ON A HYBRID PROXIMAL POINT ALGORITHM IN BANACH SPACES

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In this paper, we introduce a hybrid projection algorithm for a countable family of mappings of type (P) in Banach spaces, this class of mappings containing the classes of resolvents of maximal monotone operators in Banach spaces and the firmly nonexpansive mappings in Hilbert spaces. We prove that the generated sequence by the new algorithm converges strongly to the common fixed point of the mappings. Furthermore, we apply the result for the resolvent of a maximal monotone operator for finding a zero of it. The results obtained extend some results in this context.

Keywords: Equilibrium problem; Maximal monotone operator; Monotone bifunction; Proximal point algorithm; Resolvent operator in Banach space.
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1. Introduction

One method for approximation the zeros of maximal monotone operators is the proximal point algorithm. A well-known study for this purpose is made by Rockafellar in Hilbert spaces [1]. He proved the weak convergence of the algorithm to a zero of a maximal monotone operator; also, see [2]. Since then, several authors have studied this method and introduced modified versions: Boikanyo and Morosanu [3], Khatibzadeh *et al.* [4, 5], Rouhani and Khatibzadeh [6], Solodov and Svaiter [7, 8], Xu [9], Yao and Postolache [10] and others.

Recently, some authors introduced modified algorithms of proximal type and studied their convergence in Banach spaces: Li and Song [11], Matsushita and Xu [12, 13], Nakajo *et al.* [14], Dadashi and Khatibzadeh [15]. Very recently, Dadashi and Postolache have introduced a hybrid proximal point algorithm for special mappings in Banach spaces [16]. They proved that the generated sequence by the algorithm converges strongly to the common fixed point of the countable family of mappings and used the result to the problem of finding a zero of a maximal monotone operator in Banach space. The mappings in the main theorem have to satisfy the so called Condition (Z).

The aim of this paper is to remove Condition (Z) and prove a strong convergence theorem for the existence of the limit of mappings at an arbitrary point. The results could be applied for finding a solution of an equilibrium problem and a minimizer of a convex function. The paper is organized as follows. In Section 2, we recall some definitions on the geometry of Banach spaces and monotone operators, which will be used in what follows. In Section 3, we prove strong convergence theorems for a countable family of mappings. In

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particular, a result for firmly nonexpansive mappings has been obtained in Hilbert spaces. In Section 4, we consider the resolvent of a maximal monotone operator in Banach spaces. Using Section 3, we get strong convergence of the sequence to the zero of a maximal monotone operator.

2. Preliminaries

Throughout the present paper, let X be a real Banach space. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ weakly converges to x; as usual $x_n \to x$ will symbolize strong convergence. The *normalized duality mapping* J from X into the family of nonempty w^*-compact subsets of its dual X^* is defined by

$$J(x) = \{x^* \in X^* \ : \ \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\},$$

for each $x \in X$ [17].

Lemma 2.1 ([18])**.** *Let X be a real Banach space and J be the duality mapping. Then, for each $x, y \in X$, one has*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, J(x + y)\rangle.$$

The norm of X is said to be Gâteaux differentiable (and X is said to be smooth) if

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1}$$

exists for each $x, \ y \in U := \{z \in X : \ \|z\| = 1\}$. It is known that if X has a Gâteaux differentiable norm, J is single-valued. The norm is said to be uniformly Gâteaux differentiable if for $y \in U$, the limit is attained uniformly for $x \in U$. The space X is said to have a Fréchet differentiable norm if for each $x \in X$ the limit in (1) is attained uniformly for $y \in U$. The space X is said to have a uniformly Fréchet differentiable norm (and X is said to be uniformly smooth) if the limit in (1) is attained uniformly for $(x, y) \in U \times U$. It is known that X is smooth if and only if each duality mapping J is single-valued. It is also well known that if X has a uniformly Gâteaux differentiable norm, J is a uniformly norm to weak* continuous on each bounded subset of X.

If the norm of X is Fréchet differentiable, then J is norm to norm continuous and if X is uniformly smooth, then J is uniformly norm to norm continuous on each bounded subset of X.

