MAXIMAL INVARIANT SUBSPACES AND OBSERVABILITY OF MULTIDIMENSIONAL SYSTEMS. PART 2: THE ALGORITHM

Valeriu Prepeliţă[1], Tiberiu Vasilache[2]

The paper is connected with the Geometric Approach, a trend which enriched the field of System Theory with new notions and techniques. An algorithm is proposed, which determines the maximal invariant subspace with respect to a finite number of commuting matrices and which is included in a given subspace. The complete proof of the algorithm is provided. This algorithm can be used in the study of the observability of multidimensional (nD) linear systems and to determine the subspace of the unobservable states. A Matlab program is proposed, which implements the algorithm and computes an orthonormal basis of the maximal invariant subspace.

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1. Introduction

The Geometric Approach is a trend in System and Control Theory which has provided simpler and elegant solutions for many important problems, such as controller synthesis, decoupling, pole-assignment, controllability, observability, minimality, duality, etc. The history of the Geometric Approach started with the papers of Basile and Marro (see [3]) and was developed by Wonham and Morse [10], Silverman, Hautus, Willems et al. The cornerstone of this approach is the concept of invariance of a subspace with respect to one linear transformation.

In the past four decades a lot of published paper and books have been designed to the theory of multidimensional (nD) systems, which has become a distinct and important branch of the systems theory. The reasons for the increasing interest in this domain are on one side the important application fields (signal processing, image processing, computer tomography, gravity and magnetic field mapping, seismology etc.) and on the other side the richness and significance of the theoretical approaches, some of them being distinct from those concerning the theory of 1D systems.

[1]Department of Mathematics and Informatics, Faculty of Applied Sciences, University "Politehnica" of Bucharest, Romania, E-mail: valeriu.prepelita@upb.ro

[2] Department of Mathematics and Informatics, Faculty of Applied Sciences, University "Politehnica" of Bucharest, Romania, E-mail: tiberiu.vasilache@gmail.com

Various state space 2D discrete-time models have been proposed in literature by Roesser [9], Fornasini-Marchesini [4], Attasi [1] etc.

This paper extends the Geometric Approach techniques to present an algorithm for determining the maximal subspace which is invariant with respect to a finite number of commutative matrices and is included in a given subspace. It completes [8] by providing the proof of this algorithm and also by determining recurrently the maximal invariant subspaces w.r.t. the first one, two etc. matrices. When the matrices represent the drifts of a multidimensional linear system and the subspace is the kernel of the output matrix, this algorithm can be adapted to determine the subspace of unobservable states, with applications in the study of the properties connected with the concept of observability [6]. The dual algorithm which determines the minimal invariant subspace that includes a given subspace (e.g. the image of the input matrix) is described in [7].

Section 2 gives the description of the maximal invariant subspace with respect to $r \geq 2$ commuting matrices which is included in a given subspace. An algorithm which calculates this maximal subspace is proposed and its proof is developed. This algorithm is a generalization of the 1D method of G. Marro (see [5]). It was applied (without proof) in [8] to determine the subspace of unobservable states of a multidimensional (rD) system.

Section 3 provides a Matlab program that implements Algorithm 2.1 presented in Section 2 which computes an orthonormal basis of the maximal invariant subspace. An example illustrates the advantages of the proposed method.

In Section 4 it is shown how Algorithm 2.1 can be applied to problems concerning the concept of observability of a class of multidimensional discrete-time systems.

2. Algorithm of maximal invariant subspaces

Let $\mathbf{K}$ be a field, $\mathcal{C}$ a proper subspace of $\mathbf{K}^n$ and $A_1, \ldots, A_r \in \mathbf{K}^{n\times n}$ commuting matrices.

Definition 2.1. A subspace $\mathcal{V}$ of $\mathbf{K}^n$ is said to be $(A_1, \ldots, A_r)$-*invariant* if $A_j v \in \mathcal{V}$, $\forall v \in \mathcal{V}$, $\forall j \in \{1, 2, \ldots, r\}$.

A subspace $\mathcal{V}$ of $\mathbf{K}^n$ is said to be $(A_1, \ldots, A_r; \mathcal{C})$-*invariant* if $\mathcal{V}$ is $(A_1, \ldots, A_r)$-invariant and it is included in $\mathcal{C}$. $\mathcal{V}$ is called *maximal* if, for any subspace $\widetilde{\mathcal{V}}$ which is $(A_1, \ldots, A_r)$-invariant and included in $\mathcal{C}$, $\widetilde{\mathcal{V}} \subset \mathcal{V}$.

Let us denote by $\max I(A_1, \ldots, A_r; \mathcal{C})$ the maximal $(A_1, \ldots, A_r)$-invariant subspace included in $\mathcal{C}$.

For a subspace $\mathcal{V}$ of $\mathbf{K}^n$, we consider the following subspaces: $A_i^{-1} = \{v \in \mathbf{K}^n | A_i v \in \mathcal{V}\}$, $A_i^{-k_i} = \{v \in \mathbf{K}^n | A_i^{k_i} v \in \mathcal{V}\}$ and $(\prod_{i=1}^r A_i^{-k_i})\mathcal{V} = \{v \in \mathbf{K}^n | (\prod_{i=1}^r A_i^{k_i}) v \in \mathcal{V}\}$, $k_i \in \mathbf{N}$, where $(\prod_{i=1}^r A_i^{-0})\mathcal{V} = \mathcal{V}$. If $v \in A_i^{-j}\mathcal{V}$, then $A_i v \in A_i^{-(j-1)}\mathcal{V}, \forall i \in \{1, 2, \ldots, r\}, \forall j \geq 1$.

