*}$ be the field of dual frames. We make the following convention on the range of indices: $A, B, C, \cdots = 1, \cdots, n+p, 1^*, \cdots, (n+p)^*$; $i, j, k, \cdots = 1, \cdots, n$; $\alpha, \beta, \gamma, \cdots = n+1, \cdots, n+p, 1^*, \cdots, (n+p)^*$; $\lambda, \mu, \cdots = n+1, \cdots, n+p$.

From [6], we get for any i, j, k

$$h_{ij}^{k^*} = h_{jk}^{i^*} = h_{ik}^{j^*}. \tag{2.1}$$

The Gauss equations are

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})c + \sum_{\alpha}(h_{ik}^{\alpha}h_{jl}^{\alpha} - h_{il}^{\alpha}h_{jk}^{\alpha}), \tag{2.2}$$

$$R = n(n-1)c + n^2H^2 - S, \tag{2.3}$$

where $S = \sum_{i,j,\alpha}(h_{ij}^{\alpha})^2$, $\vec{H} = \sum_{\alpha} H^{\alpha}e_{\alpha}$, $H^{\alpha} = \frac{1}{n}\sum_i h_{ii}^{\alpha}$, $H = |\vec{H}|$ and R is the scalar curvature of M. The Codazzi equations and the Ricci identities are

$$h_{ijk}^{\alpha} = h_{ikj}^{\alpha}, \tag{2.4}$$

$$h_{ijkl}^{\alpha} - h_{ijlk}^{\alpha} = \sum_m h_{mj}^{\alpha}R_{mikl} + \sum_m h_{im}^{\alpha}R_{mjkl} + \sum_{\beta} h_{ij}^{\beta}R_{\beta\alpha kl}. \tag{2.5}$$

The Ricci equations are

$$R_{\alpha\beta kl} = K_{\alpha\beta kl} + \sum_m (h^\alpha_{km} h^\beta_{lm} - h^\beta_{km} h^\alpha_{lm}). \tag{2.6}$$

We know $h^{k^*}_{ijl}$ are totally symmetric, that is, for any i, j, k, l

$$h^{k^*}_{ijl} = h^{i^*}_{jlk} = h^{j^*}_{lki} = h^{l^*}_{kij}. \tag{2.7}$$

For the fix index α, we introduce an operator $\Box^\alpha$ due to Cheng-Yau [4] by

$$\Box^\alpha f = \sum_{i,j} (nH^\alpha \delta_{ij} - h^\alpha_{ij}) f_{i,j}. \tag{2.8}$$

When M is compact, the operator $\Box^\alpha$ is self-adjoint if and only if (see [4])

$$\int_M (\Box^\alpha f) g dv = \int_M f(\Box^\alpha g) dv, \tag{2.9}$$

where f and g are any smooth functions on M.

By the same method of [9], we may easily prove the following Lemma:

Lemma 2.1. *Let $x : M \mapsto \tilde{M}^{n+p}(4c)$ be an n-dimensional ($n \geq 2$) totally real submanifold of $\tilde{M}^{n+p}(4c)$. Then*

$$|\nabla h|^2 \geq \frac{3n^2}{n+2} |\nabla^\perp \vec{H}|^2, \tag{2.10}$$

where $|\nabla h|^2 = \sum_{i,j,k,\alpha} (h^\alpha_{ijk})^2$, $|\nabla^\perp \vec{H}|^2 = \sum_{i,\alpha} (H^\alpha_{,i})^2$.

3. Willmore equations and basic formulas

Define tensors

$$\tilde{h}^\alpha_{ij} = h^\alpha_{ij} - H^\alpha \delta_{ij}, \tag{3.1}$$

$$\tilde{\sigma}_{\alpha\beta} = \sum_{i,j} \tilde{h}^\alpha_{ij} \tilde{h}^\beta_{ij}, \quad \sigma_{\alpha\beta} = \sum_{i,j} h^\alpha_{ij} h^\beta_{ij}, \tag{3.2}$$

then the $((n+2p)\times(n+2p))$-matrix $(\tilde{\sigma}_{\alpha\beta})$ is symmetric and can be assumed to be diagonized for a suitable choice of $e_{n+1}, \cdots, e_{n+p}, e_{1^*}, \cdots, e_{(n+p)^*}$. Setting

$$\tilde{\sigma}_{\alpha\beta} = \tilde{\sigma}_\alpha \delta_{\alpha\beta}, \tag{3.3}$$

by a direct calculation, we have

$$\sum_k \tilde{h}^\alpha_{kk} = 0, \quad \tilde{\sigma}_{\alpha\beta} = \sigma_{\alpha\beta} - nH^\alpha H^\beta, \quad \rho^2 = \sum_\alpha \tilde{\sigma}_\alpha = S - nH^2, \tag{3.4}$$

$$\sum_{i,j,k,\alpha} h^\beta_{kj} h^\alpha_{ij} h^\alpha_{ik} = \sum_{i,j,k,\alpha} \tilde{h}^\beta_{kj} \tilde{h}^\alpha_{ij} \tilde{h}^\alpha_{ik} + 2\sum_{i,j,\alpha} H^\alpha \tilde{h}^\alpha_{ij} \tilde{h}^\beta_{ij} + H^\beta \rho^2 + nH^2 H^\beta. \tag{3.5}$$