The normed space X is called uniformly convex if for any $\varepsilon \in (0, 2]$ there exists a $\delta = \delta(\varepsilon) > 0$ such that if $x, y \in X$ with $\|x\| = 1$, $\|y\| = 1$ and $\|x - y\| \ge \varepsilon$, then $\|\frac{1}{2}(x + y)\| \le 1 - \delta$. A normed space X is called strictly convex if for all $x, y \in X$, $x \ne y$, $\|x\| = \|y\| = 1$, we have $\|\lambda x + (1 - \lambda)y\| < 1$, $\forall \lambda \in (0, 1)$. It is known that if X is uniformly convex, then X is strictly convex, reflexive and has the Kadec-Klee property, that is, a sequence $\{x_n\}$ in X converges strongly to x whenever $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$. If X is uniformly convex, then X is a strictly convex and reflexive Banach space, which has the Kadec-Klee property.

Lemma 2.2 ([19])**.** *Let X be a smooth, strictly convex and reflexive Banach space, $\{x_n\}$ a sequence in X, and $x \in X$. If $\langle x_n - x, Jx_n - Jx\rangle \to 0$, then $x_n \rightharpoonup x$, $Jx_n \rightharpoonup Jx$, and $\|x_n\| \to \|x\|$.*

Definition 2.1. The multifunction $A\colon X \rightrightarrows 2^{X^*}$ is called a *monotone operator* if for every $x, y \in X$,

$$\langle x^* - y^*, x - y\rangle \geq 0, \qquad \forall x^* \in A(x), \quad \forall y^* \in A(y).$$

A monotone operator $A\colon X \rightrightarrows 2^{X^*}$ is said to be *maximal monotone*, when its graph is not properly included in the graph of any other monotone operator on the same space and the *effective domain* is defined by $D(A) = \{x \in X : A(x) \neq \emptyset\}$.

Let C be a convex closed subset of X. The operator P_C is called a metric projection operator if it assigns to each $x \in X$ its nearest point $y \in C$, such that

$$\|x - y\| = \min\{\|x - z\| : z \in C\}.$$

It is known that the metric projection operator P_C is continuous in a uniformly convex Banach space X and uniformly continuous on each bounded set of X if, in addition, X is uniformly smooth. An element y is called the metric projection of X onto C and denoted by $P_C x$. It exists and is unique at any point of the reflexive strictly convex spaces.

Lemma 2.3 ([18])**.** *Let X is a reflexive and strictly convex Banach space and C is a nonempty, closed and convex subset of X. Then, for all $x \in X$, the element $z = P_C x$ if and only if*

$$\langle J(x - z), z - y\rangle \geq 0, \quad \forall y \in C.$$

For a sequence $\{C_n\}$ of nonempty, closed and convex subsets of a Banach space X, define $\mathrm{s\text{-}Li_n C_n}$ and $\mathrm{w\text{-}Ls_n C_n}$ as follows: $x \in \mathrm{s\text{-}Li_n C_n}$ if and only if there exists $\{x_n\} \subset X$ such that $\{x_n\}$ converges strongly to x and $x_n \in C_n$ for all $n \in \mathbb{N}$. Similarly, $y \in \mathrm{w\text{-}Ls_n C_n}$ if and only if there exist a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset X$ such that $\{y_i\}$ converges weakly to y and $y_i \in C_{n_i}$ for all $i \in \mathbb{N}$. If C_0 satisfies in $C_0 = \mathrm{s\text{-}Li_n C_n} = \mathrm{w\text{-}Ls_n C_n}$, it is said that $\{C_n\}$ converges to C_0 in the sense of Mosco [20] and we write $C_0 = \mathrm{M}\text{-}\lim_{n\to\infty} C_n$. It is easy to show that if $\{C_n\}$ is nonincreasing with respect to inclusion, then $\{C_n\}$ converges to $\cap_{n=1}^{\infty} C_n$ in the sense of Mosco. For more details, see [20].

The following lemma was proved by Tsukada [21].

Lemma 2.4 ([21])**.** *Let X be a uniformly convex Banach space. Let $\{C_n\}$ be a sequence of nonempty, closed and convex subsets of X. If $C_0 = \mathrm{M}\text{-}\lim_{n\to\infty} C_n$ exists and nonempty, then for each $x \in X$, $P_{C_n} x$ converges strongly to $P_{C_0} x$, where P_{C_n} and P_{C_0} are the metric projections of X onto C_n and C_0, respectively.*