Proposition 2.1. *The maximal $(A_1, \dots, A_r)$-invariant subspace included in $\mathcal{C}$ is*

$$maxI(A_1, \dots, A_r; \mathcal{C}) = \bigcap_{k_1=0}^{\infty} \cdots \bigcap_{k_r=0}^{\infty} (\prod_{i=1}^{r} A_i^{-k_i})\mathcal{C}. \tag{1}$$

Proof. Let us denote by $\mathcal{V}_1$ the subspace from the right-hand member of (1). If $v \in \mathcal{V}_1$ then particularly $v \in (\prod_{\substack{i=1 \\ i\neq j}}^{r} A_i^{-k_i})A_j^{-(k_j+1)}\mathcal{C}$, hence $A_j v \in (\prod_{i=1}^{r} A_i^{-k_i})\mathcal{C}$, $\forall k_i \in \mathbf{N}$, $j \in \{1, 2, \dots, r\}$. It follows that $A_j v \in \mathcal{V}_1$, $\forall j \in \{1, 2, \dots, r\}$ i.e. $\mathcal{V}_1$ is $(A_1, \dots, A_r)$-invariant. We can write by (1) $\mathcal{V}_1 = \mathcal{C} \cap \bigcap_{k_1=0}^{\infty} \cdots \bigcap_{k_r=0}^{\infty} (\prod_{i=1}^{r} A_i^{-k_i})\mathcal{C}$ where $(k_1, \dots, k_r) \neq (0, \dots, 0)$, hence $\mathcal{V}_1$ is included in $\mathcal{C}$.

Now, let $\mathcal{V}$ be any $(A_1, \dots, A_r)$-invariant subspace included in $\mathcal{C}$. Then, for any $v \in \mathcal{V}$, $(\prod_{i=1}^{r} A_i^{k_i})v \in \mathcal{V} \subset \mathcal{C}$, hence $v \in (\prod_{i=1}^{r} A_i^{-k_i})\mathcal{C}$, $\forall k_i \geq 0$, $\forall i \in \{1, 2, \dots, r\}$, which implies $v \in \mathcal{V}_1$. Therefore $\mathcal{V} \subset \mathcal{V}_1$, i.e. $\mathcal{V}_1$ is the maximal such subspace. □

Proposition 2.2. *The maximal $(A_1, \dots, A_r)$-invariant subspace included in $\mathcal{C}$ is*

$$maxI(A_1, \dots, A_r; \mathcal{C}) = \bigcap_{k_1=0}^{n-1} \cdots \bigcap_{k_r=0}^{n-1} (\prod_{i=1}^{r} A_i^{-k_i})\mathcal{C}. \tag{2}$$

Proof. Let us denote by $\mathcal{V}_2$ the subspace from the right-hand member of (2). Obviously, by Proposition 2.1, $\mathcal{V}_1 \subset \mathcal{V}_2$, where $\mathcal{V}_1 = \max I(A_1, \dots, A_r; \mathcal{C})$.

Now, for any $v \in \mathcal{V}_2$, $(\prod_{i=1}^{r} A_i^{k_i})v \in \mathcal{C}$, $\forall k_i \in \mathbf{N}$, $0 \leq k_i \leq n-1$, $\forall i \in \{1, 2, \dots, r\}$. Let $p_j(s) = \det(sI - A_j) = s^n + a_{n-1,j}s^{n-1} + \dots + a_{1,j}s + a_{0,j}$ be the characteristic polynomial of the matrix A_j, $j \in \{1, 2, \dots, r\}$. By Hamilton-Cayley Theorem, $p_j(A_j) = 0_n$, hence

$$A_j^n = -a_{n-1,j}A_j^{n-1} - \dots - a_{1,j}A_j - a_{0,j}I_n. \tag{3}$$

Then, for any vector $v \in \mathcal{V}_2$, $A_j^n v = -\sum_{l=0}^{n-1} a_{l,j}A_j^l v$. Since A_j are commutative matrices, we can premultiply this equality by $(\prod_{\substack{i=1 \\ i\neq j}}^{r} A_i^{k_i})$ and we obtain $(\prod_{\substack{i=1 \\ i\neq j}}^{r} A_i^{k_i})A_j^n v = -\sum_{l=0}^{n-1} a_{l,j}(\prod_{\substack{i=1 \\ i\neq j}}^{r} A_i^{k_i})A_j^l v$, hence $(\prod_{\substack{i=1 \\ i\neq j}}^{r} A_i^{k_i})A_j^n v \in \mathcal{C}$ since $(\prod_{\substack{i=1 \\ i\neq j}}^{r} A_i^{k_i})A_j^l v \in \mathcal{C}$ for $0 \leq l \leq n-1$ and $\mathcal{C}$ is a subspace. Similarly, by postmultiplying (3) by $(\prod_{\substack{i=1 \\ i\neq j}}^{r} A_i^{k_i})A_j^t v$, $t = 1, 2, \dots$, one obtains recurrently that $(\prod_{\substack{i=1 \\ i\neq j}}^{r} A_i^{k_i})A_j^{n+t}v \in \mathcal{C}$ and finally that $(\prod_{i=1}^{r} A_i^{k_i})v \in \mathcal{C}$, $\forall k_i \geq 0$, hence $v \in \mathcal{V}_1$. It follows that $\mathcal{V}_2 \subset \mathcal{V}_1$, hence $\mathcal{V}_2 = \mathcal{V}_1 = \max I(A_1, \dots, A_r; \mathcal{C})$. □

Algorithm 2.1.

Stage 1. Determine the sequence of subspaces $(S_{i_1,0,\dots,0,0})_{0 \leq i_1 \leq n}$ of the space $X = \mathbf{K}^n$:

$$S_{0,0,\dots,0,0} = \mathcal{C}; \tag{4}$$

$$S_{i_1,0,\dots,0,0} = \mathcal{C} \cap A_1^{-1} S_{i_1-1,0,\dots,0,0},\ i_1 = 1, \dots, n; \tag{5}$$

Stage 2. Determine i_1^0, the first index in $\{0,1,\dots,n-1\}$ which verifies

$$S_{i_1^0+1,0,\dots,0,0} = S_{i_1^0,0,\dots,0,0}. \tag{6}$$

% $S_{i_1^0,0,\dots,0,0}$ is the maximal A_1-invariant subspace which is included in $\mathcal{C}$.