From (2.8), (3.1), (3.4) and (3.5), we may rewrite Theorem 6.1 of [6] as follows:

Proposition 3.1. *A totally real submanifold $x : M \mapsto \tilde{M}^{n+p}(4c)$ is Willmore if and only if*

(1) *for* $\alpha \in \{1^*, \cdots, n^*\}$,

$$\begin{aligned}\Box^\alpha(\rho^{n-2}) =&(n-1)\rho^{n-2}\Delta^\perp H^\alpha + 2(n-1)\sum_i(\rho^{n-2})_i H^\alpha_{,i} \\ &+ (n-1)H^\alpha\Delta(\rho^{n-2}) + 3(n-1)c\rho^{n-2}H^\alpha \\ &+ \rho^{n-2}(\sum_\beta H^\beta\tilde{\sigma}_{\alpha\beta} + \sum_{i,j,k,\beta}\tilde{h}^\alpha_{ij}\tilde{h}^\beta_{ik}\tilde{h}^\beta_{kj}),\end{aligned} \tag{3.6}$$

(2) *for* $\alpha \in \{n+1, \cdots, n+p, (n+1)^*, \cdots, (n+p)^*\}$,

$$\begin{aligned}\Box^\alpha(\rho^{n-2}) =&(n-1)\rho^{n-2}\Delta^\perp H^\alpha + 2(n-1)\sum_i(\rho^{n-2})_i H^\alpha_{,i} \\ &+ (n-1)H^\alpha\Delta(\rho^{n-2}) + \rho^{n-2}(\sum_\beta H^\beta\tilde{\sigma}_{\alpha\beta} + \sum_{i,j,k,\beta}\tilde{h}^\alpha_{ij}\tilde{h}^\beta_{ik}\tilde{h}^\beta_{kj}).\end{aligned} \tag{3.7}$$

From (2.8), by a direct calculation, we also have

$$\begin{aligned}\sum_\alpha\Box^\alpha(nH^\alpha) =& |\nabla h|^2 - n^2|\nabla^\perp\vec{H}|^2 + \frac{1}{2}n(n-1)\Delta H^2 - \frac{1}{2}\Delta\rho^2 \\ &+ \sum_\alpha\sum_{i,j,k,l}h^\alpha_{ij}(h^\alpha_{kl}R_{lijk} + h^\alpha_{li}R_{lkjk}) + \sum_{\alpha,\beta}\sum_{i,j,k}h^\alpha_{ij}h^\beta_{ki}R_{\beta\alpha jk}.\end{aligned} \tag{3.8}$$

Multiplying (3.8) by ρ^{n-2} and taking integration, from (2.9),

$$\begin{aligned}&\sum_\alpha\int_M(nH^\alpha)\Box^\alpha(\rho^{n-2})dv = \int_M\rho^{n-2}(|\nabla h|^2 - n^2|\nabla^\perp\vec{H}|^2)dv \\ &+ \frac{1}{2}n(n-1)\int_M\rho^{n-2}\Delta H^2dv - \frac{1}{2}\int_M\rho^{n-2}\Delta\rho^2dv \\ &+ \int_M\rho^{n-2}\sum_\alpha\sum_{i,j,k,l}h^\alpha_{ij}(h^\alpha_{kl}R_{lijk} + h^\alpha_{li}R_{lkjk})dv + \int_M\rho^{n-2}\sum_{\alpha,\beta}\sum_{i,j,k}h^\alpha_{ij}h^\beta_{ki}R_{\beta\alpha jk}dv.\end{aligned} \tag{3.9}$$

Taking the Willmore equations (3.6) and (3.7) into (3.9) and making use of the same calculation in [11], we get

