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LEFT k-BI-QUASI HYPERIDEALS IN ORDERED SEMIHYPERRINGS

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In this paper, we study fundamental properties of left bi-quasi hyperideals on ordered semihyperrings and investigate some related results. We show that in a regular ordered semihyperring $(R, +, \cdot, \preceq)$, every left bi-quasi hyperideal of R is a quasi-hyperideal of R. In addition, we characterize regular ordered semihyperrings using left bi-quasi hyperideals. After this, we define left k-bi-quasi hyperideals (of type 1) of an ordered semihyperring and obtain some results. Finally, we prove that A is a k-bi-quasi hyperideal of type 1 of an ordered semihyperring R if and only if A=A —

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1. Introduction

Hyperstructure theory was born in 1934 when Marty [17] defined hypergroups based on the notion of hyperoperation. Hypergroups has played an essential role as the foundation of hyperstructure theory. There are different types of hyperrings. A special case of this type is the additive hyperring introduced by Krasner [16]. Hypermodules over a Krasner hyperring is a generalization of the classical modules over a ring. The principal notions of hyperstructure theory and its applications can be found in [3, 4, 6, 28]. Hyperstructure theory have been used in diverse branches of mathematics [10], physics [9] and etc. In [10], Farshi et al. presented some connections between graph theory and hyperstructure theory. In [9], Dehghan Nezhad et al. provided a physical example of hyperstructures associated with the elementary particle physics, Leptons.

One of the most important research areas in semihyperring theory is the investigation of k-hyperideals. Generalization of k-hyperideals in (ordered) semihyperrings is necessary for further study of (ordered) semihyperrings. k-Hyperideals play an important role in advance studies of (ordered) semihyperrings. A k-hyperideal of an ordered semihyperring was studied by Omidi and Davvaz in [19]. Using this idea, the concept of k-bi-quasi hyperideals (of type 1) of an ordered semihyperring can be introduced. In 2016, Omidi and Davvaz [20] introduced and studied the notion of pseudoorders in ordered semihyperrings. In the theory of ordered semihyperrings, pseudoorders make a connection between ordered semihyperrings and ordinary ordered semirings [20]. In 2017, Omidi and Davvaz [19] introduced the notion of 2-prime (2-prime of type 1) hyperideals of ordered semihyperrings using k-hyperideals.

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In continuity of this paper, we study 2-prime (2-prime of type 1) bi-quasi-hyperideals of ordered semihyperrings.

In 2011, Heidari and Davvaz [13] introduced the concept of ordered semihypergroups as a generalization of ordered semigroups. Since then, the study of ordered semihypergroups is one of the most exciting topics in hyperstructure theory, for example, see [8, 12, 24, 25]. In 2016, Gu and Tang [12] provided a partial answer to the open problem given by Davvaz et al. in [8]. Later on, Tang et al. [25] completely solved the open problem given by Davvaz et al. by using weak pseudoorders. According to [13], An *ordered semihypergroup* $(H, \circ, \leq)$ is a semihypergroup $(H, \circ)$ together with a partial order $\leq$ that is *compatible* with the hyperoperation $\circ$, meaning that for any x, y, a in H,

$$x \leq y \Rightarrow a \circ x \leq a \circ y \text{ and } x \circ a \leq y \circ a.$$

Here, $a \circ x \leq a \circ y$ means for any $m \in a \circ x$ there exists $n \in a \circ y$ such that $m \leq n$. The case $x \circ a \leq y \circ a$ is defined similarly.

Davvaz [5] and Vougiouklis [27] established the general notion of semihyperring where both the addition and multiplication are hyperoperation. The notion of k-hyperideals in a semihyperring was introduced and studied by Ameri and Hedayati [1]. In 2010, Ameri and Hedayati [2] investigated the behavior of fuzzy k-hyperideals under homomorphisms of semihyperrings. In [14], Huang et al. introduced the concept of $(\epsilon_\gamma, \epsilon_\gamma \vee q_\delta)$-fuzzy hyperideals of semihyperrings and investigated some of their fundamental properties. Recently, Pibaljommee and Nakkhasen [21] introduced the notion of (m, n)-bi-quasi hyperideals for semihyperrings.

The notion of a semiring was first introduced by Vandiver [26] in 1934 as a generalization of a ring. In 2011, Gan and Jiang [11] proved some results on ordered ideals in ordered semirings. Prime ordered k-bi-ideals in ordered semirings are discussed in [23]. In 2018, Rao [22] studied some properties of left bi-quasi ideals of semirings. The notion of ordered semihyperring, which is a generalization of ordered semiring, was introduced by Davvaz and Omidi in [7] and investigated in [18, 19]. In 2019, Kazanci et al. [15] introduced the concept of fuzzy ordered hyperideals of ordered semihyperrings and studied its related properties.