If $i_1^0 = n-1$, then $\max I(A_1,\dots,A_r;\mathcal{C}) = \{0\} \subset \mathbf{K}^n$. STOP

If $i_1^0 < n-1$, put $j := 2$ and GO TO Stage 3.

Stage 3. Determine by (7) the sequence of subspaces $(S_{i_1^0,i_2^0,\dots,i_{j-1}^0,i_j,0,\dots,0})_{0\le i_j\le n}$ of the space $X=\mathbf{K}^n$: for $i_j = 1,2,\dots,n$,

$$S_{i_1^0,i_2^0,\dots,i_{j-1}^0,i_j,0,\dots,0} = S_{i_1^0,i_2^0,\dots,i_{j-1}^0,i_j-1,0,\dots,0} \cap A_j^{-1} S_{i_1^0,i_2^0,\dots,i_{j-1}^0,i_j-1,0,\dots,0}. \tag{7}$$

Stage 4. Determine i_j^0, the first index i_j in $\{0,1,\dots,n-1\}$ which verifies

$$S_{i_1^0,i_2^0,\dots,i_{j-1}^0,i_j^0+1,0,\dots,0} = S_{i_1^0,i_2^0,\dots,i_{j-1}^0,i_j^0,0,\dots,0}. \tag{8}$$

% $S_{i_1^0,i_2^0,\dots,i_{j-1}^0,i_j^0,0,\dots,0}$ is the maximal $(A_1,\dots,A_j)$-invariant subspace which is included in $\mathcal{C}$

If $i_j^0 = n-1$ then $\max I(A_1,\dots,A_r;\mathcal{C}) = \{0\} \subset \mathbf{K}^n$. STOP

If $i_j^0 < n-1$ then GO TO Stage 5.

Stage 5. If $j<r$ then put $j := j+1$ and GO TO Stage 3.

If $j=r$, then $\max I(A_1,\dots,A_r;\mathcal{C}) = S_{i_1^0,i_2^0,\dots,i_{j-1}^0,i_j^0,\dots,i_r^0}$. STOP

Proof. We consider the rD chain of subspaces

$$\widetilde{S}_{i_1,i_2,\dots,i_r} = \bigcap_{k_1=0}^{i_1}\bigcap_{k_2=0}^{i_2}\cdots\bigcap_{k_r=0}^{i_r} A_1^{-k_1}A_2^{-k_2}\cdots A_r^{-k_r}\mathcal{C}, \tag{9}$$

$i_j \in \{0,1,\dots,n\}$, $j \in \{0,1,\dots,r\}$. Obviously, for $i_j \le l_j$, $\forall j \in \{0,1,\dots,r\}$,

$$\widetilde{S}_{i_1,i_2,\dots,i_r} \supseteq \widetilde{S}_{l_1,l_2,\dots,l_r}. \tag{10}$$

By (9) and Proposition 2.1 we obtain $\widetilde{S}_{i_1,i_2,\dots,i_r} \supseteq \max I(A_1,\dots,A_r;\mathcal{C})$, $\forall i_j \ge 0$, $\forall j \in \{1,2,\dots,r\}$ and it follows from Proposition 2.2 that $\widetilde{S}_{n-1,n-1,\dots,n-1} = \max I(A_1,\dots,A_r;\mathcal{C})$.

From (4) and (9) one obtains $\widetilde{S}_{0,0,\dots,0,0} = A_1^{-0}A_2^{-0}\cdots A_r^{-0}\mathcal{C} = \mathcal{C} = S_{0,0,\dots,0,0}$. Let us assume that $\widetilde{S}_{i_1-1,0,\dots,0,0} = S_{i_1-1,0,\dots,0,0}$ for some $i_1 \in \{1,2,\dots,n\}$. Applying (9), (5) and the change of the index $k_1-1=k$, we get $\widetilde{S}_{i_1,0,\dots,0,0} = \bigcap_{k_1=0}^{i_1} A_1^{-k_1}\mathcal{C} = A_1^{-0}\mathcal{C} \cap A_1^{-1}\bigcap_{k_1=1}^{i_1} A_1^{-(k_1-1)}\mathcal{C} = \mathcal{C}\cap A_1^{-1}\bigcap_{k=0}^{i_1-1} A_1^{-k}\mathcal{C} = \mathcal{C}\cap A_1^{-1}\widetilde{S}_{i_1-1,0,\dots,0,0} = S_{i_1,0,\dots,0,0}$, hence we obtained by induction and by (10) the following relations:

$$\widetilde{S}_{i_1,0,\dots,0} = S_{i_1,0,\dots,0},\ \forall i_1 \in \{1,2,\dots,n\}, \tag{11}$$

$$S_{i_1,0,\dots,0} \supset S_{i_1+1,0,\dots,0},\ \forall i_1 \in \{0,1,\dots,n-1\}. \tag{12}$$

Using Hamilton-Cayley Theorem as in the proof of Proposition 2.2, one obtains $\widetilde{S}_{n,0,\dots,0} = \widetilde{S}_{n-1,0,\dots,0}$, hence by (11), $S_{n,0,\dots,0} = S_{n-1,0,\dots,0}$, i.e. i_0 from (6) verifies $1 \le$

$i_0 \leq n-1$. Let us consider the chain of subspaces

$$\mathbf{K}^n \supseteq S_{0,0,\ldots,0,0} \supseteq S_{1,0,\ldots,0,0} \supseteq \ldots \supseteq S_{i_1,0,\ldots,0,0} \supseteq \ldots \supseteq S_{n-1,0,\ldots,0,0} = S_{n,0,\ldots,0,0}.$$

Since $S_{0,0,\ldots,0,0} = \mathcal{C}$ is a proper subspace of $\mathbf{K}^n$, $\dim S_{0,0,\ldots,0,0} \leq n-1$.

