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SETVALUED CONTRACTION MAPPING PRINCIPLE IN GENERALIZED METRIC SPACES

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In this paper we establish a multivalued contraction mapping principle in a space which is a generalization of metric spaces in which infinite distance between two points is admissible. The result is supported with an example. The space is assumed to have a partial ordering defined on it. We also assume some order conditions in our theorem with respect to this partial order. A discussion is provided in which we indicate the difference of our result with the Nadler's result on multivalued contractions in metric spaces.

Keywords: Multivalued mapping, generalized Hausdorff metric, multivalued monotone increasing mapping, partially ordered set.
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1. Introduction

The purpose of the paper is to establish a multivalued contraction mapping principle in a generalized metric space which was defined by Luxemberg [1] by allowing the metric to take up value from the extended real number system, that is, by incorporating the possibility of an infinite distance between two points. Such structures appear naturally as, for instance, in the consideration sets of functions defined on arbitrary domains. We cite an example later. Banach's fixed point theorem, which is widely recognized as the starting point of metric fixed point theory, was successfully extended to generalized metric spaces in the work of Diaz et al [2]. Like the Banach's result in metric spaces, the result of Diaz et al is also instrumental in proving many important results. We note some of its applications in the works [3, 4, 5, 6, 7].

The Banach's fixed point result was extended to the domain of set valued analysis by Nadler in 1969 [8]. The multivalued version of the contraction mapping principle proved by Nadler is also known as Nadler's theorem and is considered as one of the important results in setvalued analysis. Today the fixed point theory of mutivalued functions has a large literature of which references [9, 10, 11, 12] are some recent instances.

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Fixed point theory has deveoped in partially ordered metric spaces in recent time through a good number of papers. An early reference in this area is the work of Turinici [13] in which partial ordering was considered in uniform spaces. Later, Nieto et al [14] and Ran et al [15] worked with such structures which was followed by more recent works like [16, 17, 18, 19, 20, 21, 22]. A remarkable feature of this deveopment is a blending of analytic and order theoretic approaches in the methodology.

The purpose of the paper is to establish a multivalued contraction mapping principle in a generalized metric space which has additionally a partial ordering defined on it. Both analytic and order theoretic conditions are used in the theorem. There is an illustrative example. A comparison with Nadler's result is given in the conclusion.

The followings are the essential mathematical preliminaries for our discussion in this paper.

We recall the definition of generalized metric space by Luxemberg [1].

Definition 1.1 (**Generalized metric space**[1])**.** Let X be a nonempty set. A function $e : X \times X \to [0, \infty]$ is called a generalized metric on X if e satisfies the following properties

(1) $e(x,\ y) = 0$ if and only if $x = y$;
(2) $e(x,\ y) = e(y,\ x)$ for all $x, y \in X$;
(3) $e(x,\ z) \leq e(x,\ y) + e(y,\ z)$ for all $x, y, z \in X$;

Then the pair $(X,\ e)$ is called a generalized metric space.

A sequence $\{x_n\}$ converges to x if $e(x_n,\ x) \to 0$ as $n \to \infty$. A sequence $\{x_n\}$ in X is a e-Cauchy sequence if $\lim_{n,m \to \infty} e(x_n,\ x_m) = 0$. A generalized metric space $(X,\ e)$ is said to be complete if every e-Cauchy sequence in X is e-convergent, that is, $\lim_{n,m \to \infty} e(x_n,\ x_m) = 0$ for a sequence $x_n \in X$ implies the existence of an element $x \in X$ with $\lim_{n \to \infty} e(x,\ x_n) = 0$ [2]. By (1) and (3) in the above definition, the limit of the sequence $\{x_n\}$ is uniquely determined.