In this study, we introduce the concept of left (k-)bi-quasi hyperideals on an ordered semihyperring and investigate several related results. When we work with k-bi-quasi hyperideals (of type 1) of ordered semihyperrings, it is natural to talk about fuzzy k-bi-quasi hyperideals. According to the research results, it is suggested to define and investigate some properties of fuzzy k-bi-quasi hyperideals (of type 1) in ordered semihyperrings.

2. Basic Definitions

Definition 2.1. *A semihyperring* [27] *is an algebraic hypersructure* $(R, +, \cdot)$ *which satisfies the following axioms:*

(1) $(R, +)$ *is a commutative semihypergroup;*
(2) $(R, \cdot)$ *is a semihypergroup;*
(3) $x \cdot (y + z) = x \cdot y + x \cdot z$ *and* $(x + y) \cdot z = x \cdot z + y \cdot z$ *for all* $x, y, z \in R$.

Let $(R, +, \cdot)$ be a semihyperring. If there exists an element $0 \in R$ such that $x + 0 = \{x\} = 0 + x$ and $x \cdot 0 = \{0\} = 0 \cdot x$ for all $x \in R$; then 0 is called the zero element of R. If there exists an element $1 \in R$ such that $a \cdot 1 = \{a\} = 1 \cdot a$ for all $a \in R$, then 1 is called the identity element of R. Throughout this paper we consider a semihyperring $(R, +, \cdot)$ with zero element 0.

Let $(R, +, \cdot)$ be a semihyperring. A non-empty subset A of R is said to be a *subsemihyperring* of R if for all $x, y \in A$, we have $x + y \subseteq A$ and $x \cdot y \subseteq A$.

In the following, we recall the basic concepts of ordered semihyperring that are used in this study.

Definition 2.2. *An algebraic hypersructure* $(R, +, \cdot, \leq)$ *is called an ordered semihyperring* [7, 18] *if* $(R, +, \cdot)$ *is a semihyperring and the (partial) order relation* $\leq$ *is compatible with the hyperoperations* $+$ *and* $\cdot$*, i.e.,*

(1) *If* $a \leq b$*, then* $a + c \leq b + c$ *for all* $a, b, c \in R$*, meaning that for any* $x \in a + c$*, there exists* $y \in b + c$ *such that* $x \leq y$.
(2) *If* $a \leq b$ *and* $c \in R$*, then* $a \cdot c \leq b \cdot c$*, meaning that for any* $x \in a \cdot c$*, there exists* $y \in b \cdot c$ *such that* $x \leq y$*. The case* $c \cdot a \leq c \cdot b$ *is defined similarly.*

Let H be a non-empty subset of an ordered semihyperring $(R, +, \cdot, \leq)$. Then the subset $\{x \in R \mid x \leq h \text{ for some } h \in H\}$ is denoted by $(H]$. For $H = \{h\}$, we write $(h]$ instead of $(\{h\}]$. If A and B are non-empty subsets of R, then we have

(1) $A \subseteq (A]$;
(2) $((A]] = (A]$;
(3) $(A] \cdot (B] \subseteq (A \cdot B]$;
(4) $((A] \cdot (B]] = (A \cdot B]$;
(5) If $A \subseteq B$, then $(A] \subseteq (B]$.

Definition 2.3. *Let* $(R, +, \cdot, \leq)$ *be an ordered semihyperring. A non-empty subset* A *of* R *is called a left (resp. right) hyperideal* [19] *of* R *if it satisfies the following conditions:*

(1) $x + y \subseteq A$ *for all* $x, y \in A$*;*
(2) $R \cdot A \subseteq A$ *(resp.* $A \cdot R \subseteq A$*);*
(3) $A = (A]$*, that is, for any* $a \in A$ *and* $x \in R$*,* $x \leq a$ *implies* $x \in A$.

If A *is both a left and a right hyperideal of* R*, then* A *is called a two-sided hyperideal, or simply a hyperideal of* R.

Definition 2.4. *A non-empty subset* Q *of an ordered semihyperring* $(R, +, \cdot, \leq)$ *is called a quasi-hyperideal of* R *if the following conditions hold:*

(1) $Q + Q \subseteq Q$*;*
(2) $(Q \cdot R] \cap (R \cdot Q] \subseteq Q$*;*
(3) *When* $q \in Q$ *and* $x \in R$ *such that* $x \leq q$*, imply that* $x \in Q$.