If $i_0 = n-1$ is the first index which verifies (6), it follows that

$$n-1 \geq \dim S_{0,0,\ldots,0,0} > \dim S_{1,0,\ldots,0,0} > \ldots > \dim S_{n-1,0,\ldots,0,0} \geq 0,$$

hence $\dim S_{n-1,0,\ldots,0,0} = 0$ and we have

$$\{0\} = S_{n-1,0,\ldots,0,0} = \widetilde{S}_{n-1,0,\ldots,0,0} \supseteq \max I(A_1,\ldots,A_r;\mathcal{C}) \supseteq \{0\}.$$

Therefore $\max I(A_1,\ldots,A_r;\mathcal{C}) = \{0\}$, which proves the instruction in Stage 2.

If $i_1^0 < n-1$, one obtains by (5) and (6)

$$S_{i_1^0+2,0,\ldots,0,0} = \mathcal{C} \cap A_1^{-1} S_{i_1^0+1,0,\ldots,0,0} = \mathcal{C} \cap A_1^{-1} S_{i_1^0,0,\ldots,0,0} = S_{i_1^0+1,0,\ldots,0,0} = S_{i_1^0,0,\ldots,0,0}.$$

Let us assume that $S_{i_1,0,\ldots,0,0} = S_{i_1^0,0,\ldots,0,0}$ for some $i_1 \in \{i_1^0+2,\ldots,n\}$. Then, applying again (5) and (6), we get

$$S_{i_1+1,0,\ldots,0,0} = \mathcal{C} \cap A_1^{-1} S_{i_1,0,\ldots,0,0} = \mathcal{C} \cap A_1^{-1} S_{i_1^0,0,\ldots,0,0} = S_{i_1^0+1,0,\ldots,0,0} = S_{i_1^0,0,\ldots,0,0},$$

hence we proved by induction that $S_{i_1,0,\ldots,0,0} = S_{i_1^0,0,\ldots,0,0}$, $\forall i_1 \in \{i_1^0+1,\ldots,n\}$.

Now, let us assume that $\widetilde{S}_{i_1^0,\ldots,i_{j-1}^0,i_j-1,0,\ldots,0} = S_{i_1^0,\ldots,i_{j-1}^0,i_j-1,0,\ldots,0}$ for some $j \in \{2,\ldots,r\}$ and $i_j \in \{1,\ldots,n\}$.

Consider some subspaces $\mathcal{V}_k \in \mathbf{K}^n$, $k = 0,1,\ldots,i$, $i \in \mathbb{N}^*$. We have

$$\bigcap_{k=0}^{i} \mathcal{V}_k = \mathcal{V}_0 \cap (\bigcap_{k=1}^{i-1} \mathcal{V}_k) \cap \mathcal{V}_i = [\mathcal{V}_0 \cap (\bigcap_{k=1}^{i-1} \mathcal{V}_k)] \cap [(\bigcap_{k=1}^{i-1} \mathcal{V}_k) \cap \mathcal{V}_i] = (\bigcap_{k=0}^{i-1} \mathcal{V}_k) \cap (\bigcap_{k=1}^{i} \mathcal{V}_k).$$

Therefore, by replacing $\mathcal{V}_k$ by $A_j^{-k_j}\mathcal{C}$, we have

$$\bigcap_{k_j=0}^{i_j} A_j^{-k_j}\mathcal{C} = (\bigcap_{k_j=0}^{i_j-1} A_j^{-k_j}\mathcal{C}) \cap (\bigcap_{k_j=1}^{i_j} A_j^{-k_j}\mathcal{C}).$$

Using this equality and (6), we get

$$\widetilde{S}_{i_1^0,\ldots,i_{j-1}^0,i_j,0\ldots,0} = \bigcap_{k_1=0}^{i_1^0} \cdots \bigcap_{k_{j-1}=0}^{i_{j-1}^0} A_1^{-k_1} \cdots A_{j-1}^{-k_{j-1}} \bigcap_{k_j=0}^{i_j} A_j^{-k_j}\mathcal{C} =$$

$$(\bigcap_{k_1=0}^{i_1^0} \cdots \bigcap_{k_{j-1}=0}^{i_{j-1}^0} A_1^{-k_1} \cdots A_{j-1}^{-k_{j-1}} \bigcap_{k_j=0}^{i_j-1} A_j^{-k_j}\mathcal{C}) \cap (\bigcap_{k_1=0}^{i_1^0} \cdots \bigcap_{k_{j-1}=0}^{i_{j-1}^0} A_1^{-k_1} \cdots A_{j-1}^{-k_{j-1}} \bigcap_{k_j=1}^{i_j} A_j^{-k_j}\mathcal{C}),$$

which becomes by the backward movement of A_j^{-1} and the change of the index $k_j - 1 = k$ in the last term

$$(\bigcap_{k_1=0}^{i_1^0} \cdots \bigcap_{k_{j-1}=0}^{i_{j-1}^0} A_1^{-k_1} \cdots A_{j-1}^{-k_{j-1}} \bigcap_{k_j=0}^{i_j-1} A_j^{-k_j}\mathcal{C}) \cap A_j^{-1}(\bigcap_{k_1=0}^{i_1^0} \cdots \bigcap_{k_{j-1}=0}^{i_{j-1}^0} A_1^{-k_1} \cdots A_{j-1}^{-k_{j-1}} \bigcap_{k_j=1}^{i_j} A_j^{-(k_j-1)}\mathcal{C}) =$$