3. **Main results**

Let X be a smooth Banach space and C a nonempty subset of X. A mapping $T\colon C \to X$ is said to be of type (P) if

$$\langle Tx - Ty, J(x - Tx) - J(y - Ty)\rangle \geq 0,$$

for all $x, y \in C$ (see [19]). A mapping $S\colon C \to X$ is said to be of type (R) if

$$\langle x - Sx - (y - Sy), J(Sx) - J(Sy)\rangle \geq 0.$$

It is easy to see that T is of type (P) if and only if $S = I - T$ is of type (R). If S is of type (R), then for each $x, y \in C$ we have the following inequality,

$$\begin{aligned}(\|Sx\| - \|Sy\|)^2 &\leq \langle Sx - Sy, J(Sx) - J(Sy)\rangle \\ &\leq \langle x - y, J(Sx) - J(Sy)\rangle \\ &\leq \|x - y\|(\|Sx\| + \|Sy\|). \qquad (2)\end{aligned}$$

Theorem 3.1. *Let X be a smooth, strictly convex, and reflexive Banach space, C a nonempty subset of X and $S_n : C \to X$ a family of mappings of type* (R). *Then the following hold:*

(1) *If $\{x_n\}$ is a bounded sequence in C and $\{S_n x\}$ is bounded for some $x \in X$ or $\cap_{n=1}^{\infty} S_n^{-1}(0) \neq \emptyset$, then $\{S_n x_n\}$ is bounded;*

(2) *Let the norm of X is Fréchet differentiable. If $\{x_n\}$ be a sequence in C such that $x_n \to x \in C$ and $S_n x \to y$, then $S_n x_n \rightharpoonup y$, $J(S_n x_n) \rightharpoonup J(y)$ and $\|S_n x_n\| \to \|y\|$;*

(3) *If X has the Kadec-Klee property, then $S_n x_n \to y$.*

Proof. (1) Suppose that $\{x_n\}$ and $\{S_n x\}$ are bounded sequences in C for some $x \in X$ but $\{S_n x_n\}$ is not. Then, there exist $M, N > 0$ such that $\|S_n x\| \leq M$ and $\|S_n x_n\| > M$ for all $n \geq N$. Since S_n is of type (R) for each $n \in \mathbb{N}$, it follows from (2) that

$$\|x_n - x\| \geq \frac{(\|S_n x_n\| - \|S_n x\|)^2}{\|S_n x_n\| + \|S_n x\|} > \frac{(\|S_n x_n\| - M)^2}{\|S_n x_n\| + M},$$

for each $n \in \mathbb{N}$, and hence we get $\|x_n\| \to \infty$, which is a contradiction. Suppose that $\{x_n\}$ is a bounded sequence and $z \in \cap_{n=1}^{\infty} S_n^{-1}(0)$. So, we have $S_n z = 0$ for each $n \in \mathbb{N}$. From the definition of the mapping of type (R), we obtain

$$\begin{aligned}0 &\leq \langle x_n - S_n x_n - (z - S_n z), J(S_n x_n) - J(S_n z)\rangle \\ &= \langle x_n - S_n x_n - z, J(S_n x_n)\rangle \\ &= \langle x_n - z, J(S_n x_n)\rangle - \|S_n x_n\|^2,\end{aligned}$$

and hence $\|S_n x_n\|^2 \leq \langle x_n - z, J(S_n x_n)\rangle \leq \|x_n - z\|\|S_n x_n\|$ for each $n \in \mathbb{N}$, which shows that the sequence $\{S_n x_n\}$ is bounded.

(2) Let $\{x_n\}$ be a sequence in C such that $x_n \to x \in C$ and $S_n x \to y$. Since the norm of X is Frechet differentiable, then J is norm to norm continuous and hence $J(S_n x) \to J(y)$. It follows from (2) and part (1) that

$$\begin{aligned}0 &\leq \langle S_n x_n - y, J(S_n x_n) - J(y)\rangle \\ &= \langle S_n x_n - S_n x, J(S_n x_n) - J(S_n x)\rangle + \langle S_n x_n - S_n x, J(S_n x) - J(y)\rangle \\ &\quad + \langle S_n x - y, J(S_n x_n) - J(S_n x)\rangle + \langle S_n x - y, J(S_n x) - J(y)\rangle \\ &\leq \|x_n - x\|(\|S_n x_n\| + \|S_n x\|) + (\|S_n x_n\| + \|S_n x\|)\|J(S_n x) - J(y)\| \\ &\quad + \|S_n x - y\|(\|S_n x_n\| + \|S_n x\|) + \langle S_n x - y, J(S_n x) - J(y)\rangle \to 0,\end{aligned}$$

as $n \to \infty$. Then Lemma 2.2 implies that $S_n x_n \rightharpoonup y$, $J(S_n x_n) \rightharpoonup J(y)$ and $\|S_n x_n\| \to \|y\|$.