Obviously, every left and right hyperideal of an ordered semihyperring R is a quasi-hyperideal of R. Moreover, each quasi-hyperideal of R is a subsemihyperring of R. indeed: $Q \cdot Q \subseteq (Q \cdot Q] \subseteq (Q \cdot R] \cap (R \cdot Q] \subseteq Q$.

Definition 2.5. *A non-empty subset* A *of an ordered semihyperring* R *is called a bi-hyperideal of* R *if it satisfies:*

(1) $A + A \subseteq A$ *and* $A \cdot A \subseteq A$*;*
(2) $A \cdot R \cdot A \subseteq A$*;*
(3) *When* $a \in A$ *and* $x \in R$ *such that* $x \leq a$*, imply that* $x \in A$.

Definition 2.6. *An element* a *in an ordered semihyperring* $(R, +, \cdot, \leq)$ *is called regular* [7] *if there exists an element* $x \in R$ *such that* $a \leq a \cdot x \cdot a$*. An ordered semihyperring* R *is called regular if each element of* R *is regular.*

Equivalent definitions*:*

(1) $a \in (a \cdot R \cdot a]$*,* $\forall\ a \in R$.
(2) $A \subseteq (A \cdot R \cdot A]$*,* $\forall\ A \subseteq R$.

3. Main Results

In this section, we study the notion of left (k-)bi-quasi hyperideals (of type 1) on ordered semihyperrings and then we obtain some related results.

Definition 3.1. *Let $(R,+,\cdot,\leq)$ be an ordered semihyperring. A non-empty subset A of R is said to be a left (resp. right) bi-quasi hyperideal of R if it satisfies the following conditions:*

(1) *A is a subsemihyperring of R;*
(2) *$(R\cdot A]\cap(A\cdot R\cdot A]\subseteq A$ (resp. $(A\cdot R]\cap(A\cdot R\cdot A]\subseteq A$);*
(3) *When $a\in A$ and $r\in R$ such that $r\leq a$, imply that $r\in A$.*

If A is both a left and a right bi-quasi hyperideal of R, then A is called a bi-quasi hyperideal of R.

Theorem 3.1. *Every left hyperideal of an ordered semihyperring $(R,+,\cdot,\leq)$ is a bi-quasi hyperideal of R.*

Proof. Let A be a left hyperideal of an ordered semihyperring R. Then $R\cdot A\subseteq A$. So, we have

$$\begin{aligned}(R\cdot A]\cap(A\cdot R\cdot A] &\subseteq (A\cdot R\cdot A]\\ &\subseteq (A\cdot A]\\ &\subseteq (A]\\ &= A\end{aligned}$$

and

$$\begin{aligned}(A\cdot R]\cap(A\cdot R\cdot A] &\subseteq (A\cdot R\cdot A]\\ &\subseteq (A\cdot A]\\ &\subseteq (A]\\ &= A.\end{aligned}$$

Hence, A is a bi-quasi hyperideal of R. □

Example 3.1. *Let $R=\{0,a,b,c\}$. Define the hyperoperations $\oplus$, $\odot$ and (partial) order relation $\leq$ on R as follows:*

$\oplus$	0	a	b	c
0	0	a	b	c
a	a	$\{a,b\}$	b	c
b	b	b	$\{0,b\}$	c
c	c	c	c	$\{0,c\}$

$\odot$	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	b	b	b
c	0	c	c	c

$$\leq:=\{(0,0),(a,a),(b,b),(c,c),(0,a),(0,b),(0,c),(a,b),(a,c),(b,c)\}.$$

Then $(R,\oplus,\odot,\leq)$ is an ordered semihyperring [7]. *Put $A=\{0,a,b\}$. Clearly, A is a subsemihyperring of R. We have*

$$(R\odot A]\cap(A\odot R\odot A]=R\cap\{0,a,b\}\subseteq A,$$
$$(A\odot R]\cap(A\odot R\odot A]=\{0,a,b\}\cap\{0,a,b\}\subseteq A$$

and $(A]=A$. Hence, A is a bi-quasi hyperideal of R, but is not left hyperideal, because $R\odot A=\{0,a,b,c\}\nsubseteq A$.