$$=(\bigcap_{k_1=0}^{i_1^0}\cdots\bigcap_{k_{j-1}=0}^{i_{j-1}^0} A_1^{-k_1}\cdots A_{j-1}^{-k_{j-1}}\bigcap_{k_j=0}^{i_j-1} A_j^{-k_j}\mathcal{C})\cap A_j^{-1}(\bigcap_{k_1=0}^{i_1^0}\cdots\bigcap_{k_{j-1}=0}^{i_{j-1}^0} A_1^{-k_1}\cdots A_{j-1}^{-k_{j-1}}\bigcap_{k=0}^{i_j-1} A_j^{-k}\mathcal{C}),$$

which is equal by (9), by the induction assumption and by (7) to

$$\widetilde{S}_{i_1^0,\ldots,i_{j-1}^0,i_j-1,0\ldots,0}\cap A_j^{-1}\widetilde{S}_{i_1v,\ldots,i_{j-1}^0,i_j-1,0\ldots,0}=$$

$$=S_{i_1^0,\ldots,i_{j-1}^0,i_j-1,0\ldots,0}\cap A_j^{-1}S_{i_1^0,\ldots,i_{j-1},i_j-1^0,0\ldots,0}=S_{i_1^0,\ldots,i_{j-1}^0,i_j,0\ldots,0},$$

hence we proved by induction that

$$\widetilde{S}_{i_1^0,\ldots,i_{j-1}^0,i_j,0,\ldots,0}=S_{i_1^0,\ldots,i_{j-1}^0,i_j,0,\ldots,0},\ \forall j\in\{1,\ldots,r\},\forall i_j\in\{1,\ldots,n\}, \tag{13}$$

therefore

$$\widetilde{S}_{i_1^0,\ldots,i_j^0,\ldots,i_r^0}=S_{i_1^0,\ldots,i_j^0,\ldots,i_r^0}. \tag{14}$$

By (9),(13) and (12),we obtain, for $i_1^0,\ldots,i_{j-1}^0$ determined in Stage 4 and $i_j\in\{1,\ldots,n\}$

$$S_{i_1^0,\ldots,i_{j-1}^0,i_j,0,\ldots,0}=\bigcap_{k_1=0}^{i_1^0}\cdots\bigcap_{k_{j-1}=0}^{i_{j-1}^0}\bigcap_{k_j=0}^{i_j-1} A_1^{-k_1}\cdots A_{j-1}^{-k_{j-1}}A_j^{-k_j}\mathcal{C} \tag{15}$$

and

$$S_{i_1^0,\ldots,i_{j-1}^0,i_j,0,\ldots,0}\supset S_{i_1^0,\ldots,i_{j-1}^0,i_j+1,0,\ldots,0}.$$

By (15) and by applying Hamilton-Cayley Theorem to the matrix A_j, one obtains $S_{i_1^0,\ldots,i_{j-1}^0,n,0,\ldots,0}=S_{i_1^0,\ldots,i_{j-1}^0,n-1,0,\ldots,0}$.

Let us consider the chain of subspaces

$$\mathbf{K}^n\supseteq S_{0,0,\ldots,0,0}\supseteq S_{i_1^0,\ldots,i_{j-1}^0,0,0,\ldots,0}\supseteq\ldots\supseteq S_{i_1^0,\ldots,i_{j-1}^0,1,0,\ldots,0}\supseteq\ldots\supseteq S_{i_1^0,\ldots,i_{j-1}^0,n-1,0,\ldots,0}$$
$$=S_{i_1^0,\ldots,i_{j-1}^0,n,0,\ldots,0}.$$

If $i_j^0=n-1$ is the first index which verifies (8), since $\dim S_{0,0,\ldots,0,0}\le n-1$, it follows that

$$n-1\ge\dim S_{i_1^0,\ldots,i_{j-1}^0,0,0,\ldots,0}>\dim S_{i_1^0,\ldots,i_{j-1}^0,1,0,\ldots,0}>\ldots>\dim S_{i_1^0,\ldots,i_{j-1}^0,n-1,0,\ldots,0}\ge 0,$$

which implies $\dim S_{i_1^0,\ldots,i_{j-1}^0,n-1,0,\ldots,0}=0$, hence $S_{i_1^0,\ldots,i_{j-1}^0,n-1,0,\ldots,0}=\{0\}$. By Proposition 2.2 and (15), $\max I(A_1,\ldots,A_r;\mathcal{C})\subseteq S_{i_1^0,\ldots,i_{j-1}^0,n-1,0,\ldots,0}$, hence $\max I(A_1,\ldots,A_r;\mathcal{C})=\{0\}$, which proves the instruction from Stage 4.

Consider the case $i_j^0<n-1$ (condition which includes $i_k^0<n-1$, $\forall k\in\{1,\ldots,j-1\}$).

We have by (8) $S_{i_1^0,\ldots,i_{j-1}^0,i_j^0+1,0,\ldots,0}=S_{i_1^0,\ldots,i_{j-1}^0,i_j^0,0,\ldots,0}$.

Let us assume that $S_{i_1^0,\ldots,i_{j-1}^0,i_j,0,\ldots,0}=S_{i_1^0,\ldots,i_{j-1}^0,i_j^0,0,\ldots,0}$ for some $i_j\ge i_j^0$. Then $S_{i_1^0,\ldots,i_{j-1}^0,i_j+1,0,\ldots,0}=S_{i_1^0,\ldots,i_{j-1}^0,i_j,0,\ldots,0}\cap A_j^{-1}S_{i_1^0,\ldots,i_{j-1}^0,i_j,0,\ldots,0}=S_{i_1^0,\ldots,i_{j-1}^0,i_j^0,0,\ldots,0}\cap A_j^{-1}S_{i_1^0,\ldots,i_{j-1}^0,i_j^0,0,\ldots,0}=S_{i_1^0,\ldots,i_{j-1}^0,i_j^0+1,0,\ldots,0}=S_{i_1^0,\ldots,i_{j-1}^0,i_j^0,0,\ldots,0}$. We proved by induction

$$S_{i_1^0,\ldots,i_{j-1}^0,i_j,0,\ldots,0}=S_{i_1^0,\ldots,i_{j-1}^0,i_j^0,0,\ldots,0},\ \forall i_j\in\{i_j^0+1,\ldots,n-1\}. \tag{16}$$