Proposition 3.2. *For any n-dimensional compact Willmore totally real submanifold in $\tilde{M}^{n+p}(4c)$, the following integral equality holds*

$$\begin{aligned}&\int_M\rho^{n-2}(|\nabla h|^2 - n|\nabla^\perp\vec{H}|^2)dv + (n-2)\int_M\rho^{n-2}|\nabla\rho|^2dv \\ &- 3n(n-1)c\int_M\rho^{n-2}\sum_{\alpha=1^*}^{n^*}(H^\alpha)^2dv - \int_M\rho^{n-2}\sum_{\alpha,\beta}nH^\alpha(H^\beta\tilde{\sigma}_{\alpha\beta} + \sum_{i,j,k}\tilde{h}^\alpha_{ij}\tilde{h}^\beta_{ik}\tilde{h}^\beta_{kj})dv\end{aligned} \tag{3.10}$$

$$+\int_M \rho^{n-2}\sum_{\alpha}\sum_{i,j,k,l} h^{\alpha}_{ij}(h^{\alpha}_{kl}R_{lijk}+h^{\alpha}_{li}R_{lkjk})dv+\int_M \rho^{n-2}\sum_{\alpha,\beta}\sum_{i,j,k} h^{\alpha}_{ij}h^{\beta}_{ki}R_{\beta\alpha jk}dv=0.$$

From (2.6) and (3.1),

$$\sum_{\alpha,\beta}\sum_{i,j,k} h^{\alpha}_{ij}h^{\beta}_{ki}R_{\beta\alpha jk}=c\sum_{\alpha=1^*}^{n^*}\tilde{\sigma}_{\alpha}-n(n-1)c\sum_{\alpha=1^*}^{n^*}(H^{\alpha})^2-\frac{1}{2}\sum_{\alpha,\beta}N(\tilde{A}_{\alpha}\tilde{A}_{\beta}-\tilde{A}_{\beta}\tilde{A}_{\alpha}), \tag{3.11}$$

where $\tilde{A}_{\alpha}:=(\tilde{h}^{\alpha}_{ij})=(h^{\alpha}_{ij}-H^{\alpha}\delta_{ij})$ and $N(A)$ denote the square of the norm of matrix $A=(a_{ij})$. From (2.2), (3.2), (3.4), (3.5) and (3.11), by a direct calculation,

$$\begin{aligned}\sum_{\alpha}\sum_{i,j,k,l} h^{\alpha}_{ij}(h^{\alpha}_{kl}R_{lijk}+h^{\alpha}_{li}R_{lkjk})&=nc\rho^2-\sum_{\alpha,\beta}\sum_{i,j,k,l} h^{\alpha}_{ij}h^{\beta}_{ij}h^{\alpha}_{lk}h^{\beta}_{lk}\\&\quad+n\sum_{\alpha,\beta}\sum_{i,j,k} H^{\beta}h^{\beta}_{kj}h^{\alpha}_{ij}h^{\alpha}_{ik}+\sum_{\alpha,\beta}\sum_{i,j,k,l} h^{\alpha}_{ij}h^{\beta}_{ki}(h^{\beta}_{jl}h^{\alpha}_{lk}-h^{\beta}_{kl}h^{\alpha}_{lj})\\&=nc\rho^2-\sum_{\alpha,\beta}\tilde{\sigma}^2_{\alpha\beta}+nH^2\rho^2+n\sum_{\alpha,\beta}\sum_{i,j,k} H^{\beta}\tilde{h}^{\beta}_{kj}\tilde{h}^{\alpha}_{ij}\tilde{h}^{\alpha}_{ik}-\frac{1}{2}\sum_{\alpha,\beta}N(\tilde{A}_{\alpha}\tilde{A}_{\beta}-\tilde{A}_{\beta}\tilde{A}_{\alpha}).\end{aligned} \tag{3.12}$$

Putting (3.11) and (3.12) into (3.10), we obtain

Proposition 3.3. *For any n-dimensional compact Willmore totally real submanifold in $\tilde{M}^{n+p}(4c)$, the following integral equality holds*

$$\begin{aligned}&\int_M \rho^{n-2}(|\nabla h|^2-n|\nabla^{\perp}\vec{H}|^2)dv+(n-2)\int_M \rho^{n-2}|\nabla\rho|^2dv\\&-4n(n-1)c\int_M \rho^{n-2}\sum_{\alpha=1^*}^{n^*}(H^{\alpha})^2dv+n\int_M \rho^{n-2}(H^2\rho^2-\sum_{\alpha,\beta}H^{\alpha}H^{\beta}\tilde{\sigma}_{\alpha\beta})dv\\&+c\int_M \rho^{n-2}\sum_{\alpha=1^*}^{n^*}\tilde{\sigma}_{\alpha}dv+nc\int_M \rho^n dv\\&-\int_M \rho^{n-2}\sum_{\alpha,\beta}(N(\tilde{A}_{\alpha}\tilde{A}_{\beta}-\tilde{A}_{\beta}\tilde{A}_{\alpha})+\tilde{\sigma}^2_{\alpha\beta})dv=0.\end{aligned} \tag{3.13}$$