Example 3.2. *Let $R=\{0,a,b\}$. Define the hyperoperations $\oplus$, $\odot$ and (partial) order relation $\leq$ on R as follows:*

$\oplus$	0	a	b
0	0	a	b
a	a	a	$\{a,b\}$
b	b	$\{a,b\}$	b

$\odot$	0	a	b
0	0	0	0
a	0	$\{0,a\}$	$\{0,a\}$
b	0	$\{0,b\}$	$\{0,b\}$

$$\leq:=\{(0,0),(a,a),(b,b),(0,a),(0,b),(a,b)\}.$$

Then $(R,\oplus,\odot,\leq)$ is an ordered semihyperring [7]. *It is not difficult to verify that $A=\{0,a\}$ is a bi-quasi hyperideal of R, but is not a left hyperideal of R.*

Theorem 3.2. *Every right hyperideal of an ordered semihyperring $(R,+,\cdot,\leq)$ is a bi-quasi hyperideal of R.*

Proof. This proof is straightforward. □

The following example shows that the converse of Theorem **3**.2 is not true in general.

Example 3.3. *Let* $R = \{0, a, b, c\}$. *Define the hyperoperations* $\oplus$, $\odot$ *and (partial) order relation* $\leq$ *on* R *as follows:*

$\oplus$	0	a	b	c
0	0	a	b	c
a	a	a	a	a
b	b	a	$\{0,b\}$	$\{0,b,c\}$
c	c	a	$\{0,b,c\}$	$\{0,c\}$

$\odot$	0	a	b	c
0	0	0	0	0
a	0	a	$\{0,b\}$	0
b	0	0	0	0
c	0	$\{0,c\}$	0	0

$$\leq := \{(0,0),(a,a),(b,b),(c,c),(0,b),(0,c)\}.$$

Then $(R, \oplus, \odot, \leq)$ *is an ordered semihyperring. Put* $A = \{0, a\}$. *It is easy to see that* A *is a bi-quasi hyperideal of* R, *but is not a left (right) hyperideal of* R.

Lemma 3.1. *Let* $(R, +, \cdot, \leq)$ *be an ordered semihyperring. The intersection of a left hyperideal and a right hyperideal of* R *is a left bi-quasi hyperideal of* R.

Proof. Let A be a left hyperideal and B a right hyperideal of R. Then $R \cdot A \subseteq A$ and $B \cdot R \subseteq B$. We have

$$\begin{aligned}(R\cdot(A\cap B)]\cap((A\cap B)\cdot R\cdot(A\cap B)] &\subseteq (R\cdot A]\cap(B\cdot R\cdot B]\\ &\subseteq (R\cdot A]\cap(B\cdot B]\\ &\subseteq (A]\cap(B]\\ &= A\cap B.\end{aligned}$$

Now, let $x \in A \cap B$ and $y \in R$ such that $y \leq x$. By condition (3) of Definition **2**.3, we get $y \in A$ and $y \in B$. So, $y \in A \cap B$. Therefore, $A \cap B$ is a left bi-quasi hyperideal of R. □

Let a be an element of an ordered semihyperring $(R, +, \cdot, \leq)$. We denote by $l_1(a)$ (resp. $r_2(a)$, $i_R(a)$) the left (resp. right, two-sided) hyperideal of R generated by a. $l_1(a)$ is the intersection of all left hyperideals of R containing a. The right hyperideal $r_2(a)$ and hyperideal $i_R(a)$ generated by a are defined similarly.

Lemma 3.2. *Let* a *be an element of an ordered semihyperring* $(R, +, \cdot, \leq)$. *Then,*

(1) $l_1(a) = (a \cup R \cdot a]$;
(2) $r_2(a) = (a \cup a \cdot R]$;
(3) $i_R(a) = (a \cup R \cdot a \cup a \cdot R \cup R \cdot a \cdot R]$.

Proof. Since $a \in l_1(a)$ and $R \cdot a \subseteq l_1(a)$, it follows that $(a \cup R \cdot a] \subseteq l_1(a)$. Clearly, $(a \cup R \cdot a] \neq \emptyset$. We have

$$\begin{aligned}R\cdot(a\cup R\cdot a] &\subseteq (R]\cdot(a\cup R\cdot a]\\ &\subseteq (R\cdot(a\cup R\cdot a)]\\ &\subseteq (R\cdot a]\\ &\subseteq (a\cup R\cdot a].\end{aligned}$$

On the other hand, we have $(l_1(a)] = l_1(a)$. Thus, $l_1(a) = (a \cup R \cdot a]$ is a left hyperideal of R containing a. Now, we show that $l_1(a)$ is the smallest left hyperideal of R containing a. Suppose that A is a left hyperideal of R containing a. We have

$$l_1(a) = (a \cup R \cdot a] \subseteq (A \cup R \cdot A] \subseteq (A] = A.$$

This proves that (1) holds. The conditions (2) and (3) are proved similarly. □

Theorem 3.3. *Let* $(R, +, \cdot, \leq)$ *be a regular ordered semihyperring. Then every left bi-quasi hyperideal of* R *is a quasi-hyperideal of* R.