By (13) and (16) we obtain

$$\widetilde{S}_{i_1^0,\ldots,i_{j-1}^0,i_j,0,\ldots,0} = \widetilde{S}_{i_1^0,\ldots,i_{j-1}^0,i_j^0,0,\ldots,0},\ \forall i_j \in \{i_j^0+1,\ldots,n-1\}. \tag{17}$$

For $j=1$ we proved that if $i_1^0 < n-1$, then $S_{i_1,0,\ldots,0} = S_{i_1^0,0,\ldots,0},\ \forall i_1 \in \{i_1^0+1,\ldots,n-1\}$. Using (11), we get $\widetilde{S}_{i_1,0,\ldots,0} = \widetilde{S}_{i_1^0,0,\ldots,0},\ \forall i_1 \in \{i_1^0+1,\ldots,n\}$.

Let us now assume that

$$\widetilde{S}_{i_1,\ldots,i_{j-1},0,0,\ldots,0} = \widetilde{S}_{i_1^0,\ldots,i_{j-1}^0,0,0,\ldots,0}, \forall i_k \in \{i_k^0+1,\ldots,n-1\}, k \in \{1,2,\ldots,j-1\},$$

i.e. (see (9))

$$\bigcap_{k_1=0}^{i_1} \cdots \bigcap_{k_{j-1}=0}^{i_{j-1}} A_1^{-k_1}\cdots A_{j-1}^{-k_{j-1}}\mathcal{C} = \bigcap_{k_1=0}^{i_1^0} \cdots \bigcap_{k_{j-1}=0}^{i_{j-1}^0} A_1^{-k_1}\cdots A_{j-1}^{-k_{j-1}}\mathcal{C}.$$

Then

$$\bigcap_{k_j=0}^{i_j} A_j^{-k_j} \bigcap_{k_1=0}^{i_1} \cdots \bigcap_{k_{j-1}=0}^{i_{j-1}} A_1^{-k_1}\cdots A_{j-1}^{-k_{j-1}}\mathcal{C} = \bigcap_{k_j=0}^{i_j} A_j^{-k_j} \bigcap_{k_1=0}^{i_1^0} \cdots \bigcap_{k_{j-1}=0}^{i_{j-1}^0} A_1^{-k_1}\cdots A_{j-1}^{-k_{j-1}}\mathcal{C}$$

and using the commutativity of the matrices A_j we have

$$\bigcap_{k_1=0}^{i_1} \cdots \bigcap_{k_{j-1}=0}^{i_{j-1}} \bigcap_{k_j=0}^{i_j} A_1^{-k_1}\cdots A_{j-1}^{-k_{j-1}} A_j^{-k_j}\mathcal{C} = \bigcap_{k_1=0}^{i_1^0} \cdots \bigcap_{k_{j-1}=0}^{i_{j-1}^0} \bigcap_{k_j=0}^{i_j} A_1^{-k_1}\cdots A_{j-1}^{-k_{j-1}} A_j^{-k_j}\mathcal{C}$$

i.e. again by (9)

$$\widetilde{S}_{i_1,\ldots,i_{j-1},i_j,0,\ldots,0} = \widetilde{S}_{i_1^0,\ldots,i_{j-1}^0,i_j,0,\ldots,0},$$

and using (17) we obtain by induction, for any $j \in \{1,2,\ldots,r\}$

$$\widetilde{S}_{i_1,\ldots,i_{j-1},i_j,0,\ldots,0} = \widetilde{S}_{i_1^0,\ldots,i_{j-1}^0,i_j^0,0,\ldots,0},\ \forall i_k \in \{i_k^0+1,\ldots,n-1\}, k \in \{1,2,\ldots,j\}, \tag{18}$$

particularly

$$\widetilde{S}_{i_1,\ldots,i_j,\ldots,i_r} = \widetilde{S}_{i_1^0,\ldots,i_j^0,\ldots,i_r^0},\ \forall i_k \in \{i_k^0+1,\ldots,n-1\}, k \in \{1,2,\ldots,r\}. \tag{19}$$

It follows by Proposition 2.2 and (14) that

$$\max I(A_1,\ldots,A_r;\mathcal{C}) = \widetilde{S}_{n-1,\ldots,n-1,\ldots,n-1} = S_{i_1^0,\ldots,i_j^0,\ldots,i_r^0}, \tag{20}$$

which proves the final statement from Stage 5.

□

By (9) and (20) we obtain

Proposition 2.3. *The maximal* $(A_1,\ldots,A_r)$*-invariant subspace included in* $\mathcal{C}$ *is*

$$maxI(A_1,\ldots,A_r;\mathcal{C}) = \bigcap_{k_1=0}^{i_1^0} \cdots \bigcap_{k_r=0}^{i_r^0} A_1^{-k_1}\cdots A_r^{-k_r}\mathcal{C},$$

where $i_1^0,\ldots,i_r^0$ *are the numbers determined in stages 2 and 4.*

3. Matlab program

The *Matlab* program presented below and based upon the algorithm above calculates the dimension and an orthonormal basis of the maximal invariant subspace. The instructions make use of the m-functions *ints*, *invt* and *ima* included in the Geometric Approach toolbox published by G. Marro and G. Basile at
http://www3.deis.unibo.it/Staff/FullProf/GiovanniMarro/geometric.htm;
this GA toolbox works with Matlab and Control System Toolbox.