Definition 1.2. For any generalized metric space $(X,\ e)$, for $x \in X$ and $A(\neq \phi) \subset X$, we write $e(x,\ A)$ as

$e(x,\ A) = inf\{e(x,\ y) : y \in A, e(x,\ y) < \infty\}$ whenever the set $\{y : y \in A$ and $e(x,\ y) < \infty$ is non-empty$\}$,

$e(x,\ A) = \infty$, otherwise.

Definition 1.3 (**Generalized Hausdorff distance**)**.** Let $(X,\ e)$ be a generalized metric space. Then the generalized Hausdorff metric E introduce by e is defined as follows.

For each pair of nonempty subsets A and B of X,

$$E(A,\ B) = max\{\sup_{x\in A} d(x,\ B),\ \sup_{y\in B} d(y,\ A)\}.$$

We take, $E(\emptyset,\ \emptyset) = 0$ and $E(A,\ \emptyset) = E(\emptyset,\ A) = \infty$ for nonempty A.

Lemma 1.1. Let $(X,\ e)$ be a generalized metric space. $E : 2^X \times 2^X \to [0, \infty)$ is an extended real-valued function such that for all $A, B, C \in 2^X$, the following properties are satisfied:
$E(A,\ B) = 0$ if and only if $A = B$,

$E(A,\ B) = E(B,\ A)$,
$E(A,\ B) \leq E(A,\ C) + E(C,\ B)$.

Proof. Except for the triangle inequality, these properties follow from the definition. If any of A, B, C are empty, then the triangular inequality trivially follows. We assume that all A, B, Cs are non-empty.

For $a \in A$, $b \in B$, and $c \in C$, we have $e(a,\ B) \leq e(a,\ b) \leq e(a,\ c) + e(c,\ b)$,
so, $e(a,\ B) \leq e(a,\ c) + e(c,\ B) \leq e(a,\ c) + E(C,\ B)$.
Taking the infimum on the rightside with respect to $c \in C$, we get
$e(a,\ B) \leq e(a,\ C) + E(C,\ B) \leq E(A,\ C) + E(C,\ B)$.
So $\sup\limits_{a\in A} e(a,\ B) \leq E(A,\ C) + E(C,\ B)$.
Similarly, $\sup\limits_{b\in B} e(b,\ A) \leq E(A,\ C) + E(C,\ B)$.
Therefore, $E(A,\ B) \leq E(A,\ C) + E(C,\ B)$.
If we take $E(A,\ B) = \infty$, then all the above properties also satisfied.
This completes the proof of the lemma. □

Remark 1.1. It is noted that the generalized Hausdorff metric is defined for any two non-empty subsets of X, whereas a Hausdorff metric on a metric space is only defined on the set of closed and bounded subsets of the metric space. It gives more generality to the generalized Hausdorff metric. For a non-trivial example of the generalized metric space and the generalized Hausdorff metric we refer to example 3.1.

The following lemma is direct consequence of the definition of generalized Hausdorff metric.

Lemma 1.2. Let $(X,\ e)$ be a generalized metric space and A, B be two non-empty subsets of X. Then for any $a \in A$ and $\epsilon > 0$, there exists $b \in B$ such that $e(a,\ b) \leq E(A,\ B) + \epsilon$.

Definition 1.4. In a partially ordered set $(X,\ \preceq)$, for $x \in X$ and $A(\neq \phi) \subset X$, we define $x \preceq A$ as $x \preceq y$ for all $y \in A$.

Definition 1.5. A mapping $T : X \to 2^X - \{\phi\}$ is said to be monotone increasing if for all $x \in X$, $x \preceq Tx$ implies $y \preceq Ty$ for all $y \in Tx$.

Remark 1.2. It is noted that there are several other notions of monotonicity associated with multivalued mappings as, for instance, those discussed in [23, 24, 25].

A point $z \in X$ is a fixed point of a mapping $T : X \to 2^X$ if $z \in Tz$.

Definition 1.6. Let $T : X \to 2^X$ be a multivalued mapping from a non-empty set X to 2^X. A point $z \in X$ is an approximate fixed point of T if $e(z,\ Tz) = inf\{e(z,\ y) : y \in Tz\} = 0$.