Proof. Let A be a left bi-quasi hyperideal of a regular ordered semihyperring R. We prove this statement in three steps:

Step 1: $(A \cdot R]$ is a right hyperideal of R.
Since $0 \in A \cdot R \subseteq (A \cdot R]$, it follows that $\emptyset \neq (A \cdot R]$. Let $a, b \in (A \cdot R]$. Then $a \leq u$ and $b \leq v$ for some $u, v \in A \cdot R$. By assumption, R is an ordered semihyperring. So, $a + b \leq u + b$ and $u + b \leq u + v$. Hence, $a + b \leq u + v \subseteq A \cdot R$. Thus, for any $x \in a + b$, there exists $y \in A \cdot R$ such that $x \leq y$. So, $x \in (A \cdot R]$. This means that $a + b \subseteq (A \cdot R]$, and so the first condition of the definition of right hyperideal is verified. We have

$$\begin{aligned}(A \cdot R] \cdot R &= (A \cdot R] \cdot (R] \\ &\subseteq (A \cdot R \cdot R] \\ &\subseteq (A \cdot R].\end{aligned}$$

Now, let $x \in (A \cdot R]$ and $y \in R$ such that $y \leq x$. Since $x \in (A \cdot R]$, it follows that $x \leq w$ for some $w \in A \cdot R$. Since $y \leq x$ and $x \leq w$, we get $y \leq w$. Since $y \in R$, $y \leq w$ and $w \in A \cdot R$, we have $y \in (A \cdot R]$. Therefore, $(A \cdot R]$ is a right hyperideal of R. Similarly, we can prove that $(R \cdot A]$ is a left hyperideal of R.

Step 2: If R is a regular ordered semihyperring, then $I \cap J = (I \cdot J]$ for any right hyperideal I and left hyperideal J of R.
Assume that R is regular. Let I be a right hyperideal and J a left hyperideal of R. Then $I \cdot R \subseteq I$ and $R \cdot J \subseteq J$. So, we have

$$(I \cdot J] \subseteq (I \cdot R] \subseteq (I] = I$$

and

$$(I \cdot J] \subseteq (R \cdot J] \subseteq (J] = J.$$

Thus, $(I \cdot J] \subseteq I \cap J$. Now, let $a \in I \cap J$. Since R is regular, there exists an element $x \in R$ such that $a \leq a \cdot (x \cdot a) \subseteq I \cdot (R \cdot J) \subseteq I \cdot J$. Thus $a \leq u$ for some $u \in I \cdot J$. This means that $a \in (I \cdot J]$. Hence, $I \cap J \subseteq (I \cdot J]$. Therefore, we have $I \cap J = (I \cdot J]$.

Step 3: A is a quasi-hyperideal of R.
According to Step 2, we conclude that

$$\begin{aligned}(A \cdot R] \cap (R \cdot A] &= ((A \cdot R] \cdot (R \cdot A]] \\ &= (A \cdot R \cdot R \cdot A] \\ &= (A \cdot R^2 \cdot A] \\ &\subseteq (A \cdot R \cdot A]\end{aligned}$$

and

$$\begin{aligned}(A \cdot R] \cap (R \cdot A] &= (A \cdot R \cdot R \cdot A] \\ &\subseteq (R \cdot A].\end{aligned}$$

By hypothesis, A is a left bi-quasi hyperideal of R. So, we obtain

$$(A \cdot R] \cap (R \cdot A] \subseteq (R \cdot A] \cap (A \cdot R \cdot A] \subseteq A.$$

Hence, A is a quasi-hyperideal of R. □

Theorem 3.4. *Let $(R, +, \cdot, \leq)$ be an ordered semihyperring. Then, R is regular if and only if $A = (R \cdot A] \cap (A \cdot R \cdot A]$ for every left bi-quasi hyperideal A of R.*

Proof. Suppose that R is a regular ordered semihyperring. Let A be a left bi-quasi hyperideal of R. Then $(R \cdot A] \cap (A \cdot R \cdot A] \subseteq A$. Since R is regular, we have

$$\begin{aligned} A &\subseteq (A \cdot R \cdot A] \\ &\subseteq ((A \cdot R \cdot A] \cdot R \cdot A] \\ &\subseteq ((A \cdot R)(A \cdot R) \cdot A] \\ &\subseteq (R^2 \cdot A] \\ &\subseteq (R \cdot A] \end{aligned}$$

and

$$\begin{aligned} A &\subseteq (A \cdot R \cdot A] \\ &\subseteq (A \cdot R \cdot (A \cdot R \cdot A]] \\ &\subseteq (A \cdot (R \cdot A) \cdot R \cdot A] \\ &\subseteq (A \cdot R^2 \cdot A] \\ &\subseteq (A \cdot R \cdot A]. \end{aligned}$$

Thus $A \subseteq (R \cdot A] \cap (A \cdot R \cdot A]$. Hence, $A = (R \cdot A] \cap (A \cdot R \cdot A]$.