More precisely, given the matrices A_1, A_2 .... A_r that commute and the matrix C, the next commands will compute and display the dimension of a basis and an orthonormal one in the subspace $S = \max\mathrm{I}(A_1, A_2, \dots A_r; \mathcal{C})$, where $\mathcal{C} = \mathrm{Im}C$. The matrices are loaded from the m-File *GetAC.m*, where A_1, A_2 .... A_r are stored in an $1 \times r-$ dimensional cell array A as $A\{1\}, ...A\{r\}$.

```
% begin m-file
% ints(X,Y)=an orthonormal basis for Im(X) intersected with ImYB)
% invt(X,Y)=an orthonormal basis for the inverse image of Y through X
% ima(Z) = an othhonormal basis in the subspace generated by Z
Get_A_C;
% A is a 1 x n cell array, containing  A{1},... A{r}
[ ~, r] = size(A);
S = ima(C);
[n, dimMax] = size(S); % will be the dimension of the maximal invariant subspace
index = zeros(1, r); % the index of the calculated subspace
for j = 1:r % loop for index position
    for i= 1:n-1 % loop for index value
        S = ints(S, invt(A{j},S));
        [~, m1] = size(S);
        index(j) = i;
        if (m1 == dimMax)     break;
            else            dimMax = m1;
        end
    end
    if ((dimMax == 1) && (norm(S, 2) == 0))
        dimMax = 0;    break;
    end
end disp(['The dimension of the maximal invariant subspace is '])
disp([ num2str(dimMax)]) disp('and an orthonormal basis of this
subspace is:') disp(S)
% end m-file
```

For example, given the commuting matrices

$$A_1 = \begin{bmatrix} 1 & 2 & 1 & 3 \\ -2 & -2 & 2 & -2 \\ 1 & 2 & 1 & 1 \\ 2 & 0 & -2 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 2 & -1 & 5 \\ -6 & -4 & 6 & -6 \\ -1 & 2 & 1 & -1 \\ 6 & 0 & -6 & 2 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & -2 & -2 & 0 \\ 3 & 3 & 1 & 4 \\ -2 & 0 & 2 & -2 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

the m-File *GetAC* could be

```
A = cell(1, 3);
A{1} = [1,2,1,3;-2,-2,2,-2;1,2,1,1;2,0,-2,0];
A{2} = [1,2,-1,5;-6,-4,6,-6;-1,2,1,-1;6,0,-6,2];
A{3} = [1,3,3,2;2,-2,-2,0;3,3,1,4;-2,0,2,-2];
C = [ 1 0 0 ;0 1 0   ;0 0 1 ; 0 0 0 ];
```

and the above Matlab program will give the answers:

```
The dimension of the maximal invariant subspace is  2 and an
orthonormal basis of this subspace is:
   -0.7071          0
         0     1.0000
   -0.7071          0
         0          0 .
```

4. Application to the observability of a class of discrete-time rD systems

In this section we will show how Algorithm 2.1 can be applied to problems concerning the concept of observability of a class of multidimensional systems.

We shall use the following notations: $\overline{r} := \{1, 2, \ldots, r\}$ where $r \in \mathbf{N}^*$. A function $x(t_1, \ldots, t_r)$ is denoted by $x(t)$, where $t = (t_1, \ldots, t_r)$, and $t_i \in \mathbf{Z}^+$ are the discrete-time variables.

For a subset $\delta = \{i_1, \ldots, i_l\}$ of $\overline{r}$, we consider the notations $|\delta| := l$, $\tilde{\delta} := \overline{r} \setminus \delta$ and $|\varnothing| := 0$; for $i \in \overline{r}$, $\tilde{i} := \overline{r}\{i\}$. The notation $\delta \subset \overline{r}$ means that δ is $\varnothing$ or δ is a subset of $\overline{r}$ and $\delta \neq \overline{r}$. For $\delta = \{i_1, \ldots, i_l\}$, the shift operator σ_δ is defined by $\sigma_\delta x(t) = x(t + e_\delta)$ where $e_\delta = e_{i_1} + \cdots + e_{i_l}$, $e_j = (\underbrace{0, \ldots, 0}_{j-1}, 1, 0, \ldots, 0) \in \mathbf{Z}^r$; when $\delta = \overline{r}$ we denote $\sigma_\delta = \sigma$, hence $\sigma x(t_1, t_2, \ldots, t_r) = x(t_1 + 1, t_2 + 1, \ldots, t_r + 1)$.

Definition 4.1. An *rD discrete-time linear system* is an ensemble $\Sigma = (A_1, \ldots, A_r; B; C; D)$ where A_i, $i \in \overline{r}$ are commuting $n \times n$ matrices over a field $\mathbf{K}$ and B, C, D are respectively

$n \times m$, $p \times n$ and $p \times m$ matrices over $\mathbf{K}$. The $\mathbf{K}$-spaces $X = \mathbf{K}^n$, $U = \mathbf{K}^m$, $Y = \mathbf{K}^p$ are called respectively the *state space, input space* and *output space* and the *time set* is $T = \mathbf{N}^r$. The following equations are called respectively the *state equation* and the *output equation:*

$$\sigma x(t) = \sum_{\delta \subset \overline{r}} (-1)^{r-|\delta|-1} \left(\prod_{i \in \tilde{\delta}} A_i \right) \sigma_\delta x(t) + Bu(t), \tag{21}$$

$$y(t) = Cx(t) + Du(t), \tag{22}$$

where $x(t) = x(t_1, \dots, t_r) \in X$ is the *state*, $u(t) \in U$ is the *input* and $y(t) \in Y$ is the *output* of the system Σ at the moment $t \in T$.

For any set $\delta = \{i_1, \dots, i_l\} \subset \overline{r}$ and for $t_i \in \mathbf{Z}^+$, $i \in \delta$, we use the notation

$$x(t_\delta, 0_{\tilde{\delta}}) := x(0, \dots, 0, t_{i_1}, 0, \dots, 0, t_{i_l}, 0, \dots, 0).$$

Definition 4.2. The vector $x_0 \in \mathbf{K}^n$ is called an *initial state* of the system Σ if

$$x(t_\delta, t^0_{\tilde{\delta}}) = \left(\prod_{i \in \delta} A_i \right) x_0 \tag{23}$$

for any $\delta \subset \overline{r}$; equalities (23) are called *initial conditions of* Σ.