(3) follows directly from (1). □

Theorem 3.2. *Let X be a smooth, strictly convex, and reflexive Banach space, C a nonempty subset of X and $T_n : C \to X$ a family of mappings of type (P). Then the following hold:*

(1) *If* $\{x_n\}$ *is a bounded sequence in* C *and* $\{T_n x\}$ *is bounded for some* $x \in X$ *or* $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$, *then* $\{T_n x_n\}$ *is bounded;*

(2) *Suppose that the norm of* X *is Fréchet differentiable. If* $\{x_n\}$ *is a sequence in* C *such that* $x_n \to x \in C$ *and* $T_n x \to Tx$, *then* $T_n x_n \rightharpoonup Tx$, $J(x_n - T_n x_n) \rightharpoonup J(x - Tx)$ *and* $\|x_n - T_n x_n\| \to \|x - Tx\|$;

(3) *If* X *has the Kadec-Klee property, then* $T_n x_n \to Tx$.

Proof. Let $S_n : C \to X$ be a mapping defined by $S_n = I - T_n$. It clear that S_n is of type (R). Then, Theorem 3.1 implies the conclusions. □

Let X be a smooth Banach space and C a nonempty closed convex subset of X. Suppose that $T_n : C \to C$ be a family of mappings of type (P) with $F := \bigcap_{n=0}^{\infty} F(T_n) \neq \emptyset$, then

$$\langle T_n x - z, J(x - T_n x)\rangle \geq 0, \quad \forall n \in \mathbb{N}, \quad \forall z \in F, \quad \forall x \in C, \tag{3}$$

which is equivalent to

$$\|x - T_n x\|^2 \leq \langle x - z, J(x - T_n x)\rangle, \quad \forall n \in \mathbb{N}, \quad \forall z \in F, \quad \forall x \in C. \tag{4}$$

Algorithm 3.1. We consider the sequence $\{x_n\}$ generated by the following formulas:

$$\begin{cases} x_n = P_{C_n}(x), \\ y_n = T_n(x_n), \\ C_{n+1} = \{z \in C_n : \ \langle y_n - z, J(x_n - y_n)\rangle \geq 0\}, \end{cases}$$

where $C_1 = C$ and $x \in X$.

We first prove that the sequence $\{x_n\}$ generated by Algorithm 3.1 is well-defined. Then, we prove that $\{x_n\}$ converges strongly to $P_F(x)$, where $P_F(x)$ is the metric projection from X onto F.

Lemma 3.1. *Let* X *be a smooth, strictly convex and reflexive Banach space,* C *a nonempty, closed and convex subset of* X *and* $T_n : C \to X$ *a family of mappings of type* (P) *such that* $F := \bigcap_{n=0}^{\infty} F(T_n) \neq \emptyset$. *Then, the sequence* $\{x_n\}$ *generated by Algorithm* 3.1 *is well-defined.*

Proof. It is easy to check that C_n is closed and convex for each $n \in \mathbb{N}$. Clearly, we have $F \subset C = C_1$. Assume that $F \subset C_n$ for some $n \in \mathbb{N}$. Since X is strictly convex and reflexive, there exists a unique element $x_n \in C_n$ such that $x_n = P_{C_n}(x)$. Let $p \in F$. Since T_n is a mapping of type (P), we have $\langle T_n x_n - p, J(x_n - T_n x_n)\rangle \geq 0$ by (3), which implies that $p \in C_{n+1}$. Then $F \subset C_{n+1}$. By induction on n, we see that $F \subset C_n$ for every $n \in \mathbb{N}$. Therefore, $\{x_n\}$ is well-defined. □

Theorem 3.3. *Let assumptions in Lemma* 3.1 *are satisfied. Suppose that the norm of* X *is Frechet differentiable and* T *is a mapping of* C *into* X *defined by* $Tz = \lim_{n\to\infty} T_n z$ *for all* $z \in C$ *such that* $F(T) = F$. *Then the sequence* $\{x_n\}$ *generated by Algorithm* 3.1 *converges strongly to the element* $P_F(x)$ *of* F, *where* $P_F(x)$ *is the metric projection from* X *onto* F.