The following are some features of the present work.

- We define generalized Hausdorff metric between two arbitrary subsets of a generalized metric space.
- We assume a partial order on the generalized metric space.

- The contraction we define in a setvalued mapping which can assume any non empty subset of X.
- Under certain circumstances it is shown that there is an approximate fixed point of the mapping.
- Our result is an extension of the result of [2] in generalized metric space with a partial order. On the other hand, it is also an extension of Nadler's theorem [8] in partially ordered generalized metric spaces.
- The result is illustrated with an example.

2. **Main Result**

Theorem 2.1. *Let $(X,\ e)$ be a complete generalized metric space with an additional structure of partial order $\preceq$ defined on it. Suppose $T : X \to 2^X - \{\phi\}$ is a multivalued mapping which satisfies*

$$E(Tx,\ Ty) \leq ke(x,\ y) \text{ where } 0 < k < 1, \tag{2.1}$$

whenever $x \preceq y$ and $x, y \in X$.

Let T be a monotone increasing operator. It is assumed that there exists an element $x_0 \in X$ such that $x_0 \preceq Tx_0$. Further, we assume that for any sequence $\{x_n\}$, $x_n \to x$ and $x_n \preceq x_{n+1}$ for all n jointly imply that $x_n \preceq x$ for all $n \geq 1$.

Then corresponding to x_0, there exists a sequence $\{x_n\}$ such that either of the following two holds:

(1) *$E(Tx_n,\ Tx_{n+1}) = \infty$, for all $n \geq 0$,*
(2) *$\{x_n\}$ converges to an approximate fixed point of T.*

Proof. Let $x_1 \in Tx_0$ be arbitrary. By the conditions of the theorem that $x_0 \preceq Tx_0$, and T is monotone increasing we have, $x_0 \preceq x_1$. By virtue of lemma 1.2 we next choose $x_2 \in Tx_1$ such that

$$e(x_1,\ x_2) \leq E(Tx_0,\ Tx_1) + k. \tag{2.2}$$

Since $x_0 \preceq Tx_0$ and $x_1 \in Tx_0$ by the monotone increasing property of T, we have $x_1 \preceq Tx_1$. Since $x_2 \in Tx_1$, we have $x_1 \preceq x_2$. Again, by lemma 1.2, we choose $x_3 \in Tx_2$ such that

$$e(x_2,\ x_3) \leq E(Tx_1,\ Tx_2) + k^2. \tag{2.3}$$

Since $x_1 \preceq Tx_1$, and since $x_2 \in Tx_1$, it follows from the monotone increasing property of T that $x_2 \preceq Tx_2$. It then follows that $x_2 \preceq x_3$ since $x_3 \in Tx_2$.

Proceeding in the above manner we have a sequence $\{x_n\}$ for which

$$x_{n+1} \in Tx_n, x_n \preceq x_{n+1} \text{ for all } n \geq 0 \tag{2.4}$$

and that

$$e(x_n,\ x_{n+1}) \leq E(Tx_{n-1},\ Tx_n) + k^n \text{ for all } n \geq 1. \tag{2.5}$$

We have the following two cases.

$E(Tx_n,\ Tx_{n+1}) = \infty$, for all n which is one of the two alternative conclusions of the theorem.

Alternatively, we can have that $E(Tx_N,\ Tx_{N+1}) < \infty$ for some N.