Proposition 3.4. *Let M be an n-dimensional compact Willmore totally real submanifold in $\tilde{M}^{n+p}(4c)$. Then for any real number a,*

$$\begin{aligned}&\int_M \rho^{n-2}(|\nabla h|^2-n|\nabla^{\perp}\vec{H}|^2)dv+(n-2)\int_M \rho^{n-2}|\nabla\rho|^2dv\\&-4n(n-1)c\int_M \rho^{n-2}\sum_{\alpha=1^*}^{n^*}(H^{\alpha})^2dv+n\int_M \rho^{n-2}(H^2\rho^2-\sum_{\alpha,\beta}H^{\alpha}H^{\beta}\tilde{\sigma}_{\alpha\beta})dv\end{aligned} \tag{3.14}$$

$$- (1+a)n\int_M H^2\rho^n dv + (1+a)\int_M \rho^{n-2}\sum_\alpha\sum_{i,j,k,l} h^\alpha_{ij}(h^\alpha_{kl}R_{lijk} + h^\alpha_{li}R_{lkjk})dv$$

$$- (1+a)n\int_M \rho^{n-2}\sum_{\alpha,\beta}\sum_{i,j,k} H^\alpha\tilde{h}^\alpha_{ij}\tilde{h}^\beta_{ik}\tilde{h}^\beta_{kj}dv - anc\int_M \rho^n dv + c\int_M \rho^{n-2}\sum_{\alpha=1^*}^{n^*}\tilde{\sigma}_\alpha dv$$

$$+ a\int_M \rho^{n-2}\sum_{\alpha,\beta}\tilde{\sigma}^2_{\alpha\beta}dv - \frac{1-a}{2}\int_M \rho^{n-2}\sum_{\alpha,\beta}N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha)dv = 0.$$

4. Inequalities of Simons' type and characterization theorems

We shall prove the following theorems.

Theorem 4.1. *Let M be an n-dimensional $(n \geq 2)$ compact Willmore totally real submanifold in $CP^{n+p}(4c)$, $p > 0$. Then*

$$\int_M \rho^{n-2}\Big\{\frac{3}{2}\rho^4 - nc\rho^2 + 4n(n-1)cH^2\Big\}dv \geq 0. \tag{4.1}$$

In particular, if

$$\frac{3}{2}\rho^4 - nc\rho^2 + 4n(n-1)cH^2 \leq 0, \tag{4.2}$$

then

(i) when $c > 0$, M is totally geodesic or a minimal totally real submanifold with parallel second fundamental form and $S = \frac{2}{3}nc$;

(ii) when $c \leq 0$, M is totally umbilical.

Proof of Theorem 4.1. When $c = 0$, our theorem is trivial. From the well-known algebraic inequality of Li-Li [7] (see Theorem 1 of [7]), we see that

$$-\sum_{\alpha,\beta}N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha) - \sum_{\alpha,\beta}\tilde{\sigma}^2_{\alpha\beta} \geq -\frac{3}{2}\rho^4, \tag{4.3}$$

$$\sum_{\alpha,\beta}H^\alpha H^\beta\tilde{\sigma}_{\alpha\beta} = \sum_\alpha (H^\alpha)^2\tilde{\sigma}_\alpha \leq \sum_\alpha (H^\alpha)^2\sum_\beta\tilde{\sigma}_\beta = H^2\rho^2, \tag{4.4}$$

$$\sum_{\alpha=1^*}^{n^*}(H^\alpha)^2 \leq H^2, \quad \sum_{\alpha=1^*}^{n^*}\tilde{\sigma}_\alpha \geq 0. \tag{4.5}$$

By making use of Lemma 2.1, (3.13), (4.3) and (4.4),

$$\begin{aligned} 0 \geq & \int_M \rho^{n-2}(|\nabla h|^2 - \frac{3n^2}{n+2}|\nabla^\perp\vec{H}|^2)dv + \int_M \rho^{n-2}(\frac{3n^2}{n+2} - n)|\nabla^\perp\vec{H}|^2 dv \\ & - 4n(n-1)c\int_M \rho^{n-2}H^2 dv + nc\int_M \rho^n dv - \int_M \rho^{n-2}\frac{3}{2}\rho^4 dv. \end{aligned} \tag{4.6}$$

Thus, we get (4.1). If (4.2) holds, we see that either $\rho^{n-2} = 0, (n > 2)$ or $\frac{3}{2}\rho^4 - nc\rho^2 + 4n(n-1)cH^2 = 0, (n \geq 2)$. The first case implies that M is totally umbilical. If $c > 0$, from (4.2), we have $4n(n-1)cH^2 \le 0$. Thus, $H = 0$ and M is totally geodesic.