By [8, Proposition 2.2], we have

Theorem 4.1. *The output of the system Σ at the moment t, determined by the initial state x_0 and the output $u : T \to U$ is (with $l = (l_1, \dots, l_r)$):*

$$y(t) = C \left(\prod_{j=1}^{r} A_j^{t_j} \right) x^0 + \sum_{l_1=0}^{t_1-1} \dots \sum_{l_r=0}^{t_r-1} C \left(\prod_{j=1}^{r} A_j^{t_j - l_j - 1} \right) Bu(l) + Du(t). \tag{24}$$

Definition 4.3. A state $x \in \mathbf{K}^n$ is said to be *unobservable* if, for any input $u(t)$, the initial states $x^0 = x$ and $x^0 = 0$ produce the same output $y(t)$, $\forall t \in T$.

Proposition 4.1. *The state $x \in \mathbf{K}^n$ is unobservable if and only if*

$$C \left(\prod_{j=1}^{r} A_j^{t_j} \right) x = 0, \forall t_j \in \mathbf{N}, \forall j \in \overline{r}. \tag{25}$$

Proof. We denote by $y_x(t)$ and $y_0(t)$ the outputs produced by the initial state $x^0 = x$ and $x^0 = 0$ respectively, for an arbitrary input $u(t)$. We obtain by (24) from $y_x(t) = y_0(t)$ that the state x is unobservable if and only if $y_x(t) - y_0(t) = 0$ $\forall t \in T$ which is equivalent to (25). $\square$

In the sequel we will consider the system Σ reduced to the ensemble $\Sigma = (A_1, \ldots, A_r; C)$ which is involved in formulas concerning observability.

Using Hamilton-Cayley Theorem as in the proof of Proposition 2.2, we deduce from (25) the following result.

Proposition 4.2. *The state $x \in \mathbf{K}^n$ is unobservable if and only if*

$$C\left(\prod_{j=1}^{r} A_j^{t_j}\right) x = 0, \forall t_j \in \mathbf{N},\ t_j \leq n-1, \forall j \in \overline{r}. \tag{26}$$

Let us denote by X_{uo} the set of the unobservable states of Σ and by $\mathcal{C}$ the subspace $\mathcal{C} = \mathrm{Ker}C$. The next theorem gives the geometric characterization of the set of unobservable states of Σ.

Theorem 4.2. *X_{uo} is the maximal $(A_1, \ldots, A_r; \mathcal{C})$-invariant subspace of $\mathbf{K}^n$.*

Proof. Let x be an unobservable state of Σ. Obviously, one obtains from (25), for $t_j = 0$, $\forall j \in \overline{r}$, that $Cx = 0$, hence $X_{uo} \subset \mathrm{Ker}C$.

For arbitrary $i \in \overline{r}$ and $t_j \in \mathbf{N}$, $j \in \overline{r}$ one obtains from (25): $0 = C(\prod_{\substack{j=1 \\ j\neq i}}^{r} A_j^{t_j}) A_i^{t_i+1} x = C(\prod_{j=1}^{r} A_j^{t_j}) A_i x$, hence $A_i x \in X_{uo}, \forall i \in \overline{r}$, i.e. X_{uo} is $(A_1, \ldots, A_r; \mathcal{C})$-invariant.

Now, consider an arbitrary $(A_1, \ldots, A_r; \mathcal{C})$-invariant subspace $\mathcal{V}$ of $\mathbf{K}^n$ and let v be an element of $\mathcal{V}$. Since $\mathcal{V}$ is $(A_1, \ldots, A_r)$-invariant and it is included in $\mathrm{Ker}C$ one obtains $(\prod_{j=1}^{r} A_j^{t_j}) v \in \mathcal{V}$ and $C(\prod_{j=1}^{r} A_j^{t_j}) v = 0$, $\forall t_j \geq 0$, $\forall j \in \overline{r}$. By (25), $v \in X_{uo}$, hence $\mathcal{V} \subset X_{uo}$, i.e. X_{uo} is the maximal such space. $\square$

Definition 4.4. The system Σ is said to be *completely observable* if there is no unobservable state $x \neq 0$.

Therefore, the system Σ is completely observable if and only if $X_{uo} = \{0\}$. From Theorem 4.2 we obtain

Theorem 4.3. *The system Σ is completely observable if and only if $\{0\}$ is the maximal $(A_1, \ldots, A_r; \mathcal{C})$-invariant subspace of $\mathbf{K}^n$.*

By Theorem 4.2 we can use Algorithm 2.1 to determine the subspace X_{uo} of the unobservable states of a multidimensional system Σ.

If one gets in Stage 2 the value $i_1^0 = n-1$ or in Stage 4 $i_j^0 = n-1$, it follows that $X_{uo} = \{0\}$, hence, by Theorem 4.3, Algorithm 2.1 can be used to check if the system Σ is completely observable.

To this aim we modify Algorithm 2.1 by replacing $\mathcal{C}$ by Ker$C$ in Stage 1 (see (4) and (5)) and in Stages 2 and 4. We also replace max$I(A_1, \ldots, A_r; \mathcal{C})$ by X_{uo} in Stages 2,4 and 5. We write "The system is completely observable" in Stages 2 and 4 and "The system is not completely observable" in Stage 5.

5. **Conclusions**

An algorithm is proposed and its proof is given, in the lines of the Geometric Approach. This algorithm determines the maximal invariant subspace with respect to a finite number of commuting matrices and which is included in a given subspace. This algorithm can be used in the study of the observability of multidimensional (nD) linear systems, especially to find the canonical form of the unobservable systems and (combined with the dual algorithm for the controllable subspace), the Kalman canonical decomposition of a multidimensional system.

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