Then, by (2.1) and (2.5), we have

$$e(x_{N+1},\ x_{N+2}) \leq E(Tx_N,\ Tx_{N+1}) + k^{N+1} < \infty \tag{2.6}$$

and

$$E(Tx_N,\ Tx_{N+1}) \leq ke(x_N,\ x_{N+1}) < \infty. \tag{2.7}$$

It follows from (2.6) and (2.7) that

$$e(x_{N+1},\ x_{N+2}) \leq E(Tx_N,\ Tx_{N+1}) + k^{N+1} \leq ke(x_N,\ x_{N+1}) + k^{N+1} < \infty. \tag{2.8}$$

Again it follows from (2.1) and (2.5) that

$$e(x_{N+2},\ x_{N+3}) \leq E(Tx_{N+1},\ Tx_{N+2}) + k^{N+2} \leq ke(x_{N+1},\ x_{N+2}) + k^{N+2} < \infty.$$

Proceeding as in the above, we have that for all $i \geq 1$,

$$e(x_{N+i+1},\ x_{N+i+2}) \leq E(Tx_{N+i},\ Tx_{N+i+1}) + k^{N+i+1} \leq ke(x_{N+i},\ x_{N+i+1}) + k^{N+i+1}. \tag{2.9}$$

By the following successive applications of (2.9), we obtain, for all $i \geq 1$,

$$\begin{aligned} e(x_{N+i+1},\ x_{N+i+2}) &\leq ke(x_{N+i},\ x_{N+i+1}) + k^{N+i+1} \\ &\leq k[ke(x_{N+i-1},\ x_{N+i}) + k^{N+i}] + k^{N+i+1} \\ &\leq k^2 e(x_{N+i-1},\ x_{N+i}) + 2k^{N+i+1} \\ &\quad \ldots\ldots \\ &\leq k^i e(x_{N+1},\ x_{N+2}) + ik^{N+i+1}. \end{aligned}$$

Let $q > p > N$. Then $p = N + i + 1$ and $q = N + j + 1$. Then

$$\begin{aligned} e(x_p,\ x_q) &= e(x_{N+i+1},\ x_{N+j+1}) \\ &\leq e(x_{N+i+1},\ x_{N+i+2}) + e(x_{N+i+2},\ x_{N+i+3}) + \ldots + e(x_{N+j},\ x_{N+j+1}) \\ &= e(x_{N+i+1},\ x_{N+i+2}) + e(x_{N+(i+1)+1},\ x_{N+(i+1)+2}) + \ldots + e(x_{N+(j-1)+1},\ x_{N+(j-1)+2}) \\ &\leq (k^i e(x_{N+1},\ x_{N+2}) + ik^{N+i+1}) + (k^{i+1} e(x_{N+1},\ x_{N+2}) + (i+1)k^{N+(i+1)+1}) + \ldots \\ &+ (k^{j-1} e(x_{N+1},\ x_{N+2}) + (j-1)k^{N+j}) \\ &= e(x_{N+1},\ x_{N+2}) \sum_{r=i}^{j-1} k^r + \sum_{r=i}^{j-1} rk^{N+r+1}. \end{aligned}$$

Since $0 < k < 1$, $\sum_{r=0}^{\infty} k^r$ and $\sum_{r=0}^{\infty} rk^r$ are convergent, it follows that

$$\lim_{p,q\to\infty} e(x_p,\ x_q) = 0.$$

We conclude that $\{x_n\}$ is a Cauchy sequence, which, by virtue of the fact that $(X,\ e)$ is a complete generalized metric spaces, converges to some point $\bar{x} \in X$, that is, $x_n \to \bar{x}$ as $n \to \infty$.

In view of (2.1), by a condition of our theorem, $x_n \preceq \bar{x}$ for all n. Then

$$E(Tx_n,\ T\bar{x}) \leq ke(x_n,\ \bar{x}) \to 0 \text{ as } n \to \infty. \tag{2.10}$$

Now,

$$e(\bar{x},\ T\bar{x}) \leq e(\bar{x},\ x_n) + e(x_n,\ T\bar{x}) \leq e(\bar{x},\ x_n) + E(Tx_n,\ T\bar{x}).$$

Taking $n \to \infty$ in the above inequality, and using (2.10), we have that $e(\bar{x},\ T\bar{x}) = 0$, that is,

$\inf\{e(\bar{x},\ w) : w \in T\bar{x}\} = 0$, that is, $\bar{x}$ is an approximate fixed point of T.