If the second case holds, we see that the equalities in (4.6), (4.5) and (4.4) hold. Thus,

$$\sum_{\alpha=1^*}^{n^*}(H^\alpha)^2 = H^2, \sum_{\alpha=1^*}^{n^*}\tilde{\sigma}_\alpha = 0, \sum_{\alpha,\beta}H^\alpha H^\beta\tilde{\sigma}_{\alpha\beta} = H^2\rho^2. \tag{4.7}$$

From (4.7), $H^\alpha = 0$, $\alpha \in \{n+1, \cdots, n+p, (n+1)^*, \cdots, (n+p)^*\}$, $\tilde{\sigma}_\alpha = 0$, $\alpha \in \{1^*, \cdots, n^*\}$. Combining (3.3) and (4.7), we get $H^2\rho^2 = \sum_\alpha (H^\alpha)^2\tilde{\sigma}_\alpha = 0$. Thus, $H = 0$ or $\rho^2 = 0$.

(i) If $H = 0$, for $c > 0$, from $\frac{3}{2}\rho^4 - nc\rho^2 + 4n(n-1)cH^2 = 0$, we get $S = \rho^2 = 0$ and M is totally geodesic or $S = \rho^2 = \frac{2}{3}nc$, in this case, from the equality of (2.10), $|\nabla h|^2 = 0$, thus, M is a minimal totally real submanifold with parallel second fundamental form.

For $c < 0$, this contradicts the well-known fact that there is no compact minimal submanifolds in a simply connected manifold with nonpositive sectional curvature.

(ii) If $\rho^2 = 0$, from $\frac{3}{2}\rho^4 - nc\rho^2 + 4n(n-1)cH^2 = 0$, we see that $H^2 = 0$. Thus, for $c > 0$, M is totally geodesic. For $c < 0$, from above assertion, a contradiction. This completes the proof of Theorem 4.1.

Theorem 4.2. *Let M be an n-dimensional $(n \ge 2)$ compact Willmore totally real submanifold in $\tilde{M}^{n+p}(4c)$, $p > 0$ and let K be the function which assigns to each point of M the infimum of the sectional curvature at that point. Then*

(i) *when $c > 0$,*

$$\int_M \rho^{n-2}\{[(K - \frac{n-2}{\sqrt{n(n-1)}}H\rho - H^2) - \frac{(b-1)c}{2b-1}]\rho^2 - \frac{4(n-1)bc}{2b-1}H^2\}dv \le 0, \tag{4.8}$$

where $b = n + 2p$.

In particular, if

$$[(K - \frac{n-2}{\sqrt{n(n-1)}}H\rho - H^2) - \frac{(b-1)c}{2b-1}]\rho^2 - \frac{4(n-1)bc}{2b-1}H^2 \ge 0, \tag{4.9}$$

then M is totally geodesic, or a minimal totally real submanifold with parallel second fundamental form and $K = \frac{(b-1)c}{2b-1}$;

(ii) *when $c = 0$,*

$$\int_M \rho^n\{K - \frac{n-2}{\sqrt{n(n-1)}}H\rho - H^2\}dv \le 0. \tag{4.10}$$

In particular, if

$$K \geq \frac{n-2}{\sqrt{n(n-1)}} H\rho + H^2, \tag{4.11}$$

then M is totally umbilical;

(iii) when $c < 0$,

$$\int_M \rho^n \{K - \frac{n-2}{\sqrt{n(n-1)}} H\rho - H^2 - \frac{(n-1)b-n}{n(2b-1)} c\} dv \leq 0, \tag{4.12}$$

where $b = n + 2p$.

In particular, if

$$K \geq \frac{n-2}{\sqrt{n(n-1)}} H\rho + H^2 + \frac{(n-1)b-n}{n(2b-1)} c, \tag{4.13}$$

then M is totally umbilical.

Proof of Theorem 4.2. For a fixed α, we take a local orthonormal frame field $\{e_1, \cdots, e_n\}$ such that $h^\alpha_{ij} = \lambda^\alpha_i \delta_{ij}$, then $\tilde{h}^\alpha_{ij} = \mu^\alpha_i \delta_{ij}$ with $\mu^\alpha_i = \lambda^\alpha_i - H^\alpha$, $\sum\limits_i \mu^\alpha_i = 0$. Thus,

$$\sum_{\alpha,i,j,k,l} h^\alpha_{ij}(h^\alpha_{kl} R_{lijk} + h^\alpha_{li} R_{lkjk}) = \frac{1}{2} \sum_{\alpha,i,j} (\mu^\alpha_i - \mu^\alpha_j)^2 R_{ijij} \geq nK\rho^2, \tag{4.14}$$

and the equality in (4.14) holds if and only if $R_{ijij} = K$ for any $i \neq j$.