Conversely, suppose that $A = (R \cdot A] \cap (A \cdot R \cdot A]$ for every left bi-quasi hyperideal A of R. Let I be a right hyperideal and J a left hyperideal of R. By Lemma **3**.1, $I \cap J$ is a left bi-quasi hyperideal of R. By assumption, we have

$$\begin{aligned} I \cap J &= (R \cdot (I \cap J)] \cap ((I \cap J) \cdot R \cdot (I \cap J)] \\ &\subseteq ((I \cap J) \cdot R \cdot (I \cap J)] \\ &\subseteq (I \cdot R \cdot J] \\ &\subseteq (I \cdot J]. \end{aligned}$$

On the other hand, $I \cdot J \subseteq I \cdot R \subseteq I$. So, we get $(I \cdot J] \subseteq I$. Similarly, $(I \cdot J] \subseteq J$ and this implies that $(I \cdot J] \subseteq I \cap J$. Thus, $I \cap J = (I \cdot J]$ for any right hyperideal I and left hyperideal J of R. Now, let $a \in R$. By Lemma **3**.2, we have

$$a \in l_1(a) \cap r_2(a) = (l_1(a) \cdot r_2(a)] \subseteq (a \cdot R \cdot a].$$

By Definition **2**.6, R is a regular ordered semihyperring. □

Theorem 3.5. *Let $(R, +, \cdot, \leq)$ be an ordered semihyperring. Then, R is regular if and only if $A = (A \cdot R] \cap (A \cdot R \cdot A]$ for every right bi-quasi hyperideal A of R.*

Proof. This proof is straightforward. □

Definition 3.2. *Let $(R, +, \cdot, \leq)$ be an ordered semihyperring. A non-empty subset A of R is called a left k-bi-quasi hyperideal of R, if A is a left bi-quasi hyperideal of R and for any $a \in A$ and $x \in R$, from $a + x \approx A$ it follows $x \in A$, where we say that $A \approx B$ if $A \cap B \neq \emptyset$. A right k-bi-quasi hyperideal is defined similarly. If a hyperideal A is both left and right k-bi-quasi hyperideal, then A is known as a k-bi-quasi hyperideal of R.*

We note that every right k-hyperideal or left k-hyperideal is a k-bi-quasi hyperideal of R.

Example 3.4. *In Example **3**.1, $A = \{0, a, b\}$ is a k-bi-quasi hyperideal of R.*

Clearly, every k-bi-quasi hyperideal of an ordered semihyperring R is a bi-quasi hyperideal of R. The converse is not true, in general, that is, a bi-quasi hyperideal may not be a k-bi-quasi hyperideal as the following example shows.

Example 3.5. *In Example **3**.2, $A = \{0, a\}$ is not a k-bi-quasi hyperideal of R. Indeed:*

$$b \oplus a = \{a, b\} \approx \{0, a\} \text{ and } a \in \{0, a\} \text{ but } b \notin \{0, a\}.$$

Definition 3.3. *Let $(R, +, \cdot, \leq)$ be an ordered semihyperring. A non-empty subset A of R is called a left k-bi-quasi hyperideal of type 1 of R, if A is a left bi-quasi hyperideal of R and for any $a \in A$ and $x \in R$, from $a + x \subseteq A$ it follows $x \in A$. A right k-bi-quasi hyperideal of type 1 is defined similarly. If a hyperideal A is both left and right k-bi-quasi hyperideal of type 1, then A is known as a k-bi-quasi hyperideal of type 1 of R.*

Clearly, every k-bi-quasi hyperideal of an ordered semihyperring R is a k-bi-quasi hyperideal of type 1 of R. The converse is not true, in general, that is, a k-bi-quasi hyperideal of type 1 may not be a k-bi-quasi hyperideal of R.

Example 3.6. *Let $(R, \oplus, \odot, \leq)$ be the ordered semihyperring defined as in Example **3**.2. It is easy to see that $A = \{0, a\}$ is a k-bi-quasi hyperideal of type 1 of R, but it is not a k-bi-quasi hyperideal of R.*

Lemma 3.3. *Let $(R, +, \cdot, \leq)$ be an ordered semihyperring and $\{I_k \mid k \in \Lambda\}$ be a family of left k-bi-quasi hyperideals of R. Then $\bigcap_{k \in \Lambda} I_k$ is a left k-bi-quasi hyperideal of R.*

Proof. This proof is straightforward. □

The k-closure of a nonempty subset A of an ordered semihyperring R is defined by $\overline{A} = \{x \in R \mid \exists\, a, b \in A, a + x \leq b\}$.