This completes the proof of the theorem. □

3. **Illustration**

In this section we illustrate our result with the help of a nontrivial example.

Example 3.1. *We consider the set* $X = \{h : h : [0,\ \infty)^2 \to [0,\ \infty)\}$*, that is, the set of all non-negative real valued functions defined on* $[0, \infty)^2$*. We define a function* $e : X \times X \to [0,\ \infty)$ *as follows. Let* $S = \{\lambda \geq 0 : |f(x,\ y) - g(x,\ y)| \leq \lambda|x - y|$ *for all* $x, y \in [0,\ \infty)\}$*,*

$$e(f,\ g) = \begin{cases} inf\, S, & if\ S\ is\ non\text{-}empty \\ \infty & if\ S\ is\ empty. \end{cases}$$

Then e *is a generalized metric on* X*. We defined a relation* $\preceq$ *on* X *as* $f \preceq g$ *whenever* $f(x,\ y) \geq g(x,\ y)$ *for all* $x, y \in [0,\ \infty)$*. Then* $\preceq$ *is a partial ordering on* X*. The corresponding generalized Hausdorff metric is given by the following:*

For every pair of non-empty subsets A, B *of* X*,*

$$E(A,\ B) = \begin{cases} inf\, W, & if\ W\ is\ non\text{-}empty \\ \infty, & if\ W\ is\ empty, \end{cases}$$

where

$$\begin{aligned} W &= \{\lambda \geq 0 : max\{\sup_{f\in A} \inf_{g\in B} |f(x,\ y) - g(x,\ y)|, \sup_{g\in B} \inf_{f\in A} |f(x,\ y) - g(x,\ y)|\} \\ &\leq \lambda|x - y|\}\ for\ all\ x, y \in [0,\ \infty)\}. \end{aligned}$$

Let $T : X \to 2^X$ *be defined as* $Tg = \{h : 0 \leq h \leq \frac{g}{2}\}$ *for* $g \in X$*.*

Then for $g_1 \preceq g_2$*, that is, for the case* $g_1(x,\ y) \geq g_2(x,\ y)$ *for all* $x, y \in [0,\ \infty)$*, we have*

$$\begin{aligned} E(Tg_1,\ Tg_2) &= inf\, \{\lambda \geq 0 : \frac{1}{2}|g_1(x,\ y) - g_2(x,\ y)| \leq \lambda|x - y|\ for\ all\ x, y \in [0,\ \infty)\} \\ &= \frac{1}{2}e(g_1,\ g_2), provided\ e(g_1,\ g_2)\ is\ finite. \end{aligned}$$

If $e(g_1,\ g_2) = \infty$*, then (2.1) is satisfied with* $k = \frac{1}{2}$*. The theorem 2.1 is applicable to this example.*

4. Conclusions

The present result differs from the result of Nadler's theorem [8] in the following ways. Firstly, the Hausdorff metric is defined between two closed and bounded sets in a metric space. But in our theorem the generalized Hausdorff metric is defined for any two arbitrary subsets. Secondly, the conclusion in the case (ii) of our theorem is the convergence to an approximate fixed point. It is a fixed point in Nadler's theorem since the sets are closed sets in the metric topology. But the metric topology is not applicable to the generalized metric space. In fact we have not considered any topology either generated by the generalized metric or otherwise on the space. Furthermore, there is a partial ordering defined on the space with respect to which the mapping is monotone increasing. When the generalized metric is a metric in the special case, and the mapping is from X to closed and bounded subsets of X, then we have the result of Nadler [8] in partially ordered metric space. The result is also a extension of the result of Diaz et al [2]. The example 3.1 is not applicable to either of the two above mentioned cases, that is, the theorem is an actual extension of these results.

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