Since $\sum\limits_i \tilde{h}^\beta_{ii} = 0$, $\sum\limits_i \mu^\alpha_i = 0$, $\sum\limits_i (\tilde{h}^\beta_{ii})^2 = \tilde{\sigma}_\beta$ and $\sum\limits_i (\mu^\alpha_i)^2 = \tilde{\sigma}_\alpha$, from the algebraic Lemmas in [3] (see Lemma 3.3 and Lemma 3.4 in [3]) and (3.3),

$$\sum_{\alpha,\beta} \sum_{i,j,k} H^\alpha \tilde{h}^\alpha_{ij} \tilde{h}^\beta_{kj} \tilde{h}^\beta_{ik} = \sum_{\alpha,\beta} H^\beta \sum_i \tilde{h}^\beta_{ii} (\mu^\alpha_i)^2 \leq \frac{n-2}{\sqrt{n(n-1)}} H\rho^3, \tag{4.15}$$

$$\sum_{\alpha,\beta} \tilde{\sigma}^2_{\alpha\beta} = \sum_\alpha \tilde{\sigma}^2_\alpha \geq \frac{1}{n+2p} (\sum_\alpha \tilde{\sigma}_\alpha)^2 = \frac{1}{n+2p} \rho^4. \tag{4.16}$$

From Lemma 1 in [5], (3.2) and (3.3),

$$\sum_{\alpha,\beta} N(\tilde{A}_\alpha \tilde{A}_\beta - \tilde{A}_\beta \tilde{A}_\alpha) \leq 2 \sum_{\alpha \neq \beta} \tilde{\sigma}_\alpha \tilde{\sigma}_\beta = 2(\sum_\alpha \tilde{\sigma}_\alpha)^2 - 2 \sum_\alpha \tilde{\sigma}^2_\alpha \leq 2(1 - \frac{1}{n+2p}) \rho^4. \tag{4.17}$$

(i) When $c > 0$, from (3.14), (4.4), (4.5), Lemma 2.1, (4.14)–(4.17),

$$\begin{aligned} 0 \geq & \int_M \rho^{n-2} (|\nabla h|^2 - \frac{3n^2}{n+2} |\nabla^\perp \vec{H}|^2) dv + \int_M \rho^{n-2} (\frac{3n^2}{n+2} - n) |\nabla^\perp \vec{H}|^2 dv \\ & - 4n(n-1)c \int_M \rho^{n-2} H^2 dv - (1+a)n \int_M H^2 \rho^n dv + (1+a) \int_M \rho^{n-2} nK\rho^2 dv \end{aligned} \tag{4.18}$$

$$- (1+a)n\int_M \rho^{n-2}\frac{n-2}{\sqrt{n(n-1)}}H\rho^3 dv - anc\int_M \rho^n dv$$
$$+ [\frac{a}{n+2p} - (1-a)(1-\frac{1}{n+2p})]\int_M \rho^{n-2}\rho^4 dv, \quad (0<a<1),$$

Putting $a = \frac{n+2p-1}{n+2p}$, we get (4.8). If (4.9) holds, we see that either $\rho^{n-2} = 0, (n > 2)$ or $[(K - \frac{n-2}{\sqrt{n(n-1)}}H\rho - H^2) - \frac{(b-1)c}{2b-1}]\rho^2 - \frac{4(n-1)bc}{2b-1}H^2 = 0$, $n \geq 2$. The first case implies that M is totally umbilical, since $c > 0$, from (4.9), we have $H = 0$ and M is totally geodesic. If the second case holds, we see that the equalities in (4.18), (4.5) and (4.4) hold. Thus, we also get (4.7). By the same proof of Theorem 4.1, we see that $H = 0$ or $\rho^2 = 0$.

If $H = 0$, we get $S = \rho^2 = 0$ and M is totally geodesic or $K = \frac{(b-1)c}{2b-1}$, in this case, from the equality of (2.10), we get $|\nabla h|^2 = 0$, thus, M is a minimal totally real submanifold with parallel second fundamental form. If $\rho^2 = 0$, we see that $H^2 = 0$, that is M is totally geodesic.

(*ii*) When $c = 0$, from (3.14), (4.4), Lemma 2.1, (4.14)–(4.17),

$$0 \geq \int_M \rho^{n-2}(|\nabla h|^2 - \frac{3n^2}{n+2}|\nabla^{\perp}\vec{H}|^2)dv + \int_M \rho^{n-2}(\frac{3n^2}{n+2} - n)|\nabla^{\perp}\vec{H}|^2 dv \tag{4.19}$$
$$+ (1+a)n\int_M \rho^n(K - \frac{n-2}{\sqrt{n(n-1)}}H\rho - H^2)dv$$
$$+ [\frac{a}{n+2p} - (1-a)(1-\frac{1}{n+2p})]\int_M \rho^{n+2}dv, \quad (0<a<1),$$