Proof. Suppose that $w = P_F(x)$. Since $x_n = P_{C_n}(x)$ and $F \subset C_n$, we have

$$\|x_n - x\| \leq \|w - x\|, \tag{5}$$

hence the sequence $\{x_n\}$ is bounded. From (4), we get

$$\|x_n - y_n\| = \|x_n - T_n x_n\| \leq \|x_n - w\|,$$

which implies that the sequence $\{y_n\}$ is bounded too.

Let $D = \cap_{n=1}^{\infty} C_n$. Since $\emptyset \neq F \subset D$, we observe that $D \neq \emptyset$. By Lemma 2.4, we get $x_n = P_{C_n} x \to P_D x = w_0$. By assumptions and Theorem 3.2 (part (2)), it follows that $y_n \rightharpoonup T w_0$, $J(x_n - y_n) \rightharpoonup J(w_0 - T w_0)$ and

$$\|x_n - y_n\| \to \|w_0 - T w_0\|. \tag{6}$$

Taking into account that $w_0 \in C_{n+1}$, we have

$$0 \leq \langle y_n - w_0, J(x_n - y_n)\rangle = -\|x_n - y_n\|^2 + \langle x_n - w_0, J(x_n - y_n)\rangle,$$

therefore,

$$\|x_n - y_n\|^2 \leq \langle x_n - w_0, J(x_n - y_n)\rangle \to \langle w_0 - w_0, J(w_0 - T w_0)\rangle = 0. \tag{7}$$

From the uniqueness of the limit, (6) and (7), we get $T w_0 = w_0$, hence the sequence $\{x_n\}$ converges strongly to $w_0 \in F$.

Now, we show that $w_0 = P_F(x)$. From (5), we get $\lim_{n\to\infty} \|x_n - x\| \leq \|w - x\|$. Therefore, from $w = P_F(x)$, $w_0 \in F$ we obtain

$$\|w - x\| \leq \|w_0 - x\| = \lim_{n\to\infty} \|x_n - x\| \leq \|w - x\|.$$

This together with the uniqueness of $P_F(x)$, implies that $w_0 = w = P_F(x)$. Hence $\{x_n\}$ converges strongly to $P_F(x)$, and this completes the proof. □

Corollary 3.1. *Let X be a smooth, strictly convex, and reflexive Banach space such that the norm of X is Fréchet differentiable. Suppose that C is a nonempty closed convex subset of X and $T\colon C \to X$ be a mapping of type* (P) *such that $F(T) \neq \emptyset$. Let the sequence $\{x_n\}$ generated by*

$$\begin{cases} x_n = P_{C_n}(x), \\ y_n = T(x_n), \\ C_{n+1} = \{z \in C_n : \ \langle y_n - z, J(x_n - y_n)\rangle \geq 0\}, \end{cases}$$

where $C_1 = C$ and $x \in X$. Then the sequence $\{x_n\}$ converges strongly to the element $P_{F(T)}(x)$.

A mapping $T\colon C \to H$ is said to be firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle, \quad \text{for all } x, y \in C.$$

Obviously, if a mapping $T\colon C \to H$ is firmly nonexpansive, then

$$\langle Tx - Ty, (x - Tx) - (y - Ty)\rangle \geq 0,$$

holds for all $x, y \in C$, hence T is of type (P).

From Theorem 3.3, we have the following results in Hilbert spaces.

Theorem 3.4. *Let H be a Hilbert space, C a nonempty closed convex subset of H, $\{T_n\}$ a sequence of firmly nonexpansive mappings of C into H such that $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Consider T be a mapping of C into X defined by $Tz = \lim_{n\to\infty} T_n z$ for all $z \in C$ such that $F(T) = F$. Let $x \in H$, $\{x_n\}$ be a sequence in C and $\{C_n\}$ a sequence of closed convex subsets of H defined by $C_1 = C$ and*

$$\begin{cases} x_n = P_{C_n}(x), \\ y_n = T_n x_n, \\ C_{n+1} = \{z \in C_n : \ \langle y_n - z, x_n - y_n\rangle \geq 0\}, \end{cases}$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $P_F(x)$.