Theorem 3.6. *Let A be a bi-quasi hyperideal of an ordered semihyperring R. Then, the following assertions are equivalent:*

(1) *A is a k-bi-quasi hyperideal of type 1 of R.*
(2) *$\overline{A} = A$.*

Proof. (1) ⇒ (2): Assume that (1) holds. Clearly, $A \subseteq (A] \subseteq \overline{A}$. Let $x \in \overline{A}$. Then $a + x \leq b$ for some $a, b \in A$. Hence, for any $u \in a + x$, we have $u \leq b$. Since A is a bi-quasi hyperideal of R, it follows that $u \in A$. Thus, $a + x \subseteq A$. Since A is a k-bi-quasi hyperideal of type 1 of R, we have $x \in A$. So $\overline{A} \subseteq A$ and thus $\overline{A} = A$.

(2) ⇒ (1): Assume that $\overline{A} = A$. Let $x \in R$ such that $a + x \subseteq A$ for some $a \in A$. Then, for any $u \in a + x$, we have $u \in A$. Thus $a + x \leq u$ for some $a, u \in A$. So, $x \in \overline{A}$. By assumption, we have $x \in A$. Therefore, A is a k-bi-quasi hyperideal of type 1 of R. □

Theorem 3.7. *Let A be a left k-bi-quasi hyperideal of type 1 of an ordered semihyperring $(R, +, \cdot, \leq)$. Then $(A]$ is a left k-bi-quasi hyperideal of type 1 of R generated by A.*

Proof. First of all, we show that $(A]$ is closed under hyperaddition $+$. Let $x, y \in (A]$. Then there exist $a, b \in A$ such that $x \leq a$ and $y \leq b$. Since R is an ordered semihyperring, we have $x + y \leq a + b \subseteq A$. Hence, for any $u \in x + y$, there exists $v \in A$ such that $u \leq v$. Thus $u \in (A]$ and so $x + y \subseteq (A]$. Also, we have

$$(A] \cdot (A] \subseteq (A \cdot A] \subseteq (A] = A.$$

Since A is a bi-quasi hyperideal of R, it follows $(A] = A$. So, we have

$$(R \cdot (A]] \cap ((A] \cdot R \cdot (A]] = (R \cdot A] \cap (A \cdot R \cdot A] \subseteq A.$$

Since A is a left bi-quasi hyperideal of R, we have $(A] \subseteq A$. So, $((A]] \subseteq (A]$ and hence $(A]$ is a left bi-quasi hyperideal of R. If B is a hyperideal of R such that $A \subseteq B$, then $(A] \subseteq (B] \subseteq B$. So, $(A] \subseteq B$. Finally, we prove that $\overline{(A]} = (A]$. It is clear that $A \subseteq (A] \subseteq \overline{A} \subseteq \overline{(A]}$; Now, let $x \in \overline{(A]}$. Then there exist $a, b \in (A]$ such that $a + x \leq b$. So, for any $u \in a + x$, $u \leq b$ for some $b \in A$. Since A is a left bi-quasi hyperideal of R, we have $b \in A$. This means that $a + x \subseteq A$. Since A is a left k-bi-quasi hyperideal of type 1 of R, it follows that $x \in A$. Theus, $\overline{(A]} \subseteq A = (A]$ and so $\overline{(A]} = (A]$. By Theorem **3**.6, $(A]$ is a left k-bi-quasi hyperideal of type 1 of R. □

4. Conclusions

In this study, we extended the concept of a left bi-quasi ideal of a semiring to an ordered semihyperring. Also, we introduced the notion of left k-bi-quasi hyperideal (of type 1) and then we obtained some related basic results. When we deal with k-bi-quasi hyperideals of ordered semihyperrings, it is natural to talk about fuzzy k-bi-quasi hyperideal. According to the research results, it is suggested to define and investigate some properties of fuzzy k-bi-quasi hyperideals in ordered semihyperrings. In continuity of this paper, we study 2-prime (2-prime of type 1) bi-quasi-hyperideals of ordered semihyperrings.

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REFERENCES

[1] R. Ameri and H. Hedayati, *On k-hyperideals of semihyperrings*, J. Discrete Math. Sci. Cryptogr. **10** (2007), No. 1, 41–54.

[2] R. Ameri and H. Hedayati, *Homomorphism and quotient of fuzzy k-hyperideals*, Ratio Mathematica, **20** (2010), 148–165.