Putting $a = \frac{n+2p-1}{n+2p}$ in (4.19), we get (4.10). If (4.11) holds, we see that either $\rho^n = 0$ and M is totally umbilical or $K - \frac{n-2}{\sqrt{n(n-1)}}H\rho - H^2 = 0$, in this case, we see that the equalities in (4.19), (4.4), (4.16) and (4.17) hold. If $\rho = 0$, we know that M is totally umbilical. If $\rho \neq 0$, we see that $\nabla^{\perp}\vec{H} = 0$, $\nabla h = 0$ and

$$\tilde{\sigma}_{n+1} = \cdots = \tilde{\sigma}_{n+2p}, \quad \sum_{\alpha,\beta} H^{\alpha}H^{\beta}\tilde{\sigma}_{\alpha\beta} = H^2\rho^2. \tag{4.20}$$

From Lemma 1 in [5], at most two of $\tilde{A}_{\alpha} = (\tilde{h}^{\alpha}_{ij})$ are different from zero. If all of $\tilde{A}_{\alpha}$ are zero, which contradicts $\rho \neq 0$. If only one of them, say, $\tilde{A}_{\alpha}$, is different from zero, which contradicts (4.20). Thus, we may assume $\tilde{A}_{n+1} = \lambda'\tilde{A}$, $\tilde{A}_{n+2} = \mu'\tilde{B}$, $\lambda', \mu' \neq 0$, $\tilde{A}_{\alpha} = 0$, $\alpha \neq n+1, n+2$, where $\tilde{A}$ and $\tilde{B}$ are defined by Lemma 1 in [5]. From (4.20), $\lambda'^2(H^{n+1})^2 + \mu'^2(H^{n+2})^2 = (\lambda'^2 + \mu'^2)[(H^{n+1})^2 + (H^{n+2})^2]$. Since λ', $\mu' \neq 0$, we infer that $H^{\alpha} = 0$ for all α, that is, $H = 0$, this contradicts the well-known fact that there is no compact minimal submanifolds in a simply connected manifold with nonpositive sectional curvature.

(iii) When $c<0$, from (3.14), (4.4), Lemma 2.1, (4.14)–(4.17),

$$0\geq\int_M\rho^{n-2}(|\nabla h|^2-\frac{3n^2}{n+2}|\nabla^{\perp}\vec{H}|^2)dv+\int_M\rho^{n-2}(\frac{3n^2}{n+2}-n)|\nabla^{\perp}\vec{H}|^2dv$$
$$+(1+a)n\int_M\rho^n(K-\frac{n-2}{\sqrt{n(n-1)}}H\rho-H^2)dv-(an-1)c\int_M\rho^ndv$$
$$+[\frac{a}{n+2p}-(1-a)(1-\frac{1}{n+2p})]\int_M\rho^{n+2}dv,\quad(0<a<1),\tag{4.21}$$

where the following inequalities are used

$$\sum_{\alpha=1^*}^{n^*}(H^\alpha)^2\geq 0,\quad\sum_{\alpha=1^*}^{n^*}\tilde{\sigma}_\alpha\leq\rho^2.\tag{4.22}$$

Putting $a=\frac{n+2p-1}{n+2p}$ in (4.21), we get (4.12). If (4.13) holds, either $\rho^n=0$ and M is totally umbilical or $K-\frac{n-2}{\sqrt{n(n-1)}}H\rho-H^2-\frac{(n-1)b-n}{n(2b-1)}c=0$, in this case, the equalities in (4.21), (4.4), (4.22) hold. Thus, $H^\alpha=0$, $\alpha\in\{1^*,\cdots,n^*\}$, $\tilde{\sigma}_\alpha=0$, $\alpha\in\{n+1,\cdots,n+p,(n+1)^*,\cdots,(n+p)^*\}$ and $H^2\rho^2=\sum_\alpha(H^\alpha)^2\tilde{\sigma}_\alpha=0$. Therefore, $H=0$, from above assertion, this is a contradiction, or $\rho^2=0$ and M is totally umbilical. This completes the proof of Theorem 4.2.

Theorem 4.3. *Let M be an n-dimensional $(n\geq 4)$ compact Willmore totally real submanifold in $\tilde{M}^{n+p}(4c)$, $p>0$ and let Q be the function which assigns to each point of M the infimum of the Ricci curvature at that point. Then*

(i) *when $c>0$,*

$$\int_M\rho^{n-2}\{[Q-\frac{4(n-2)}{n}H\rho-\frac{n^2-5n+8}{n}H^2-(n-2)c]\rho^2-4(n-1)cH^2\}dv\leq 0.\tag{4.23}$$

In particular, if

$$[Q-\frac{4(n-2)}{n}H\rho-\frac{n^2-5n+8}{n}H^2-(n-2)c]\rho^2-4(n-1)cH^2\geq 0,\tag{4.24}$$

then M is totally geodesic, or a minimal totally real submanifold with parallel second fundamental form and $Q=(n-2)c$;