Corollary 3.2. *Let H be a Hilbert space, C a nonempty closed convex subset of H, and T a firmly nonexpansive mapping of C into H such that $F(T) \neq \emptyset$. Let $x \in H$, $\{x_n\}$ a sequence in C and $\{C_n\}$ a sequence of closed and convex subsets of H defined by $C_1 = C$ and*

$$\begin{cases} x_n = P_{C_n}(x), \\ y_n = T x_n, \\ C_{n+1} = \{z \in C_n : \quad \langle y_n - z, x_n - y_n\rangle \geq 0\}, \end{cases}$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $P_{F(T)}(x)$.

4. **Proximal Point Method**

A well-known method for solving the equation problem $0 \in A(p)$, in a Hilbert space H, is the proximal-point algorithm (please see [1]) in which $x_1 = x \in H$ is arbitrary and

$$x_{n+1} = J_{r_n} x_n + e_n, \ n = 1, 2, 3, \cdots, \tag{8}$$

where e_n is an error vector, $\{r_n\} \subset (0, \infty)$ and $J_r = (I + rA)^{-1}$ for all $r > 0$ is the resolvent operator for A.

Definition 4.1. Let X be a smooth, strictly convex and reflexive Banach space. Suppose that $A\colon X \rightrightarrows 2^{X^*}$ is a maximal monotone operator. The operator $J_\lambda^A\colon X \to D(A)$ given by $J_\lambda^A(x) = x_\lambda$ is called the *resolvent* of A, which x_λ satisfies $\frac{1}{\lambda} J(x - x_\lambda) \in A(x_\lambda)$.

In the following, we denote the resolvent operator J_λ^A by J_λ.

Algorithm 4.1. Let $A\colon X \to 2^{X^*}$ be maximal monotone and J_{β_n} is the resolvent of A for $\beta_n > 0$. Suppose that the sequence $\{x_n\}$ generated by:

$$\begin{cases} y_n = J_{\beta_n}(x_0), \\ x_n = \alpha_n u + (1 - \alpha_n) y_n + e_n, \qquad n = 1, 2, 3, \cdots. \end{cases}$$

Theorem 4.1. *Let X is uniformly convex $A\colon X \rightrightarrows 2^{X^*}$ is maximal monotone, $F := A^{-1}(0) \neq \emptyset$ and the sequence $\{x_n\}$ generated by Algorithm* 4.1. *If $\alpha_n \to 0$, $\beta_n \to \infty$ and $e_n \to 0$, then $x_n \to q = P_F(x_0)$.*

Proof. Let $p \in A^{-1}(0)$ and the sequence $\{x_n\}$ generated by Algorithm 4.1. Then $0 \in Ap$, $\frac{1}{\beta_n} J(x_0 - y_n) \in A(y_n)$ and, from the monotonicity of A, we get:

$$0 \leq \langle J(x_0 - y_n), y_n - p\rangle \ = -\|x_0 - y_n\|^2 + \|x_0 - y_n\|\|x_0 - p\|.$$

Therefore, $\|y_n - x_0\| \leq \|x_0 - p\|$, and hence the sequence $\{y_n\}$ is bounded. Since

$$\|x_n - x_0\| \leq \alpha_n \|u - x_0\| + (1 - \alpha_n)\|y_n - x_0\| + \|e_n\|, \tag{9}$$

and $\{y_n\}$ and $\{e_n\}$ are bounded, we know that $\{x_n\}$ is bounded. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to some point $p \in X$. By Algorithm 4.1, and replacing n by n_k we obtain

$$y_{n_k} = \frac{1}{1 - \alpha_{n_k}}(x_{n_k} - \alpha_{n_k} u - e_{n_k}),$$

and hence $\{y_{n_k}\}$ converges weakly to p. From boundedness of $\{y_n\}$ and $\beta_n \to \infty$, we get

$$A(y_{n_k}) \ni \frac{1}{\beta_{n_k}} J(x_0 - y_{n_k}) \to 0.$$

Since A is demiclosed, we have $p \in A^{-1}(0)$.

Finally, we show that $\{x_n\}$ converges strongly to $q = P_F(x_0)$.