[3] P. Corsini, *Prolegomena of Hypergroup Theory*, Second edition, Aviani editore, Italy, 1993.

[4] P. Corsini and V. Leoreanu, *Applications of Hyperstructure Theory*, Advances in Mathematics, Kluwer Academic Publishers, Dordrecht, 2003.

[5] B. Davvaz, *Rings derived from semihyperrings*, Algebras Groups Geom. **20** (2003), 245–252.

[6] B. Davvaz and V. Leoreanu-Fotea, *Hyperring Theory and Applications*, International Academic Press, Palm Harbor, USA, 2007.

[7] B. Davvaz and S. Omidi, *Basic notions and properties of ordered semihyperrings*, Categ. General Alg. Structures Appl. **4** (2016), No. 1, 43–62.

[8] B. Davvaz, P. Corsini and T. Changphas, *Relationship between ordered semihypergroups and ordered semigroups by using pseudoorder*, European J. Combinatorics, **44** (2015), 208–217.

[9] A. Dehghan Nezhad, S. M. Moosavi Nejad, M. Nadjafikhah and B. Davvaz, *A physical example of algebraic hyperstructures: Leptons*, Indian Journal of Physics, **86** (2012), No. 11, 1027–1032.

[10] M. Farshi, B. Davvaz and S. Mirvakili, *Degree hypergroupoids associated with hypergraphs*, Filomat. **28** (2014), 119–129.

[11] A. P. Gan and Y. L. Jiang, *On ordered ideals in ordered semirings*, J. Math. Res. Exposition, **31** (2011), No. 6, 989–996.

[12] Z. Gu and X. Tang, *Ordered regular equivalence relations on ordered semihypergroups*, J. Algebra, **450** (2016), 384–397.

[13] D. Heidari and B. Davvaz, *On ordered hyperstructures*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.**73** (2011), No. 2, 85–96.

[14] X. Huang, Y. Yin and J. Zhan, *Characterizations of semihyperrings by their* $(\epsilon_\gamma, \epsilon_\gamma \vee q_\delta)$*-fuzzy hyperideals*, J. Appl. Math. 2013 (2013), 13–pages.

[15] O. Kazanci, S. Yilmaz and B. Davvaz, *On fuzzy ordered hyperideals in ordered semihyperrings*, Advances in Fuzzy Systems , 2019 (2019), 7–pages.

[16] M. Krasner, *A class of hyperrings and hyperfields*, International J. Math. and Math. Sci. **6** (1983), 307–312.

[17] F. Marty, *Sur une generalization de la notion de groupe*, 8^{iem} Congres Math. Scandinaves, Stockholm, Sweden, 1934, 45–49.

[18] S. Omidi and B. Davvaz, *Construction of ordered regular equivalence relations on ordered semihyperrings*, Honam Mathematical J. **40** (2018), No. 4, 601–610.

[19] S. Omidi and B. Davvaz, *Contribution to study special kinds of hyperideals in ordered semihyperrings*, J. Taibah Univ. Sci. **11** (2017), No. 6, 1083–1094.

[20] S. Omidi and B. Davvaz, *Foundations of ordered (semi)hyperrings*, J. Indones. Math. Soc. **22(2)** (2016), 131–150.

[21] B. Pibaljommee and W. Nakkhasen, *Connections of* (m, n)*-bi-quasi hyperideals in semihyperrings*, Thai J. Math. Special Issue (2019), 39–48.

[22] M. M. K. Rao, *Left bi-quasi ideals of semirings*, Bull. Int. Math. Virtual Inst. **8** (2018), 45–53.

[23] P. Senarat and B. Pibaljommee, *Prime ordered k-bi-ideals in ordered semirings*, Quasigroups and Related Systems, **25** (2017), 121–132.

[24] J. Tang, B. Davvaz and X. Y. Xie, *An investigation on hyper S-posets over ordered semihypergroups*, Open Mathematics, **15** (2017), 37–56.

[25] J. Tang, X. Feng, B. Davvaz and X. Y. Xie, *A further study on ordered regular equivalence relations in ordered semihypergroups*, Open Mathematics, **16** (2018), 168–184.

[26] H. S. Vandiver, *Note on a simple type of algebra in which cancellation law of addition does not hold*, Bull. Amer. Math. Soc.(N.S.), **40** (1934), 914–920.

[27] T. Vougiouklis, *On some representation of hypergroups*, Ann. Sci. Univ. Clermont-Ferrand II Math. **26** (1990), 21–29.

[28] T. Vougiouklis, *Hyperstructures and Their Representations*, Hadronic Press, Florida, 1994.