(ii) *when $c=0$,*

$$\int_M\rho^n\{Q-\frac{4(n-2)}{n}H\rho-\frac{n^2-5n+8}{n}H^2\}dv\leq 0.\tag{4.25}$$

In particular, if

$$Q\geq\frac{4(n-2)}{n}H\rho+\frac{n^2-5n+8}{n}H^2,\tag{4.26}$$

then M is totally umbilical;

(iii) when $c<0$*,*

$$\int_M \rho^n\{Q-\frac{4(n-2)}{n}H\rho-\frac{n^2-5n+8}{n}H^2-\frac{n^2-2n-1}{n}c\}dv \le 0. \quad (4.27)$$

In particular, if

$$Q \ge \frac{4(n-2)}{n}H\rho+\frac{n^2-5n+8}{n}H^2+\frac{n^2-2n-1}{n}c, \quad (4.28)$$

then M *is totally umbilical.*

Proof of Theorem 4.3. Firstly, by making use of the same method in [11], we have

$$\sum_{\alpha,\beta} N(\tilde{A}_\alpha\tilde{A}_\beta-\tilde{A}_\beta\tilde{A}_\alpha) \le 4\{(n-1)c+(n-2)H\rho+H^2-Q\}\rho^2-\frac{4}{n}\rho^4. \quad (4.29)$$

(i) When $c>0$, for $n\ge 4$, from (2.3), (3.13), Lemma 2.1, (3.3), (4.4), (4.5), (4.29) and

$$\sum_{\alpha,\beta}\tilde{\sigma}^2_{\alpha\beta}=\sum_\alpha \tilde{\sigma}^2_\alpha \le (\sum_\alpha \tilde{\sigma}_\alpha)^2=\rho^4, \quad (4.30)$$

we obtain (4.23). If (4.24) holds, we see that either $\rho^{n-2}=0$ or $[Q-\frac{4(n-2)}{n}H\rho-\frac{n^2-5n+8}{n}H^2-(n-2)c]\rho^2-4(n-1)cH^2=0$. The first case and (4.24) imply that M is totally geodesic. If the second case holds, by the same proof of Theorem 4.1, we see that $H=0$ or $\rho^2=0$.

If $H=0$, we get $[Q-(n-2)c]\rho^2=0$. Thus $\rho^2=0$ and M is totally geodesic or $Q=(n-2)c$, in this case, from the equality of (2.10), we get $|\nabla h|^2=0$, thus, M is a minimal totally real submanifold with parallel second fundamental form. If $\rho^2=0$, we see that $H^2=0$ and M is totally geodesic.

(ii) When $c=0$, for $n\ge 4$, from (2.3), (3.13), Lemma 2.1, (3.3), (4.4), (4.29) and (4.30), we get (4.25).

If (4.26) holds, we obtain $\rho=0$, that is M is totally umbilical, or $Q-\frac{4(n-2)}{n}H\rho-\frac{n^2-5n+8}{n}H^2=0$, in this case, if $\rho^2=0$, then M is totally umbilical; if $\rho^2\ne 0$, we see that the equality in (4.30) holds. From $\sum_\alpha\tilde{\sigma}^2_\alpha=(\sum_\alpha\tilde{\sigma}_\alpha)^2$, $\sum_{\alpha\ne\beta}\tilde{\sigma}_\alpha\tilde{\sigma}_\beta=0$, this implies that $(n+2p-1)$ of $\tilde{\sigma}_\alpha$ must be zero. Since $\rho^2=\sum_{\alpha,i,j}(\tilde{h}^\alpha_{ij})^2\ne 0$ and $\tilde{\sigma}_\alpha=\sum_{i,j}(\tilde{h}^\alpha_{ij})^2$, we infer that $(n+2p-1)$ of $\tilde{A}_\alpha=(\tilde{h}^\alpha_{ij})$ must be zero so that $n+2p=1$, i.e. $p=-\frac{n-1}{2}<0$, this contradicts the assumption $p>0$.

(ii) When $c<0$, for $n\ge 4$, from (2.3), (3.13), Lemma 2.1, (3.3), (4.4), (4.22), (4.29) and (4.30), we get (4.27). If (4.28) holds, by the same proof of (iii) in Theorem 4.2, M is totally umbilical. This completes the proof of Theorem 4.3.

5. Conclusions

Let M be an n-dimensional compact Willmore totally real submanifold of complex space forms $\tilde{M}^{n+p}(4c)$, $p \geq 0$. When $p = 0$, i.e., the Lagrangian case, we obtained some important results in [11]. In this paper, we continue study the general totally real case, i.e., $p > 0$, we obtain some important results, see Theorem 4.1–Theorem 4.3. These theorems, including the theorems in [11], give a complete integral inequalities of Simons'type and characterization theorems of such submanifolds.

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