By (9) and $\|y_n - x_0\| \leq \|x_0 - p\|$

$$\|x_n - x_0\| \leq \alpha_n \|u - x_0\| + (1 - \alpha_n)\|x_0 - p\| + \|e_n\|,$$

and then, $\limsup_{n\to\infty} \|x_n - x_0\| \leq \|x_0 - p\|$. From the weakly lower semicontinuity of the norm and the assumptions, we obtain

$$\begin{aligned} \|x_0 - q\| &\leq \|x_0 - p\| \\ &\leq \liminf \|x_0 - x_{n_k}\| \\ &\leq \limsup \|x_0 - x_{n_k}\| \\ &\leq \|x_0 - q\|. \end{aligned}$$

This together with the uniqueness of $P_F(x_0)$, implies $q = p$, and hence $\{x_{n_k}\}$ converges weakly to q. Therefore, we obtain that $\{x_n\}$ converges weakly to q. Furthermore, we have that

$$\lim_{n\to\infty} \|x_0 - x_n\| = \|x_0 - q\|.$$

Since X is uniformly convex, we have that $x_0 - x_n \to x_0 - q$. It follows that $x_n \to q$, and this completes the proof. □

Corollary 4.1. *Let X be uniformly convex, $A\colon X \rightrightarrows 2^{X^*}$ be maximal monotone, $F := A^{-1}(0) \neq \emptyset$ and the sequence $\{x_n\}$ generated by $x_n = J_{\beta_n}(x_0) + e_n$. If $\beta_n \to \infty$ and $e_n \to 0$, then $x_n \to q = P_F(x_0)$.*

Corollary 4.2. *Let X be uniformly convex, $A\colon X \rightrightarrows 2^{X^*}$ be maximal monotone, $F := A^{-1}(0) \neq \emptyset$ and the sequence $\{x_n\}$ generated by $x_n = J_{\beta_n}(x_0)$. If $\beta_n \to \infty$, then $x_n \to q = P_F(x_0)$.*

From Theorem 3.3 and Corollary 4.2, we have the following result.

Theorem 4.2. *Let X is a smooth, strictly convex, and reflexive Banach space such that the norm of X is Fréchet differentiable. Suppose that $A\colon X \rightrightarrows 2^{X^*}$ is a maximal monotone* *operator such that* $F := A^{-1}(0) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence in* X *and* $\{C_n\}$ *a sequence of closed and convex subset of* X *defined by* $C_1 = X$ *and*

$$\begin{cases} x_n = P_{C_n}(x), \\ y_n = J_{\beta_n}(x_n), \\ C_{n+1} = \{z \in C_n : \ \langle y_n - z, J(x_n - y_n)\rangle \geq 0\}, \end{cases}$$

where $x \in X$, $\{\beta_n\} \subset (0, +\infty)$ *with* $\beta_n \to \infty$ *and* J_{β_n} *be the resolvent of* A. *Then* $\{x_n\}$ *converges strongly to the element* $P_F(x)$ *of* F.

Proof. Put $T_n = J_{\beta_n}$ for $n \in \mathbb{N}$. Then $F = \bigcap_{n=0}^{\infty} F(T_n) = A^{-1}(0) \neq \emptyset$. For each $x, y \in X$, we have $\frac{1}{\beta_n} J(x - T_n x) \in A(T_n x)$ and $\frac{1}{\beta_n} J(y - T_n y) \in A(T_n y)$. By the monotonicity of A, it follows that

$$0 \leq \langle T_n x - T_n y, J(x - T_n x) - J(y - T_n y)\rangle,$$

and then T_n is of type (P) for all $n \in \mathbb{N}$. By Corollary 4.2, it follows that $T_n(z) = J_{\beta_n}(z)$ implies $P_F(z) = T(z)$, for all $z \in C$. Therefore, all conditions of Theorem 3.3 are satisfied, and we obtain the conclusion. □

Corollary 4.3. *Let* H *be a Hilbert space,* $A \subset H \times H$ *a maximal monotone operator such that* $F := A^{-1}(0) \neq \emptyset$, $\{\beta_n\}$ *a sequence of positive real numbers such that* $\beta_n \to \infty$, *and* $x \in H$. *Let* $\{x_n\}$ *be a sequence in* H *and* $\{C_n\}$ *a sequence of closed and convex subsets of* H, *defined by* $C_1 = H$ *and*

$$\begin{cases} x_n = P_{C_n}(x), \\ y_n = T_n x_n, \\ C_{n+1} = \{z \in C_n : \ \langle y_n - z, x_n - y_n\rangle \geq 0\}, \end{cases}$$

for $n \in \mathbb{N}$. *Then* $\{x_n\}$ *converges strongly to* $P_F(x)$.

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